# WORKSHEET SOLUTIONS: INFINITE LIMITS

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall: We say $\lim_{x\to a} f(x) = \infty$ (even though $\infty$ is not a number) if one can force $f(x)$ to be as large as one wants by requiring that $x$ be close enough to $a$. In particular, $f$ has a vertical asymptote at $x = a$, near which the graph goes up on both sides. Example: $\lim_{x\to o} \frac{1}{x^2} = \infty$.

Plot this on your calculator to see, moving the screen around to show the parts of the function near $x = 0$.

Also look at the following table of values. You see that when $x$ is close to zero (with either $x > 0$ or $x < 0$), then $x^2$ is positive and close to zero, and $\frac{1}{x^2}$ is ;positive and large.

| $x$ | $x^2$ | $\frac{1}{x^2}$ |
|---|---|---|
| 1 | 1 | 1 |
| 0.1 | 0.01 | 100 |
| 0.01 | 0.0001 | 10,000 |
| 0.0001 | 0.00000001 | 100,000,000 |
| $-1$ | 1 | 1 |
| $-0.1$ | 0.01 | 100 |
| $-0.01$ | 0.0001 | 10,000 |
| $-0.0001$ | 0.00000001 | 100,000,000 |

Similar considerations give meanings to $\lim_{x\to a} f(x) = -\infty$, $\lim_{x\to a^-} f(x) = \pm\infty$, $\lim_{x\to\infty} f(x) = \pm\infty$, etc.

1. What are the following? Try computing some values if needed. After you have answered, look at calculator graphs to check.

*Solution.* a. $\lim_{x\to 0} \left(-\frac{1}{x^2}\right) = -\infty$.

Some values:

| $x$ | $x^2$ | $-\frac{1}{x^2}$ |
|---|---|---|
| 1 | 1 | $-1$ |
| 0.1 | 0.01 | $-100$ |
| 0.01 | 0.0001 | $-10{,}000$ |
| 0.0001 | 0.00000001 | $-100{,}000{,}000$ |
| $-1$ | 1 | $-1$ |
| $-0.1$ | 0.01 | $-100$ |
| $-0.01$ | 0.0001 | $-10{,}000$ |
| $-0.0001$ | 0.00000001 | $-100{,}000{,}000$ |

When $x$ is close to zero, then $x^2$ is very close to zero and positive, so $-\frac{1}{x^2}$ is very far from zero and **negative**. □

*Solution.* b. $\lim_{x\to 0^-} \frac{1}{x} = -\infty$.

Some values:

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| $x$ | $\frac{1}{x}$ |
|---|---|
| $-1$ | $-1$ |
| $-0.1$ | $-10$ |
| $-0.01$ | $-100$ |
| $-0.0001$ | $-10{,}000$ |

When $x$ is close to zero **and negative**, then $\frac{1}{x}$ is very far from zero and **negative**. □

*Solution.* c. $\lim_{x\to 0^-} \left(\frac{1}{x} + 45\right) = -\infty$.

Some values:

| $x$ | $\frac{1}{x}$ |
|---|---|
| $-1$ | $-44$ |
| $-0.1$ | $-35$ |
| $-0.01$ | $-55$ |
| $-0.0001$ | $-9955$ |
| $-0.00001$ | $-99{,}955$ |

When $x$ is close to zero **and negative**, then $\frac{1}{x}$ is very far from zero and **negative**, so also $\frac{1}{x}+45$ is very far from zero and **negative**. □

*Solution.* d. $\lim_{x\to 2^+} \frac{3}{x-2} = \infty$.

Some values:

| $x$ | $x-2$ | $\frac{3}{x-2}$ |
|---|---|---|
| 3 | 1 | 3 |
| 2.1 | 0.1 | 30 |
| 2.01 | 0.01 | 300 |
| 2.0001 | 0.0001 | 30,000 |

When $x$ is close to 2 **and** $x > 2$, then $x-2$ is close to zero **and positive**, so $\frac{3}{x-2}$ is very far from zero and **positive**. □

*Solution.* e. $\lim_{x\to 2^+} \frac{x+1}{x-2} = \infty$.

Some values:

| $x$ | $x-2$ | $\frac{1}{x-2}$ | $x+1$ | $\frac{x+1}{x-2}$ |
|---|---|---|---|---|
| 3 | 1 | 1 | 4 | 3 |
| 2.1 | 0.1 | 10 | 3.1 | 31 |
| 2.01 | 0.01 | 100 | 3.01 | 301 |
| 2.0001 | 0.0001 | 10,000 | 3.0001 | 30,001 |

When $x$ is close to 2 **and** $x > 2$, then $x-2$ is close to zero **and positive**, and $x+1$ is close to 3, so $\frac{x+1}{x-2}$, like $\frac{3}{x-2}$, is very far from zero and **positive**. □

*Solution.* f. $\lim_{x\to-\infty}(7x^2+1)=\infty$.

Some values:

| $x$ | $x^2$ | $7x^2+1$ |
|---|---|---|
| $-1$ | 1 | 8 |
| $-10$ | 100 | 701 |
| $-100$ | 10,000 | 70,001 |
| $-10{,}000$ | 100,000,000 | 700,000,001 |

When $x$ is very far from zero and negative (that is, $x$ is negative and $|x|$ is very large), then $7x^2+1$ **positive** and very large. □

2. Suppose $\lim_{x\to3} f(x)=\infty$ and $\lim_{x\to3} g(x)=\infty$. What can you say about $\lim_{x\to3}(f(x)+g(x))$ and $\lim_{x\to3}(f(x)-g(x))$?

*Solution.* $\lim_{x\to3}(f(x)+g(x))=\infty$. If both $f(x)$ and $g(x)$ are positive and very large, then so is $f(x)+g(x)$.

You can't say anything about $\lim_{x\to3}(f(x)-g(x))$. As just one example, take $f(x)=\frac{1}{(x-3)^2}$ and $g(x)=\frac{1}{(x-3)^2}+3\pi^4$. Then $\lim_{x\to3}(f(x)-g(x))=-3\pi^4$. This is an **indeterminate form**. □