WORKSHEET: LIMITS AT INFINITY 1

Names and student IDs: _____

Recall: We say $\lim_{x\to\infty} f(x) = L$ if one can force $f(x)$ to be as close to L as one wants by requiring that x be large enough. In particular, f has a horizontal asymptote at $y = L$. Caution: the graph of $y = f(x)$ can cross the line $y = L$, even cross it infinitely often. See the third example below. We define $\lim_{x\to-\infty} f(x) = L$ similarly. Combining this with the meaning of $\lim_{x\to a} f(x) = \infty$ and related notions, one gets the meaning of $\lim_{x\to\infty} f(x) = \infty$ etc. Caution: $\lim_{x\to\infty} f(x) = \infty$ does not imply the existence of any asymptote. Examples:

$$\lim_{x\to\infty} \frac{1}{x} = 0, \qquad \lim_{x\to\infty} \left(2 + \frac{1}{x}\right) = 2, \qquad \lim_{x\to\infty} \left(1 + \frac{\sin(8x)}{x}\right) = 1, \qquad \text{and} \qquad \lim_{x\to\infty} x^2 = \infty.$$

All the usual laws of limits at a real number a also apply to limits at $\pm\infty$.

1. Plot the functions in the examples above, to see what the behavior looks like graphically. Draw the results on your paper. (Some of the extra coefficients were chosen to make the plots look better.)

2. If $\lim_{x\to\infty} f(x) = 8$ and $\lim_{x\to\infty} g(x) = 11$, what are

$$\lim_{x\to\infty} (f(x) + g(x)), \qquad \lim_{x\to\infty} (21 - 3g(x)), \qquad \lim_{x\to\infty} f(x)g(x), \qquad \text{and} \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)}?$$

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3. Recall that $\lim_{x\to\infty} \frac{x^2}{18} = \infty$. Given that $\lim_{x\to\infty} \frac{3}{x} = 0$ (you know this) and $\lim_{x\to\infty} \frac{290x^2 - x + 18}{7x^2 + 2x + 121} = \frac{290}{7}$ (we will see later how to find this limit), what are $\lim_{x\to\infty} \left(\frac{x^2}{18} + \frac{3}{x}\right)$ and $\lim_{x\to\infty} \left(\frac{x^2}{18} - \frac{290x^2 - x + 18}{7x^2 + 2x + 121}\right)$?

4. Recall that $\lim_{x\to\infty} x^2 = \infty$. What is $\lim_{x\to\infty} (x^2 + 5\sin(8x))$?

5. Some basic limits at infinity. (Answers may be $\pm\infty$, or that the limit does not exist and is not even ∞ or $-\infty$.)

a. What is $\lim_{x\to\infty} e^x$?

b. What is $\lim_{x\to\infty} e^{-x}$? (Remember that $e^{-x} = \frac{1}{e^x}$.)

c. What is $\lim_{x\to\infty} \ln(x)$?

d. What is $\lim_{x\to\infty} \arctan(x)$? (Look at the graph of $y = \tan(x)$.)

e. What is $\lim_{x\to\infty} \sin(x)$?

6. Find the following, after doing appropriate algebra:

$$\lim_{x\to\infty} \frac{x^2}{x^3}, \qquad \lim_{x\to\infty} \frac{x^3}{x^2}, \qquad \lim_{x\to\infty} \frac{3x^2}{8x^2}, \qquad \lim_{x\to\infty} \frac{290x^2}{2x^2}, \qquad \text{and} \qquad \lim_{x\to\infty} \frac{290x^2}{-2x^2}.$$

What does this tell you about the possibilities for $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ when $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$?