WORKSHEET SOLUTIONS: IMPLICIT DIFFERENTIATION

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall the chain rule: If g is differentiable at x and f is differentiable at $g(x)$, and if $h(x) = f(g(x))$ for all x (in a suitable open interval), then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

1. Consider the problem: use implicit differentiation to find $\frac{dy}{dx}$: $x^2 + y^2 = 49$. You **must** solve for $\frac{dy}{dx}$. Read the discussion below before trying this problem!

1a. When you differentiate $x^2 + y^2 = 49$ or $x^2 + [y(x)]^2 = 49$ with respect to x, there is one term on which you will need the chain rule. Which term is it?

Solution. The term y^2 or $[y(x)]^2$, depending on which notation you use. □

1b. Differentiate $x^2 + y^2 = 49$ or $x^2 + [y(x)]^2 = 49$ with respect to x, remembering (if you use the first notation) that y is a function of x. Show an intermediate step, using $\frac{d}{dx}(\ldots)$ notation.

Solution. Version 1: write y explicitly as a function of x, getting:

$$x^2 + y(x)^2 = 49$$

Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y(x)^2) = \frac{d}{dx}(49).$$
$$2x + 2y(x)y'(x) = 0$$

Version 2, in physicists' notation: Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y2) = \frac{d}{dx}(49).$$
$$2x + 2y\frac{dy}{dx} = 0$$

□

If your solution contains $\cancel{\frac{dy}{dx}(x^2)}$ or $\cancel{\frac{dy}{dx}(y2)}$, it is wrong!

1c. In the result you got above, solve for $\frac{dy}{dx}$ or $y'(x)$, depending on which notation you use.

Solution. Version 1, with y explicitly as a function of x:

$$2x + 2y(x)y'(x) = 0$$
$$2y(x)y'(x) = -2x$$
$$y'(x) = \frac{-2x}{2y(x)} = -\frac{x}{y(x)}.$$

Version 2, in physicists' notation:

$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}.$$

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□

2. If $y^3 = x^2 + \sin(y) + 5$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)
Steps:

2a. When you differentiate $y^3 = x^2 + \sin(y) + 5$ or $y^3 = x^2 + \sin(y(x)) + 5$ with respect to x, there are two terms on which you will need the chain rule. Which terms are they?

Solution. The terms y^3 and $\sin(y)$, or $[y(x)]^3$ and $\sin(y)$, depending on which notation you use. □

2b. Differentiate $y^3 = x^2 + \sin(y) + 5$ or $y^3 = x^2 + \sin(y(x)) + 5$ with respect to x, remembering (if you use the first notation) that y is a function of x. Show an intermediate step, using $\frac{d}{dx}(\ldots)$ notation.

Solution. Version 1, with y explicitly as a function of x:

$$y(x)^3 = x^2 + \sin(y(x)) + 5$$

Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}\big(y(x)^3\big) = \frac{d}{dx}\big(x^2\big) + \frac{d}{dx}\big(\sin(y(x))\big) = \frac{d}{dx}\big(5\big).$$
$$3y(x)^2 y'(x) = 2x + \cos(y(x))y'(x)$$

Version 2, in physicists' notation: Differentiate, **remembering to use the chain rule**:

$$\frac{d}{dx}\big(y^3\big) = \frac{d}{dx}\big(x^2\big) + \frac{d}{dx}\big(\sin(y)\big) = \frac{d}{dx}\big(5\big).$$
$$3y^2\frac{dy}{dx} = 2x + \cos(y)\frac{dy}{dx}$$

□

If your solution contains $\cancel{\frac{dy}{dx}(y3)}$, $\cancel{\frac{dy}{dx}(x^2)}$, or $\cancel{\frac{dy}{dx}(\sin(y))}$, it is wrong!

2c. In the result you got above, solve for $\frac{dy}{dx}$ or $y'(x)$, depending on which notation you use.

Solution. Version 1, with y explicitly as a function of x:

$$3y(x)^2y'(x) = 2x + \cos(y(x))y'(x)$$
$$3y(x)^2y'(x) - \cos(y(x))y'(x) = 2x$$
$$\big(3y(x)^2 - \cos(y(x))\big)y'(x) = 2x$$
$$y'(x) = \frac{2x}{3y(x)^2 - \cos(y(x))}$$

Version 2, in physicists' notation:

$$3y^2\frac{dy}{dx} = 2x + \cos(y)\frac{dy}{dx}$$
$$3y^2\frac{dy}{dx} - \cos(y)\frac{dy}{dx} = 2x$$
$$(3y^2 - \cos(y))\frac{dy}{dx} = 2x$$
$$\frac{dy}{dx} = \frac{2x}{3y^2 - \cos(y)}.$$

□