WORKSHEET SOLUTIONS: SQUEEZE THEOREM AND TRIGONOMETRIC DERIVATIVES

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Solutions have not been properly proofread. Remember that there is extra credit for reporting errors!

Recall the Squeeze Theorem: if $f(x) \leq g(x) \leq h(x)$ on an open interval containing a, except at a itself, $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$. In particular, $\lim_{x\to a} g(x)$ exists.

1. Does the Squeeze Theorem apply to $\lim_{x\to 0} x^4 \sin\left(\frac{1}{x^2}\right)$? If so, what do you take for $f(x)$, $g(x)$, and $h(x)$? If not, why not?

Solution. Yes. Since $-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$ when $x \neq 0$, we can take, for $x \neq 0$,

$$f(x) = -x^4, \qquad g(x) = x^4 \sin\left(\frac{1}{x^2}\right), \qquad \text{and} \qquad h(x) = x^4.$$

Then $f(x) \leq g(x) \leq h(x)$ for all $x \neq 0$, and $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} h(x) = 0$, so

$$\lim_{x\to 0} x^4 \sin\left(\frac{1}{x^2}\right) = \lim_{x\to 0} g(x) = 0.$$

□

2. Does the Squeeze Theorem apply to $\lim_{x\to 0}(1+x^4) \sin\left(\frac{1}{x^2}\right)$? If so, what do you take for $f(x)$, $g(x)$, and $h(x)$? If not, why not?

Solution. No. Since $-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$ when $x \neq 0$, the plausible choices for $f(x)$, $g(x)$, and $h(x)$ are

$$f(x) = -(1+x^4), \qquad g(x) = (1+x^4) \sin\left(\frac{1}{x^2}\right), \qquad \text{and} \qquad h(x) = 1+x^4.$$

Then $f(x) \leq g(x) \leq h(x)$ for all $x \neq 0$, but unfortunately $\lim_{x\to 0} f(x) = -1$ and $\lim_{x\to 0} h(x) = 1$. These are not equal, so the Squeeze Theorem doesn't apply.

In fact, $\lim_{x\to 0}(1+x^4) \sin\left(\frac{1}{x^2}\right)$ does not exist. (Try graphing the function with your calculator.)

□

3. Does the Squeeze Theorem apply to $\lim_{x\to 0} x^4\left(1 + \sin\left(\frac{1}{x^2}\right)\right)$? If so, what do you take for $f(x)$, $g(x)$, and $h(x)$? If not, why not? (Be careful!)

Solution. Yes. Since $-2 \leq 1 + \sin\left(\frac{1}{x^2}\right) \leq 2$ when $x \neq 0$, we can take, for $x \neq 0$,

$$f(x) = -2x^4, \qquad g(x) = x^4\left(1 + \sin\left(\frac{1}{x^2}\right)\right), \qquad \text{and} \qquad h(x) = 2x^4.$$

Then $f(x) \leq g(x) \leq h(x)$ for all $x \neq 0$, and $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} h(x) = 0$, so

$$\lim_{x\to 0} x^4\left(1 + \sin\left(\frac{1}{x^2}\right)\right) = \lim_{x\to 0} g(x) = 0.$$

(Actually, you could even take $f(x)$ to be the constant function 0.) □

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Recall the derivatives of $\tan(x)$ and $\sec(x)$: $\tan'(x) = \sec^2(x)$ and $\sec'(x) = \sec(x)\tan(x)$.

4. Find $\frac{d}{dx}\big(\tan(x^3 + 5x)\big)$.

Solution. Use the chain rule:

$$\frac{d}{dx}\big(\tan(x^3 + 5x)\big) = \tan'(x^3 + 5x)\frac{d}{dx}\big(x^3 + 5x\big) = \sec^2(x^3 + 5x)(3x^2 + 5).$$

(All parentheses in this solution are essential.) □

5. Find $\frac{d}{dt}\big(\sec(8t)\big)$.

Solution. Use the chain rule:

$$\frac{d}{dt}\big(\sec(8t)\big) = \sec'(8t)\frac{d}{dt}\big(8t\big) = \sec(8t)\tan(8t) \cdot 8 = 8\sec(8t)\tan(8t).$$

□