WORKSHEET SOLUTIONS: DERIVATIVES FROM THE DEFINITION

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

1. Find $\displaystyle\lim_{x\to 3} \frac{x^2 - x - 6}{2x^2 + x + 7}$.

Solution. Since $2 \cdot 3^2 + 3 + 7 = 28 \neq 0$, the function $f(x) = \dfrac{x^2 - x - 6}{2x^2 + x + 7}$ is defined and continuous at $x = 3$. Therefore

$$\lim_{x\to 3} \frac{x^2 - x - 6}{2x^2 + x + 7} = \lim_{x\to 3} f(x) = f(3) = \frac{3^2 - 3 - 6}{2 \cdot 3^2 + 3 + 7} = \frac{0}{28} = 0.$$

Notation: in this computation, the expression "$\lim_{x\to 3}$" must appear everywhere it is written and nowhere else, and may **never** have an equals sign immediately **after** it. ☐

2. Find $\displaystyle\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 + x - 12}$.

Solution. Both the numerator and denominator have the value 0 when $x = 3$. Therefore the limit has the indeterminate form "$\frac{0}{0}$", so work is needed (if it can be done at all). Here, factor the numerator and denominator, and cancel common factors:

$$\lim_{x\to 3} \frac{x^2 - x - 6}{x^2 + x - 12} = \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x+4)} = \lim_{x\to 3} \frac{x+2}{x+4} = \frac{3+2}{3+4} = \frac{5}{7}.$$

Notation: in this computation, the expression "$\lim_{x\to 3}$" must appear everywhere it is written and nowhere else, and may **never** have an equals sign immediately **after** it. ☐

Recall: if f is a function defined on an open interval containing a, then the derivative of f at a is $f'(a) = \displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$, **if this limit exists**. (An alternate formulation is: the derivative of f at a is $f'(a) = \displaystyle\lim_{x\to a} \frac{f(x) - f(a)}{x - a}$, **if this limit exists**.)

3. Let $k(x) = x^2$ for all real numbers x. Find $k'(3)$ directly from the definition. (You should get $k'(3) = 6$.)

Solution. We find the limit of the difference quotient. Direct substitution gives the expression "$\frac{0}{0}$", which is an indeterminate form. Therefore we use the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} k'(3) &= \lim_{h\to 0} \frac{k(3+h) - k(3)}{h} = \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h} = \lim_{h\to 0} \frac{(3^2 + 2 \cdot 3 \cdot h + h^2) - 3^2}{h} \\ &= \lim_{h\to 0} \frac{6h + h^2}{h} = \lim_{h\to 0} \frac{(6+h)h}{h} = \lim_{h\to 0} (6+h) = 6. \end{aligned}$$

Notation: the parentheses in "$\lim_{h\to 0}(6+h)$" are **essential**. ☐

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4. Let k be as above. Let a be an arbitrary real number. Find $k'(a)$ directly from the definition. (You should get $k'(a) = 2a$.)

Solution. We find the limit of the difference quotient. Direct substitution gives the expression "$\frac{0}{0}$", which is an indeterminate form. Therefore we use the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} k'(a) &= \lim_{h\to 0} \frac{k(a+h)-k(a)}{h} = \lim_{h\to 0} \frac{(a+h)^2-a^2}{h} = \lim_{h\to 0} \frac{(a^2+2ah+h^2)-a^2}{h} \\ &= \lim_{h\to 0} \frac{2ah+h^2}{h} = \lim_{h\to 0}(2a+h) = 2a. \end{aligned}$$

□

5. If $k(x) = x^2$ for all real numbers x, then for all real numbers x, we have $k'(x) = \underline{\ 2x\ }$.

6. Expand the expression $(x+h)^3$.

Solution.

$$\begin{aligned} (x+h)^3 &= (x+h)(x+h)^2 = (x+h)(x^2+2xh+h^2) \\ &= (x^3+2x^2h+xh^2)+(x^2h+2xh^2+h^3) = x^3+3x^2h+3xh^2+h^3. \end{aligned}$$

□

7. Let $l(x) = x^3$ for all real numbers x. Let a be an arbitrary real number. Find $l'(x)$ directly from the definition. You should get $l'(x) = 3x^2$.

Solution. We find the limit of the difference quotient. Direct substitution gives the expression "$\frac{0}{0}$", which is an indeterminate form. Therefore we use the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} l'(x) &= \lim_{h\to 0} \frac{l(x+h)-l(x)}{h} = \lim_{h\to 0} \frac{(x+h)^3-x^3}{h} = \lim_{h\to 0} \frac{\left(x^3+3x^2h+3xh^2+h^3\right)-x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h+3xh^2+h^3}{h} = \lim_{h\to 0}(3x^2+3xh+h^2) = 3x^2. \end{aligned}$$

□

8. For a nonnegative integer n, let $P_n(x)$ be the function $P_n(x) = x^n$ for all real numbers x. (Take $P_0(0) = 1$.)

You have seen $P_0'(x)$, $P_1'(x)$, $P_2'(x)$, and $P_3'(x)$. What do you think $P_4'(x)$ is? What do you think $P_n'(x)$ is?

Solution. The obvious guesses are $P_4'(x) = 4x^3$ and $P_n'(x) = nx^{n-1}$. These are in fact correct, even when n isn't an integer. □