MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 4 PART 2

This homework sheet is due in class on Wednesday 23 April 2025 (week 4), in class. Write answers on a separate piece of 8.5 by 11 inch paper, well organized and well labelled, with each solution starting on the left margin of the page. Or, print this page and write on it, using the back for the second problem if needed.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

One problem: 20 points.

1. (20 points.) James Tutt Snodgrass III has finished Math 251 and gone home for the holidays to his mother's farm, only to be asked to solve the following problem:

His mother wants to fence off a rectangular enclosure for llamas. The west side of the enclosure will be a wall which is already present. The north side will be an old fence which will cost \$10 per foot to adequately reinforce. The remaining two sides will consist of new fencing costing \$20 per foot. What are the dimensions of the largest enclosure that can be made for \$6000?

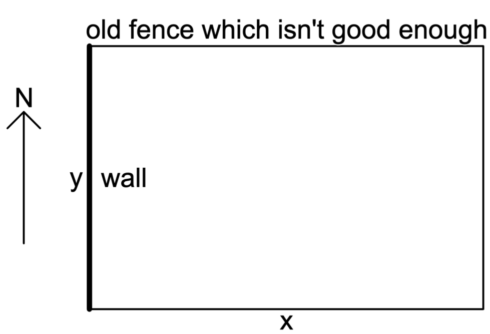
Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

Recall the list of steps on page 441 of the book, as further subdivided in class:

(1) (a) Draw a picture if possible. (Most solution files don't include pictures.)
(b) Name all the quantities which appear in the problem.
(2) (a) What quantity is to be minimized or maximized, and which of these is it?
(b) What are the restrictions on the values of the variables? (Include "degenerate" cases if possible.)
(3) Express the quantity to be minimized or maximized (from (2a)) in terms of (some of) the other variables.
(4) (a) Relate the variables in the formula in (2a) to each other.
(b) Use the relations in (4a) to write the quantity to be minimized or maximized as a function of a single variable.
(5) Determine the domain of the function in (4b) from the physical problem to be solved. (For example, lengths of fences can't be negative.)
(6) Use methods of calculus to find the absolute minimum or maximum of the function in (4b), as appropriate, on the interval in (5).

In (6), it is **not** sufficient to simply find the critical points; you must check that your function has an absolute minimum or maximum on the interval in (5) at the value of the variable you choose. (Don't use the second derivative test, which has not yet been discussed. This problem is designed to be accessible via the methods we have already seen.)

Solution. Step 1. The arrangement of fences and the wall will be a rectangle with sides running north-south and east-west. Let x be the length of the enclosure in the east-west direction, and let y be its length in the north-south direction, both measured in feet. Here is a picture:



Also, let A be the area and let C be the cost.

Step 2a. We are supposed to *maximize* the total area of the enclosure, which is A.

Step 2b. We must have $x \geq 0$ and $y \geq 0$. (The cases $x = 0$ and $y = 0$ are degenerate cases.)

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Step 3. $A = xy$.

Step 4a. The formula for A has too many variables, and ignores the fact that x and y are not independent. We must eliminate one of the variables by using the restriction on the total cost. There is nothing to be gained by using less than the allowed amount of money, so we assume the total cost is exactly 6000 (measured in dollars). The cost of the west side is 0, regardless of its length (this is where the wall is). The east side has length y feet and costs \$20 per foot, for a cost of $20y$. The south side has length x feet and costs \$20 per foot, for a cost of $20x$. The north side also has length x feet, but costs only \$10 per foot, for a cost of $10x$. Adding these together give a total cost C of

$$C = 0 + 20y + 20x + 10x = 20y + 30x.$$

Therefore

$$20y + 30x = 6000.$$

Step 4b. We ned to solve for one of the variables in terms of the other. It doesn't matter much which variable you solve for. I decided to solve for y, giving

$$y = \frac{6000 - 30x}{20} = 300 - \tfrac{3}{2}x.$$

Substitute this for y in the formula $A = xy$ and write it as a function of x:

$$A(x) = x\left(300 - \tfrac{3}{2}x\right). \qquad (1)$$

Step 5. We saw above that $x \geq 0$. Also, $y \geq 0$. Since $y = 300 - \frac{3}{2}x$, this says that $x \leq 200$. (Here is another way to see this. The total cost of the north and south sides is $30x$, and this can be at most 6000.) Therefore the correct domain is $[0, 200]$. Our problem is therefore to maximize the function $A(x) = 300x - \frac{3}{2}x^2$ for x in the closed bounded interval $[0, 200]$.

Step 6. We search for critical numbers. Multiply out the formula for $A(x)$ so that it is convenient to differentiate:

$$A(x) = 300x - \tfrac{3}{2}x^2.$$

Differentiate:

$$A'(x) = 300 - 3x.$$

Set the derivative equal to zero and solve:

$$0 = A'(x) = 300 - 3x$$
$$x = 100.$$

Since we are maximizing over the closed bounded interval $[0, 200]$ (this is why we included the "degenerate" cases $x = 0$ and $y = 0$), we need only compare the numbers $A(0)$, $A(100)$, and $A(200)$. It is easiest to use (1) for this, giving

$$A(0) = 0 \cdot \left(300 - \tfrac{3}{2} \cdot 0\right) = 0, \qquad A(100) = 100\left(300 - \tfrac{3}{2} \cdot 100\right) = 100(300 - 150) = 15000,$$

and

$$A(200) = 200\left(300 - \tfrac{3}{2} \cdot 200\right) = 200(300 - 300) = 0.$$

Clearly the largest value is at $x = 100$. Therefore the east-west length should be 100 feet, and the north-south length should be $300 - \frac{3}{2} \cdot 100 = 150$ feet. (Include the units!) □