SOLUTIONS TO MIDTERM 2 REVIEW SESSION WORKSHEET FOR SPRING 2025

Not enough proofreading has been done!

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1. (20 points) You live on level ground. Your nosy neighbor has a drone with a camera, which is flying horizontally directly towards you, at 40 feet above you. You have a shotgun which you keep aimed at the drone. (You will shoot it down as soon as it crosses your property line.) When the drone is 70 feet away horizontally, it is approaching you at a horizontal speed of 20 feet per minute. How is the angle of elevation of your shotgun changing? (Be sure to include correct units.)

Solution. There is no picture in the file.

Measure time t in minutes. Let $x(t)$ be the horizontal distance, in feet, between you and the drone. Let $\theta(t)$ be the angle of elevation of the drone (measured, of course, in radians). The drone is 40 feet above you, and this height does not change. The missing picture should therefore show a right triangle, with horizontal side labelled $x(t)$, vertical side labelled 40, and the angle between the horizontal side and the hypotenuse labelled $\theta(t)$.

Since we have a right triangle

$$\tan(\theta(t)) = \frac{40}{x(t)}. \tag{1}$$

(There are other ways the relation could be written, for example, $\cot(\theta(t)) = \frac{1}{40}x(t)$.) Rewrite (1) to make it easier to differentiate:

$$\tan(\theta(t)) = 40x(t)^{-1}$$

Now use the chain rule on both sides:

$$\sec^2(\theta(t))\theta'(t) = -40x(t)^{-2}x'(t). \tag{2}$$

Let t_0 be the time at which we are interested. Put $t = t_0$ in (2), getting

$$\sec^2(\theta(t_0))\theta'(t_0) = -40x(t_0)^{-2}x'(t_0).$$

We are given

$$x(t_0) = 70 \qquad \text{and} \qquad x'(t_0) = -20.$$

(The derivative $x'(t_0)$ is negative because the horizontal distance between you and the drone is decreasing.) So

$$\sec^2(\theta(t_0))\theta'(t_0) = -40\left(\frac{1}{70^2}\right)(-20) = \frac{10 \cdot 4 \cdot 10 \cdot 2}{10^2 \cdot 7^2} = \frac{8}{7^2}. \tag{3}$$

We still need $\sec^2(\theta(t_0))$. Since

$$\sec^2(\theta(t_0)) = \tan^2(\theta(t_0)) + 1 \qquad \text{and} \qquad \tan(\theta(t_0)) = \frac{40}{x(t_0)} = \frac{40}{70} = \frac{4}{7},$$

we get

$$\sec^2(\theta(t_0)) = \frac{4^2}{7^2} + 1 = \frac{7^2+4^2}{7^2}.$$

Substituting in (3), we get

$$\left(\frac{7^2+4^2}{7^2}\right)\theta'(t_0) = \frac{8}{7^2}.$$

Therefore

$$\theta'(t_0) = \left(\frac{7^2}{7^2+4^2}\right)\left(\frac{8}{7^2}\right) = \frac{8}{7^2+4^2} = \frac{8}{65}.$$

So the angle of elevation of your shotgun is increasing at the rate of $\frac{8}{65}$ radians per minute. (Don't forget the units!) □

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2. (16 points) If $y^2 = \arctan(3x-5y) + \ln(11)$, find $\dfrac{dy}{dx}$ by implicit differentiation. (You must solve for $\dfrac{dy}{dx}$.)

Solution. More steps are shown that is necessary.

Let's write it with y as an explicit function $y(x)$ of x:

$$y(x)^2 = \arctan(3x-5y(x)) + \ln(11).$$

Differentiate with respect to x, using the chain rule on both sides:

$$\frac{d}{dx}\big(y(x)^2\big) = \frac{d}{dx}\big(\arctan(3x-5y(x)) + \ln(11)\big)$$

$$2y(x)y'(x) = \frac{1}{1+(3x-5y(x))^2}\cdot\frac{d}{dx}(3x-5y(x)) + \frac{d}{dx}(\ln(11)).$$

Since the derivative of a constant is zero, we obtain:

$$2y(x)y'(x) = \frac{1}{1+(3x-5y(x))^2}\left(3-5y'(x)\right).$$

Rearrange and factor out $y'(x)$:

$$2y(x)y'(x) + \frac{1}{1+(3x-5y(x))^2}\cdot 5y'(x) = \frac{3}{1+(3x-5y(x))^2}$$

$$\left(2y(x) + \frac{5}{1+(3x-5y(x))^2}\right)y'(x) = \frac{3}{1+(3x-5y(x))^2}.$$

Therefore

$$y'(x) = \frac{\left(\frac{3}{1+(3x-5y(x))^2}\right)}{\left(2y(x)+\frac{5}{1+(3x-5y(x))^2}\right)} = \frac{3}{(1+(3x-5y(x))^2)\left(2y(x)+\frac{5}{1+(3x-5y(x))^2}\right)}$$

$$= \frac{3}{2y(x)\big(1+(3x-5y(x))^2\big)+5}.$$

For those who prefer the other notation, here it is written with $\dfrac{dy}{dx}$. Differentiate with respect to x, using the chain rule on both sides, just as before:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}\big(\arctan(3x-5y) + \ln(11)\big)$$

$$2y\frac{dy}{dx} = \frac{1}{1+(3x-5y)^2}\cdot\frac{d}{dx}(3x-5y) + \frac{d}{dx}(\ln(11)).$$

Since the derivative of a constant is zero, we obtain:

$$2y\frac{dy}{dx} = \frac{1}{1+(3x-5y)^2}\left(3 - 5\frac{dy}{dx}\right).$$

Rearrange and factor out $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} + \frac{1}{1+(3x-5y)^2}\left(5\frac{dy}{dx}\right) = \frac{3}{1+(3x-5y)^2}$$

$$\left(2y + \frac{5}{1+(3x-5y)^2}\right)\frac{dy}{dx} = \frac{3}{1+(3x-5y)^2}.$$

Therefore

$$\frac{dy}{dx} = \frac{\left(\frac{3}{1+(3x-5y)^2}\right)}{\left(2y+\frac{5}{1+(3x-5y)^2}\right)} = \frac{3}{(1+(3x-5y)^2)\left(2y+\frac{5}{1+(3x-5y)^2}\right)}$$

$$= \frac{3}{2y\big(1+(3x-5y)^2\big)+5}.$$

□

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3. (25 points) A jewelry container is supposed to have a top and bottom which are triangles **similar to** the right triangle with side lengths 3, 4, and 5 cm. (The side lengths need not be 3, 4, and 5 cm: these only give the proportions.) The sides of the container are vertical rectangles. If its volume is supposed to be 96 cubic centimeters, what should the height be to minimize the total surface area?

Solution. There is no picture in this file. The missing picture should show a column with a flat top and bottom, whose a cross section is a right triangle.

We take the lengths of the sides of the base triangle to be $3r$, $4r$, and $5r$ for some $r > 0$. (The degenerate case $r = 0$ does not work: it gives volume 0.) Let the height of the container be h.

Since the base is a right triangle with hypotenuse $5r$, its area is $\frac{1}{2}(3r)(4r)$. The volume of the container is therefore

$$V = \frac{1}{2}(3r)(4r)h = 6r^2h. \tag{4}$$

We want to minimize the surface area S, which is the sum of five terms: for the top, bottom, and three sides. It is

$$S = \frac{1}{2}(3r)(4r) + \frac{1}{2}(3r)(4r) + 3rh + 4rh + 5rh = 12r^2 + 12rh.$$

Use (4) and the assumption $V = 96$ to get

$$h = \frac{16}{r^2}. \tag{5}$$

Therefore

$$S(r) = 12r^2 + 12r\left(\frac{16}{r^2}\right) = 12r^2 + 12 \cdot 16 \cdot r^{-1}.$$

For the domain, first, $r > 0$. Also, $h > 0$. (Again, the degenerate case $h = 0$ does not work: it gives volume 0.) Knowing $h > 0$ gives no new information, so the domain is $(0, \infty)$.

We find the critical points. We get

$$S'(r) = 24r - 12 \cdot 16 \cdot r^{-2}.$$

This is zero when

$$24r = \frac{12 \cdot 16}{r^2}.$$

Multiply both sides by r^2:

$$24r^3 = 12 \cdot 16.$$

$$r^3 = 8.$$

$$r = 2.$$

This number is in fact in $(0, \infty)$.

Did we get an absolute minimum? We use the limit version of the endpoint test. Since $\lim_{r \to 0^+} 12r^2 = 0$ and $\lim_{r \to 0^+} 12 \cdot 16 \cdot r^{-1} = \infty$, we have $\lim_{r \to 0^+} S(r) = \infty$. Since $\lim_{r \to \infty} 12r^2 = \infty$ and $\lim_{r \to \infty} 12 \cdot 16 \cdot r^{-1} = 0$, we have $\lim_{r \to \infty} S(r) = \infty$. Whatever it is, $S(4)$ is at least finite, so $r = 4$ is an absolute minimum. □

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4. (17 points) Let $g(x) = x^3 - 12x + 2$. Identify the open intervals on which g is increasing, those on which g is decreasing, and all critical points, local minimums, and local maximums.

Solution. We have

$$g'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x - 2)(x + 2).$$

The solutions to $g'(x) = 0$ are therefore $x = 2$ and $x = -2$. These are the critical points.

For x in the interval $(-\infty, -2)$, we have $x < -2$, so $x^2 > 4$, so $g'(x) = 3(x^2 - 4) > 0$. Therefore g is increasing on $(-\infty, -2)$.

For x in the interval $(-2, 2)$, we have $-2 < x < 2$, so $x^2 < 4$, so $g'(x) = 3(x^2 - 4) < 0$. Therefore g is decreasing on $(-2, 2)$.

For x in the interval $(2, \infty)$, we have $x > 2$, so $x^2 > 4$, so $g'(x) = 3(x^2 - 4) > 0$. Therefore g is increasing on $(2, \infty)$.

Since g is continuous, g is increasing on $(-\infty, -2)$, and g is decreasing on $(-2, 2)$, it follows that g has a local maximum at -2.

Since g is continuous, g is decreasing on $(-2, 2)$, and g is increasing on $(2, \infty)$, it follows that g has a local minimum at 2. □

We can use test points instead. The point -3 is in $(-\infty, -2)$ and $g'(-3) = 3((-3)^2 - 4) = 15 > 0$. Since g' is continuous, g' does not change sign on $(-\infty, -2)$. Therefore $g'(x) > 0$ on $(-\infty, -2)$, so g is increasing on $(-\infty, -2)$.

Similarly, using the point 0 in $(-2, 2)$, for which we ~~get~~ calculate $g'(0) = -12$, we get $g'(x) < 0$ on $(-2, 2)$~~.~~, whence g is decreasing on $(-2, 2)$.

Using the point 3 in ~~$(-2, 2)$~~ $(2, \infty)$, for which we ~~get~~ calculate $g'(0) = 15$, we get $g'(x) > 0$ on $(2, \infty)$~~.~~, whence g is increasing on $(2, \infty)$.

We can also use the second derivative test to check what happens at the critical points. We have $g''(x) = 6x$. Therefore $g''(-2) = -12 < 0$, so g has a local maximum at -2. Also, $g''(2) = 12 > 0$, so g has a local minimum at 2.

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5. (11 points) Find the exact value of the limit $\displaystyle\lim_{x \to -\infty} \left(\frac{2}{x^2} + 12 + 5x^3 \right)$ (possibly ∞ or $-\infty$), or explain why it does not exist, not even as $\pm\infty$. Give reasons.

Solution. We have (showing more steps than necessary)

$$\lim_{x \to -\infty} 12 = 12 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{2}{x^2} = 2 \left(\lim_{x \to -\infty} \frac{1}{x} \right)^2 = 2 \cdot 0^2 = 0.$$

Also, $\displaystyle\lim_{x \to -\infty} x^3 = -\infty$ so $\displaystyle\lim_{x \to -\infty} 5x^3 = \infty$. Since the limits of the other two summands are finite, it follows that

$$\lim_{x \to -\infty} \left(\frac{2}{x^2} + 12 + 5x^3 \right) = -\infty.$$

□

Since ∞ and $-\infty$ are not numbers, we can't do arithmetic on them. (In this case, the correct answer is as if you could.) Therefore, any solution containing

~~∞^3~~, ~~$(-\infty)^3$~~, or ~~$12 - \infty$~~

uses incorrect notation, and will therefore lose points.

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6. (11 points) Find the exact value of the limit $\displaystyle\lim_{x \to 2} \frac{e^{-x}}{x - 2}$ (possibly ∞ or $-\infty$), or explain why it does not exist, not even as $\pm\infty$. Give reasons.

Solution. At $x = 2$, the denominator is zero but the numerator is not. Moreover, both the denominator and the numerator are continuous at ~~4~~2. Therefore the function $f(x) = \frac{e^{-x}}{x-2}$ has a vertical asymptote at $x = 2$.

For $x < 2$ but very close to 2, the denominator is negative and very close to zero. The numerator is very close to $e^{-2} = 1/e^2 > 0$, so is positive and not close to zero. Therefore the quotient is negative and very far from zero. Thus

$$\lim_{x \to 2^-} \frac{e^{-x}}{x - 2} = -\infty.$$

Similarly,

$$\lim_{x \to 2^+} \frac{e^{-x}}{x - 2} = \infty.$$

Therefore $\lim_{x \to 2^-} \frac{e^{-x}}{x-2}$ does not exist, and is also not ∞ or $-\infty$. $\square$