WORKSHEET SOLUTIONS: INVERSE FUNCTIONS

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall the chain rule: If g is differentiable at x and f is differentiable at $g(x)$, and if $h(x) = f(g(x))$ for all x (in a suitable open interval), then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

You will also need $\tan'(x) = \sec^2(x)$.

1. First, just an example. Differentiate the function $q(x) = \arcsin(e^{-x})$, which is defined and differentiable for $x > 0$.

Solution. Use the chain rule twice:

$$q'(x) = \frac{d}{dx}\big(\arcsin(e^{-x})\big) = \arcsin'(e^{-x})\frac{d}{dx}\big(e^{-x}\big) = \frac{1}{\sqrt{1-(e^{-x})^2}}e^{-x}\frac{d}{dx}\big(-x\big) = -\frac{e^{-x}}{\sqrt{1-e^{-2x}}}.$$

This computation **can't be written using Leibniz notation throughout** without using extra letters, such as $v = -x$ and $u = e^{-x}$, so don't try. □

Next, let's find $\arctan'(x)$, from "scratch".

2. Is it more useful to differentiate both sides of the equation of functions $\arctan(\tan(x)) = x$ (valid when $-\frac{\pi}{2} < x < \frac{\pi}{2}$) or $\tan(\arctan(x)) = x$ (valid for all real x)? Remember that you will use the chain rule, and you want $\arctan'(x)$ somewhere in the answer.

Solution. Differentiate both sides of $\tan(\arctan(x)) = x$. □

3. Carry out the differentiation from the previous step, and solve for $\arctan'(x)$.

Solution. Since $\tan(\arctan(x)) = x$ is an equation of **functions**, we can differentiate both sides with respect to x. Use the chain rule on the left:

$$\begin{aligned}
\frac{d}{dx}\big(\tan(\arctan(x))\big) &= \frac{d}{dx}\big(x\big) \\
\tan'(\arctan(x))\arctan'(x) &= 1 \\
\sec^2(\arctan(x))\arctan'(x) &= 1 \\
\arctan'(x) &= \frac{1}{\sec^2(\arctan(x))}.
\end{aligned}$$

□

4. Use a trigonometric identity to eliminate all trigonometric functions in the previous answer. (The identity is less commonly used than the one needed for $\arcsin'(x)$, but the other steps are less complicated.)

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Solution. The identity to use is $\sec^2(\theta) = 1 + \tan^2(\theta)$. Put $\theta = \arctan(x)$, and remember that $\tan(\arctan(x)) = x$ for **all** real x, to get:

$$\sec^2(\arctan(x)) = 1 + \tan^2(\arctan(x)) = 1 + x^2.$$

Therefore

$$\arctan'(x) = \frac{1}{\sec^2(\arctan(x))} = \frac{1}{1+x^2}.$$

□

5. Now repeat for the inverse function Q (defined for all real x) of the function $h(x) = x^7 + x + 6$. You won't be able to simplify the way we did with $\arcsin'(x)$ and $\arctan'(x)$.

Solution. We have $h(Q(x)) = x$ for all real x. Use the chain rule to differentiate the functions on each side of the equation $h(Q(x)) = x$:

$$\frac{d}{dx}\big(h(Q(x))\big) = \frac{d}{dx}\big(x\big)$$

$$1 = \frac{d}{dx}\big(h(Q(x))\big) = h'(Q(x))Q'(x) = \big(7Q(x)^6 + 1\big)Q'(x).$$

$$Q'(x) = \frac{1}{7Q(x)^6 + 1}.$$

Using $h(Q(x)) = x$ and the formula for h, one can rewrite this as

$$Q'(x) = \frac{Q(x)}{7x - 42 - 6Q(x)},$$

but that isn't much of an improvement. □