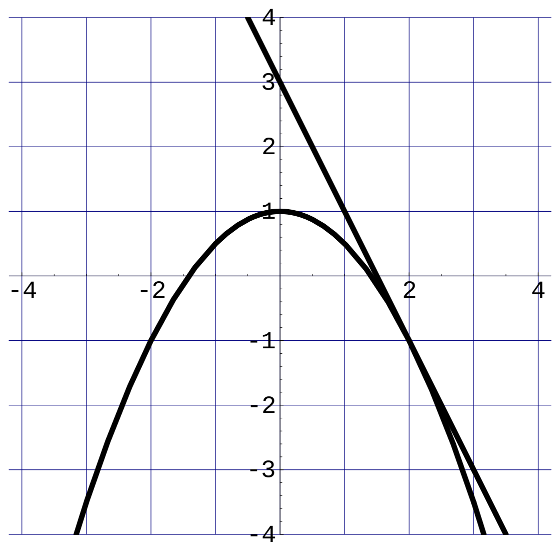
MATH 251 (PHILLIPS): SOLUTIONS TO WRITTEN HOMEWORK 4 PART 1

This homework sheet is due in class on Tuesday 22 April 2025 (week 4), in class. Write answers on a separate piece of 8.5 by 11 inch paper, well organized and well labelled, with each solution starting on the left margin of the page.

All the requirements in the sheet on general instructions for homework apply. In particular, show your work (unlike WeBWorK), give exact answers (not decimal approximations), and **use correct notation.** (See the course web pages on notation.) Some of the grade will be based on correctness of notation in the work shown.

Point values as indicated, total 46 points.

1. (10 points.) The picture below shows the graph of a function $y = f(x)$ and the tangent line to the graph at $x = 2$.



Let g be the function $g(x) = \dfrac{f(x)}{x^2+7}$. Find $g'(2)$. **Do not try to guess a formula for $f(x)$; use only the information provided.**

Solution. By the quotient rule, we have

$$g'(x) = \frac{f'(x)(x^2+7) - f(x)\frac{d}{dx}\big(x^2+7\big)}{(x^2+7)^2} = \frac{(x^2+7)f'(x) - 2xf(x)}{(x^2+7)^2}.$$

Therefore

$$g'(2) = \frac{(2^2+7)f'(2) - 2\cdot 2f(2)}{(2^2+7)^2} = \frac{11f'(2) - 4f(2)}{11^2}.$$

We can read off $f(2) = -1$ from the graph.

We now need $f'(2)$. Examining the graph, we see that the tangent line goes through the points $(2, -1)$ and $(0, 3)$. Therefore its slope is $f'(2) = \frac{3-(-1)}{0-2} = -2$.

Now substitute:

$$g'(2) = \frac{11f'(2) - 4f(2)}{11^2} = \frac{11(-2) - 4(-1)}{121} = -\frac{18}{121}.$$

For reference, the function in the graph is $f(x) = 1 - \frac{1}{2}x^2$. □

2. (10 points.) Let f and g be functions which are differentiable at -2 and which satisfy

$$f(-2) = -5, \qquad f'(-2) = -3, \qquad g(-2) = 4, \qquad \text{and} \qquad g'(-2) = 2.$$

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Let $w(x) = x - f(x)g(x)$ for all x. Find $w'(-2)$.

Solution. Using the product rule on the second part, we get:

$$w'(x) = 1 - [f'(x)g(x) + f(x)g'(x)] = 1 - f'(x)g(x) - f(x)g'(x).$$

This gives

$$w'(-2) = 1 - f'(-2)g(-2) - f(-2)g'(-2) = 1 - (-3)(4) - (-5)(2) = 23.$$

□

3. (10 points.) Use the methods of calculus to find the exact values of x at which the function $f(x) = \left(3x^2 - 5x - 5\right) e^{-x-7}$ takes its absolute minimum and maximum values on the interval $[2, 10]$.

Hint: $f'(x) = \left(-3x^2 + 11x\right) e^{-x-7}$.

Also, you will probably want a calculator to evaluate f at the critical points etc.

Solution. We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate f at all critical numbers and at the endpoints, and compare values.

To find the critical numbers, we solve the equation $f'(x) = 0$. We are already given $f'(x)$, and to solve this equation we factor it:

$$f'(x) = \left(-3x^2 + 11x\right) e^{-x-7} = -x\,(3x - 11)\, e^{-x-7}.$$

This last expression is zero when $x = 0$ and when $x = \frac{11}{3}$. (The factor e^{-x-7} is never zero.)

We now have two critical numbers, namely 0 and $\frac{11}{3}$. Of these, 0 is not in the interval under consideration, so we ignore it. (**I must see you reject this value**, since otherwise I don't know that you correctly solved the equation $f'(x) = 0$.) So we must compare the values of f at $\frac{11}{3}$ and at the endpoints 2 and 10. We evaluate:

$$f(2) = \left(3 \cdot 2^2 - 5 \cdot 2 - 5\right) e^{-9} = -3e^{-9} \approx -0.000370229,$$

$$f\left(\tfrac{11}{3}\right) = \left(3\left(\tfrac{11}{3}\right)^2 - 5\left(\tfrac{11}{3}\right) - 5\right) e^{-\frac{11}{3}-7} = \left(\tfrac{121}{3} - \tfrac{55}{3} - 5\right) e^{-32/3} = 17e^{-32/3} \approx 0.000396255,$$

and

$$f(10) = \left(3 \cdot 10^2 - 5 \cdot 10 - 5\right) e^{-10-7} = 245e^{-17} \approx 0.0000101428.$$

(You will probably use your calculator to get the decimal approximations.) The smallest of these is $f(2)$ and the largest is $f\left(\frac{11}{3}\right)$, so the absolute minimum on the interval $[2, 10]$ occurs at $x = 2$ and the absolute maximum on the interval $[2, 10]$ occurs at $x = \frac{11}{3}$. □

There is another way to find the maximum and minimum, which does not require the calculator at all. From the formula for $f'(x)$, we see that $f'(x) > 0$ for x in the interval $\left[2, \frac{11}{3}\right)$, and $f'(x) < 0$ for x in the interval $\left(\frac{11}{3}, 10\right]$. Therefore f is increasing on the interval $\left[2, \frac{11}{3}\right)$, and f is decreasing on the interval $\left(\frac{11}{3}, 10\right]$. It follows that the absolute maximum on the interval $[2, 10]$ occurs at $x = \frac{11}{3}$. The absolute minimum must be at one of the endpoints, and no calculator is needed to show that

$$f(2) = \left(3 \cdot 2^2 - 5 \cdot 2 - 5\right) e^{-9} = -3e^{-9} < 0$$

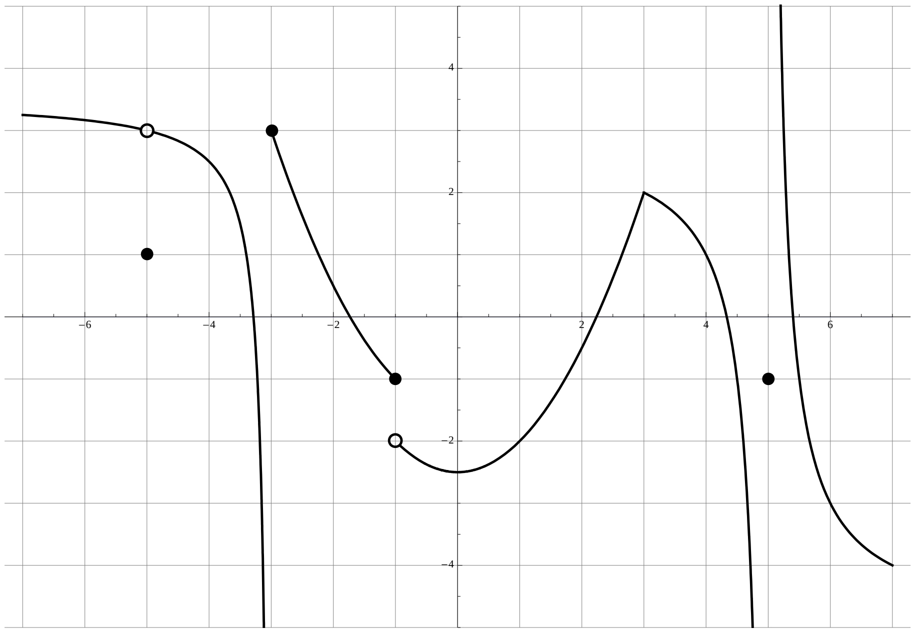
and

$$f(10) = \left(3 \cdot 10^2 - 5 \cdot 10 - 5\right) e^{-10-7} = 245e^{-17} > 0.$$

Therefore the absolute minimum on the interval $[2, 10]$ occurs at $x = 2$.

Note that $x = 0$ is not correct for the minimum, even though $f(0) = -5e^{-7} \approx -0.00455941$ is less than $f(2)$, because 0 is not in the interval $[2, 10]$.

4. (2 points/part) For the function $y = g(x)$ graphed below, answer the following questions:



(To make this faster to do and grade, the problems do **not** ask for reasons, but you should be able to give reasons. Reasons will be included in the solutions.)

(a) List all points a in $(-7, 7)$ at which g is neither continuous nor differentiable.

Solution. $a = -5$, $a = -3$, $a = -1$, and $a = 5$.
Reasons (not asked for in the problem): $\lim_{x \to -5} g(x)$ exists but is not equal to $g(-5)$. $\lim_{x \to -3} g(x)$, $\lim_{x \to -1} g(x)$, and $\lim_{x \to 5} g(x)$ don't exist.
Also, if g is not continuous at a, then g is not differentiable at a. ☐

(b) List all points a in $(-7, 7)$ at which g is continuous but not differentiable.

Solution. $a = 3$.
Reason (not asked for in the problem): There is a corner in the graph at $x = 3$, so there is no tangent line. (It is true that there are right and left tangent lines, but they have different slopes.) ☐

(c) List all points a in $(-7, 7)$ at which g is differentiable but not continuous.

Solution. None.
Reason (not asked for in the problem): for any function f, if f is differentiable at a, then f is continuous at a. ☐

(d) List all points a in $(-7, 7)$ for which $\lim_{x \to a} g(x)$ does not exist.

Solution. $a = -3$, $a = -1$, and $a = 5$.
Reasons (not asked for in the problem): at each of these points, the one sided limits disagree. At $a = -3$, the left and right limits are $-\infty$ and 3, at $a = -1$, the left and right limits are -1 and -2, and at $a = 5$, the left and right limits are $-\infty$ and ∞. ☐

(e) List all points a in $(-7, 7)$ for which $\lim_{x \to a} g(x)$ does exist but is not equal to $g(a)$. For each of these points, find $\lim_{x \to a} g(x)$.

Solution. $a = -5$, where $\lim_{x \to -5} g(x) = 3$ but $g(-5) = 1$. □

(f) List all points a in $(-7, 7)$ such that $g'(a) = 0$.

Solution. $a = 0$, or at least somewhere near there.

Reasons (not asked for in the problem): the tangent line to the graph at $x = 0$ is horizontal. □

(g) Give a rough estimate of $g'(1)$, or state that $g'(1)$ does not exist.

Solution. $g'(1) \approx 1$. (Draw a tangent line on the graph and see!) □

(h) Give a rough estimate of $g'(4)$, or state that $g'(4)$ does not exist.

Solution. $g'(4) \approx -2$. (Draw a tangent line on the graph and see!) □