MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 5.

This homework sheet is due in class on Wednesday 30 April 2025 (week 5), in class. Write answers on a separate piece of paper, well organized and well labelled, with **each solution starting on the left margin of the page**.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

For each of the following problems, do the following.

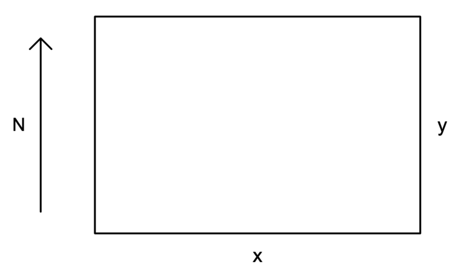
(1) Draw a picture of the situation.
(2) Name all quantities you will need to use in the solution. Include the units with the names, as in the example below. Label items in the picture with the names of the appropriate variables.
(3) Determine all the relations between the quantities in (1). Write the these relations as equations.

Do not attempt any further steps towards solving the problem.

Example.

Problem: A farmer wants to build a rectangular fenced enclosure. Because of bizarre local laws, the east fence will cost 7 florins per meter, the west fence will cost 3 florins per meter, the north fence will cost 4 florins per meter, and the south fence will cost 2 florins per meter. The farmer has 6000 florins available to build the enclosure. The farmer wants to find the lengths of the south and west fences of the enclosure with the largest area that can be built.

Solution: Here is the picture, with the appropriate quantities on it labelled:



Here, x is the length of the south and north sides of the enclosure, and y is the length of the east and west sides of the enclosure, both in meters. Also, let A be the area of the enclosure, in square meters, and let C be the total cost of the fences, in florins.

There are three relations:

$$A = xy, \qquad C = 6000, \qquad \text{and} \qquad C = 4x + 3y + 2x + 7y = 6x + 10y.$$

Comments:

(1) You don't strictly need C. Without C, you can combine the second and third relations as $6x + 10y = 6000$.
(2) It is **wrong** to just label the sides 4, 3, 2, and 7. These are not variables which are useful in the solution. It is fine to **in addition** label the sides with the costs per meter, for example 3 florins/meter etc. Do **not** just use "3", since that will be assumed to be a length.

10 points per problem, total 50 points.

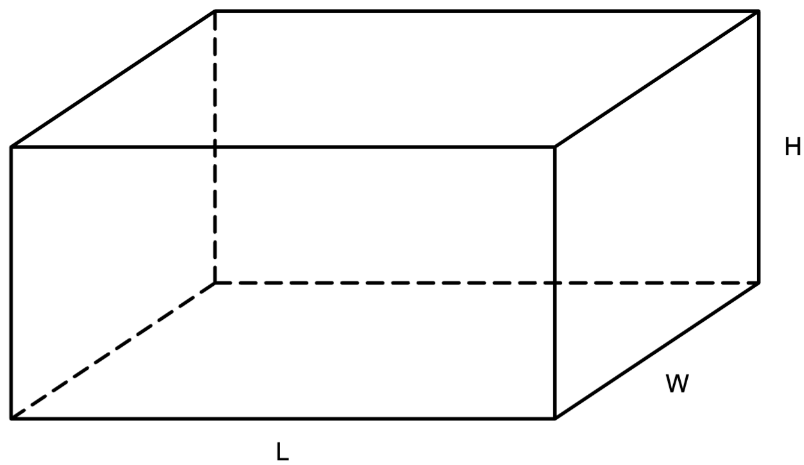
Date: 30 April 2025.

Warning: Solutions have not been properly proofread. Remember that there is extra credit for reporting errors!

Usually several solutions are possible, and in some cases multiple solutions are given.

1. A storage shed for hay will be built on an already existing raised platform. Its floor will be a rectangle which is 4 times as long as it is wide. It will have a flat roof. Material for the walls costs 2 pesos per square foot, and material for the roof costs 3 pesos per square foot. If 4000 pesos are available to purchase the materials, what is the largest possible volume the shed can have?

Solution. Here is a picture, with the appropriate quantities on it labelled:



Here, L and W are the lengths of the sides of the base, in feet, and H is the height, also in feet. We also need the volume in cubic feet; call it V, and the total cost in pesos; call it C.

Two relations are immediate: $V = LWH$ and $L = 4W$. There is a third, giving the total cost of the materials. There are four walls, each of which costs 2 pesos per square foot. The front and back (according to the picture) each have area LH, and the two ends each have area WH. Also, the roof has area LW and costs 3 pesos per square foot. Adding this up, showing one term for each of the five sides:

$$C = 2LH + 2LH + 2WH + 2WH + 3LW.$$

(The first two terms are for the front and back, the next two terms are for the two ends, and the last term is for the roof.) Simplifying the right hand side above, we get

$$C = 4LH + 4WH + 3LW.$$

Finally, $C = 4000$. □

One can combine two of these, giving the answer for the relations:

$$V = LWH, \qquad L = 4W, \qquad \text{and} \qquad 4LH + 4WH + 3LW = 4000.$$

Alternate solution. In the same picture, one can also just label the bottom front edge $4W$, using immediately the floor is 4 times as long as it is wide. Then there are two relations:

$$V = (4W) \cdot W \cdot H = 4W^2 H$$

and

$$4000 = C = 2 \cdot 4WH + 2 \cdot 4WH + 2WH + 2WH + 3 \cdot 4W \cdot W = 20WH + 12W^2,$$

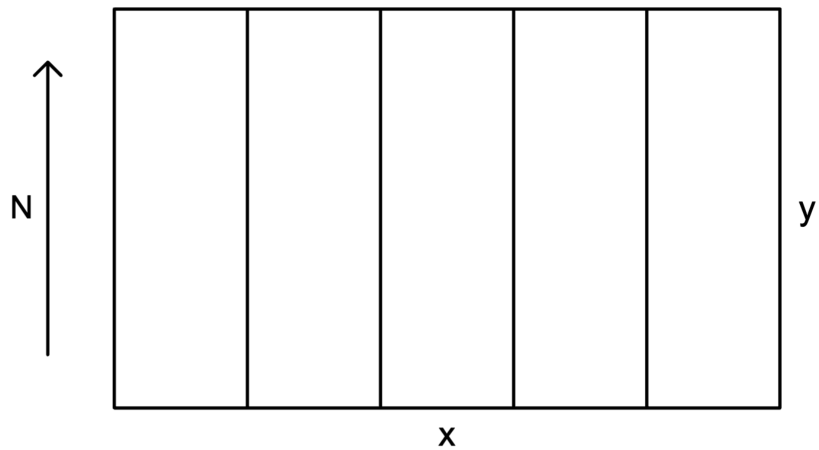
that is,

$$V = 4W^2 H \qquad \text{and} \qquad 4000 = 20WH + 12W^2.$$

□

2. A wizard wants to construct a pen for five different magical creatures, which must be kept separated from each other. (They include a crumple-horned snorkack and a spiral-horned snorkack.) The pen will be rectangular, with walls running east-west and north-south, and it will be divided into 5 parts by 4 additional north-south walls. If 3700 feet of wall can be built, what is the largest total area that can be enclosed?

Solution. Here is a picture, with the appropriate quantities on it labelled:



Here, x is the length of the south and north sides of the pen, and y is the length of the east and west sides of the pen, both in feet. Also, let A be the total area of the pen, in square feet, and let L be the total length of all the walls, in feet.

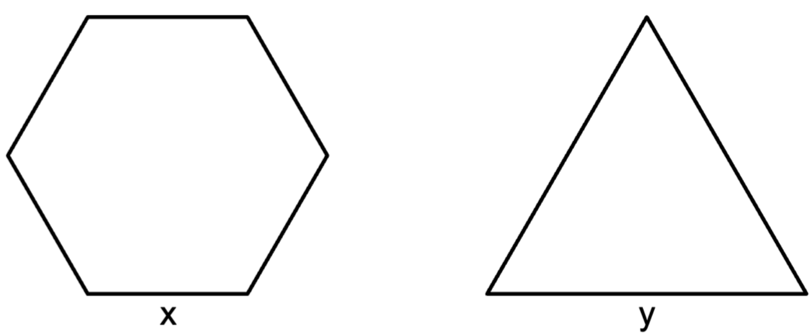
There are three relations:

$$A = xy, \qquad L = 3700, \qquad \text{and} \qquad L = 2x + 6y.$$

□

3. A piece of string 10 yards long will be cut in two sections. One section will be used to form the perimeter of a regular hexagon, and the other section will be used to form the perimeter of an equilateral triangle. How should the string be cut to minimize the combined area of the two shapes? (The area of a regular hexagon of side length x is 6 times the area of an equilateral triangle of side length x.)

Solution. Here is a picture, with the appropriate quantities on it labelled:



We let x be the length of one of the sides of the hexagon, and let y be the length of one of the sides of the triangle, both in yards. Also let H be the area of the hexagon, and let T be the area of the triangle, both in square yards. Finally let A be the total area of both figures. The perimeter of the hexagon is $6x$ and the perimeter of the triangle is $3y$, so $6x + 3y = 10$. Using the Pythagorean Theorem, the height of the triangle is

$$\sqrt{y^2 - \left(\frac{y}{2}\right)^2} = y\sqrt{1 - \frac{1}{4}} = \frac{\sqrt{3}}{2}y,$$

so

$$T = \frac{1}{2}y \cdot \frac{\sqrt{3}}{2}y = \frac{\sqrt{3}}{4}y^2.$$

Using the formula in the statement of the problem, we get

$$H = 6 \cdot \frac{\sqrt{3}}{4}x^2 = \frac{3\sqrt{3}}{2}x^2.$$

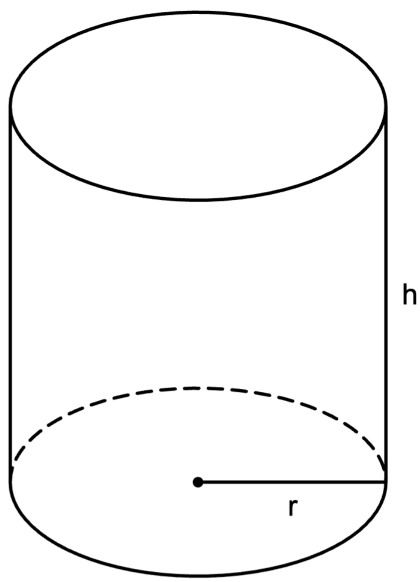
Finally, $A = H + T$. □

It is helpful, but not necessary, to name the separate areas. Without doing this, one gets the relations

$$6x + 3y = 10 \qquad \text{and} \qquad A = \frac{3\sqrt{3}}{2}x^2 + \frac{\sqrt{3}}{4}y^2.$$

4. A cylindrical can, with no top, is to have a volume of 1000 cubic inches. What dimensions minimize the total area of material needed to manufacture it?

Solution. Here is a picture, with the appropriate quantities on it labelled:



Here, r is the radius of the base of the cylinder and h is its height, both in inches. Also let V be the volume, in cubic inches, and let A be the area of the top and side combined, in square inches. The bottom has area πr^2. The side, if cut vertically and rolled out flat, is a rectangle with one side of length h and the other of length $2\pi r$ (the circumference of the circle). Therefore the relations are

$$V = \pi r^2 h \qquad \text{and} \qquad A = \pi r^2 + 2\pi r h.$$

□

It is possible to use the diameter d of the base instead of r, giving

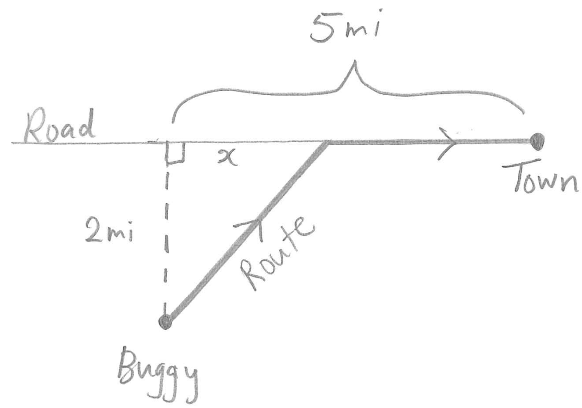
$$V = \frac{\pi d^2 h}{4} \qquad \text{and} \qquad A = \frac{\pi r^2}{4} + \pi d h.$$

One can also give separate names, say B and S, to the areas of the base and sides, both in square inches. This gives the relations

$$V = \pi r^2 h, \qquad B = \pi r^2, \qquad S = 2\pi r h, \qquad \text{and} \qquad A = B + S.$$

5. A paved road runs east through the desert into a town. A dune buggy is located 5 miles west and 2 miles south of the town, in the desert. The buggy can travel 30 mph in the desert and 30 mph on the road. Find the route that gets it into town the fastest. (It will go diagonally from its starting location to some point on the road, and then along the road to the town, or possibly directly across the desert to the town.)

Solution. Here is a picture, with the appropriate quantity on it labelled:



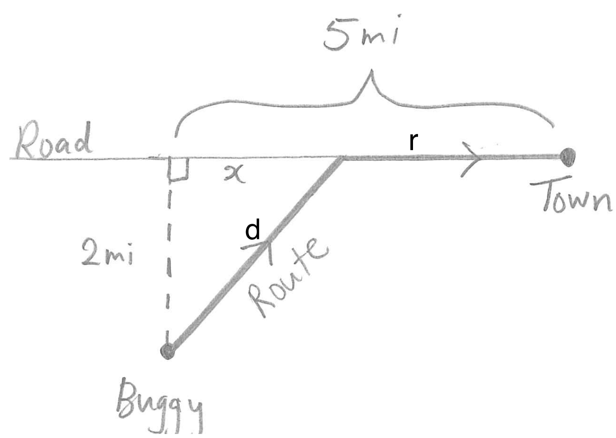
Here x is the distance from the point on the road closest to the starting location to the point at which the dune buggy meets the road, in miles. Also let T be the total time the trip takes, in hours. The relation is then

$$T = \frac{1}{30}\sqrt{2^2 + x^2} + \frac{5 - x}{50}.$$

□

It may be easier to use more names.

Alternate solution. Picture, with the appropriate quantities on it labelled:



In addition to the above, d is the distance from the starting location to point at which the dune buggy meets the road, in miles, and r is the distance it travels along the road, again in miles. The relations are now:

$$T = \frac{d}{30} + \frac{r}{50}, \qquad d = \sqrt{2^2 + x^2}, \qquad \text{and} \qquad x + r = 5.$$

□