SOLUTIONS TO MIDTERM 2 PROBLEMS

1. (1 point) True or false: Related rates problems were invented by the devil.

Solution: Well, some people think so, but wait until you see the extra credit problem on this exam.

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2. (12 points) Find the exact value of the limit $\lim_{x \to 4^+} \frac{x-8}{x-4}$ (possibly ∞ or $-\infty$), or explain why it does not exist. Give reasons.

Solution: At $x = 4$, the denominator is zero but the numerator is not. Moreover, both the denominator and the numerator are continuous at 4. Therefore the function $f(x) = \frac{x-8}{x-4}$ has a vertical asymptote at $x = 4$.

For $x > 4$ but very close to 4, the denominator is positive and very close to zero. The numerator is very close to $4 - 8 = -4 < 0$, so is negative and not close to zero. Therefore the quotient is negative and very far from zero. Thus

$$\lim_{x \to 4^+} \frac{x-8}{x-4} = -\infty.$$

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3. (20 points) If $\sin(y) = (4x - y)^3 - \ln(x)$, find $\frac{dy}{dx}$ by implicit differentiation. (You must solve for $\frac{dy}{dx}$.)

Solution. Let's write it with y as an explicit function $y(x)$ of x:

$$\sin(y(x)) = (4x - y(x))^3 - \ln(x).$$

Differentiate both sides with respect to x, using the chain rule on both sides:

$$\cos(y(x))y'(x) = 3(4x - y(x))^2(4 - y'(x)) - \frac{1}{x}.$$

Now solve for $y'(x)$:

$$\cos(y(x))y'(x) = 3(4x - y(x))^2 \cdot 4 - 3(4x - y(x))^2 y'(x) - \frac{1}{x}$$

$$\cos(y(x))y'(x) + 3(4x - y(x))^2 y'(x) = 12(4x - y(x))^2 - \frac{1}{x}$$

$$y'(x) = \frac{12(4x - y(x))^2 - \frac{1}{x}}{\cos(y(x)) + 3(4x - y(x))^2}.$$

This expression can be rearranged to

$$y'(x) = \frac{12x(4x - y(x))^2 - 1}{x\cos(y(x)) + 3x(4x - y(x))^2}.$$

However, that isn't obviously simpler and is not necessary.

For those who prefer the other notation, here it is written with $\frac{dy}{dx}$. Differentiate with respect to x, using the chain rule on both sides, just as before:

$$\cos(y)\frac{dy}{dx} = 3(4x-y)^2\left(4-\frac{dy}{dx}\right) - \frac{1}{x}.$$

Now solve for $\frac{dy}{dx}$:

$$\begin{aligned}\cos(y)\frac{dy}{dx} &= 3(4x-y)^2\cdot 4 - 3(4x-y)^2\frac{dy}{dx} - \frac{1}{x}\\ \cos(y)\frac{dy}{dx} + 3(4x-y)^2\frac{dy}{dx} &= 12(4x-y)^2 - \frac{1}{x}\\ \frac{dy}{dx} &= \frac{12(4x-y)^2-\frac{1}{x}}{\cos(y)+3(4x-y)^2}.\end{aligned}$$

As before, this expression can't be further simplified, but can be rearranged to

$$\frac{dy}{dx} = \frac{12x(4x-y)^2-1}{x\cos(y)+3x(4x-y)^2}.$$

□

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4. (27 points) A box shaped storage shed has a square base, square roof, and rectangular sides, and total volume 18 cubic yards. The floor will be just bare ground, and one of the sides will be open. What height minimizes the total area of the roof and the other three sides?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

Solution. There is no picture in this file. It should show a box shape with rectangular sides, top, and bottom.

The bottom is square. Call its side length x. Let h be the height, let A be the total area of the roof and the three sides that are not open, and let V be the volume. The top has area x^2 and each side has area xh. One of the four sides is not actually present, and neither is the base, so the total area is only $A = x^2 + 3xh$ (not ~~$2x^2 + 4xh$~~). We want to minimize this. There are too many variables.

We have $V = x^2h$ (area of base times height), and also $V = 18$. So $x^2h = 18$. It is easiest to solve for h, giving $h = \frac{18}{x^2}$. Therefore

$$A(x) = x^2 + 3x\left(\frac{18}{x^2}\right) = x^2 + \frac{3\cdot 18}{x} = x^2 + 3\cdot 18\cdot x^{-1}.$$

We must have $x > 0$. (Taking $x = 0$ gives $V = 0$.) Also $h > 0$, but this gives no new information. Therefore the domain is $(0, \infty)$.

Differentiate the last expression for $A(x)$:

$$A'(x) = 2x - 3\cdot 18\cdot x^{-2} = 2(x - 27x^{-2}).$$

This is zero when $x - 27x^{-2} = 0$. Multiplying by x^2, we get $x^3 - 27 = 0$. The only solution is $x = 3$. (~~$x = 0$~~ and ~~$x = -3$~~ are not solutions.)

Do we have a minimum at $x = 3$? Endpoint test: since $\lim_{x \to 0^+} x^2 = 0$ and $\lim_{x \to 0^+} \frac{3 \cdot 18}{x} = +\infty$, we have

$$\lim_{x \to 0^+} A(x) = \lim_{x \to 0^+} \left(x^2 + \frac{3 \cdot 18}{x} \right) = +\infty.$$

Also, since $\lim_{x \to \infty} x^2 = +\infty$ and $\lim_{x \to \infty} \frac{3 \cdot 18}{x} = 0$, we have

$$\lim_{x \to \infty} A(x) = \lim_{x \to \infty} \left(x^2 + \frac{3 \cdot 18}{x} \right) = +\infty.$$

Obviously the smallest of $\lim_{x \to 0^+} A(x)$, $\lim_{x \to \infty} A(x)$, and $A(3)$ is $A(3)$.

Now $h = \frac{18}{3^2} = 2$, in yards. $\square$

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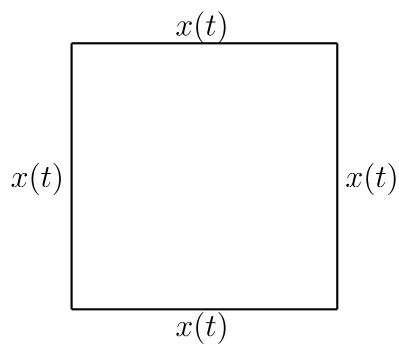
5. (20 points.) A magical square field has a side length which varies with time. On 29 February 2025, its sides were 20 meters long, and were increasing at the rate of 3 meters per week. At this time, was its area increasing or decreasing? How fast? (Be sure to include the correct units.)

Solution. Here is the picture.



As shown, let $x(t)$ be length of the sides of the square, in meters, at time t, in weeks. Let $A(t)$ be the area of the square, in square meters, at time t, again in weeks. Let t_0 represent 29 February 2025. We are given the following:

$$x(t_0) = 20 \qquad \text{and} \qquad x'(t_0) = 3.$$

(The derivative $x'(t_0)$ is positive because the side lengths are increasing.)

Our quantities are related by the equation $A(t) = x(t)^2$. Differentiating this equation with respect to t gives

$$A'(t) = 2x(t)x'(t).$$

(Don't forget to use the chain rule!) Substituting t_0 for t gives

$$A'(t_0) = 2x(t_0)x'(t_0).$$

Substituting for the known quantities gives

$$A'(t_0) = 2 \cdot 20 \cdot 3 = 120.$$

Therefore the area of the square was increasing at 120 square meters per week. (The units are required.) ☐

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6. (20 points) Let $c(x) = x^3 - 3x^2 - 24x + 5$. Identify the open intervals on which c is increasing, those on which c is decreasing, and all critical points, local minimums, and local maximums.

Solution. We have

$$c'(x) = 3x^2 - 6x - 24 = 3(x^2 - 2x - 8) = 3(x - 4)(x + 2).$$

The solutions to $c'(x) = 0$ are therefore $x = 4$ and $x = -2$. These are the critical points.

For x in the interval $(-\infty, -2)$, we have $x+2 < 0$ and $x-4 < 0$, so $c'(x) = 3(x-4)(x+2) > 0$. Therefore c is increasing on $(-\infty, -2)$.

For x in the interval $(-2, 4)$, we have $x+2 > 0$ and $x-4 < 0$, so $c'(x) = 3(x-4)(x+2) < 0$. Therefore c is decreasing on $(-2, 4)$.

For x in the interval $(4, \infty)$, we have $x+2 > 0$ and $x-4 > 0$, so $c'(x) = 3(x-4)(x+2) > 0$. Therefore c is increasing on $(4, \infty)$.

Since c is continuous, c is increasing on $(-\infty, -2)$. and c is decreasing on $(-2, 4)$, it follows that c has a local maximum at -2.

Since c is continuous, c is decreasing on $(-2, 4)$. and c is increasing on $(4, \infty)$, it follows that c has a local minimum at 4. ☐

We can use test points instead. The point -3 is in $(-\infty, -2)$ and

$$c'(-3) = 3((-3) - 4)(-3 + 2) = 3(-7)(-1) = 21 > 0.$$

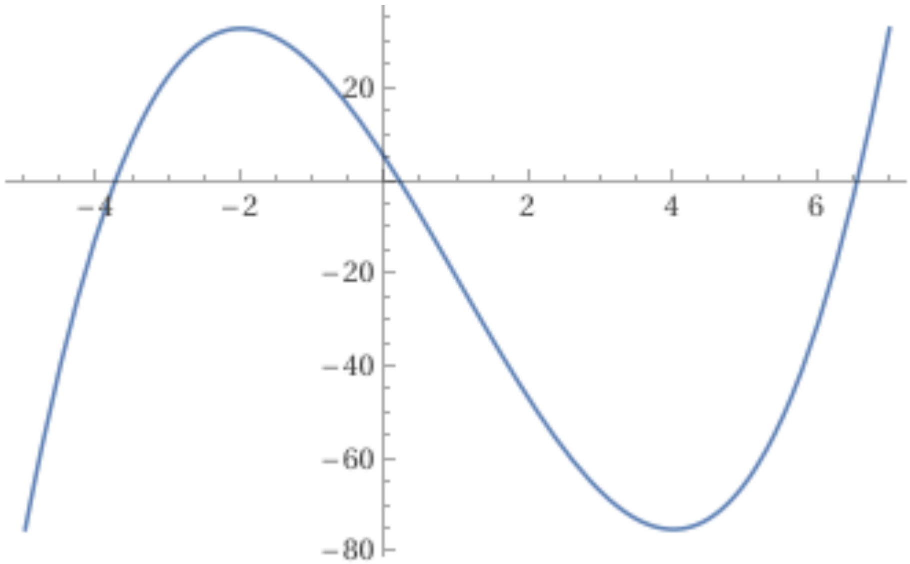
Since c' is continuous, c' does not change sign on $(-\infty, -2)$. Therefore $c'(x) > 0$ on $(-\infty, -2)$, so c is increasing on $(-\infty, -2)$.

Similarly, using the point 0 in $(-2, 4)$, for which $c'(0) = -24$, we get $c'(x) < 0$ on $(-2, 4)$, whence c is decreasing on $(-2, 4)$.

Using the point 5 in $(4, \infty)$, for which $c'(5) = 3(5 - 4)(5 + 2) = 21 > 0$, we get $c'(x) > 0$ on $(4, \infty)$. whence c is increasing on $(4, \infty)$.

We can also use the second derivative test to check what happens at the critical points. We have $c''(x) = 6x - 6$. Therefore $c''(-2) = -18 < 0$, so c has a local maximum at -2. Also, $c''(4) = 18 > 0$, so c has a local minimum at 4.

Here is a graph of the function (**not** required as part of the solution; given so you can compare your results with the actual graph):



Extra credit. (15 extra credit points; grading will be harsher than on related problems on the main exam. Do not attempt this problem until you have done and checked your answer to all the ordinary problems on this exam. It will only be counted if you get a grade of B or better on the main part of this exam.)

A four dimensional box has a cubical base and no top. Its four dimensional volume is supposed to be $8\,\mathrm{ft}^4$. What dimensions minimize the three dimensional volume of material needed to make its base and sides?

Hint: A box in four dimensional space has 8 three dimensional "faces", each of which has the shape of a three dimensional box.

Solution. Since I can't draw a four dimensional box, there is no picture in this solution.

Let x be the side length of the base, and let y be the height. The base has volume x^3. There are 6 "sides", each with a volume of x^2y. The four dimensional volume is x^3y. So we are supposed to minimize the three dimensional volume $v = x^3 + 6x^2y$ subject to the restriction $x^3y = 8$.

The restriction is $y = 8x^{-3}$, so we are minimizing

$$v(x) = x^3 + 6x^2(8x^{-3}) = x^3 + 48x^{-1}.$$

The constraints are $x > 0$ and $y > 0$. The constraint $y > 0$ adds nothing new, so we are minimizing $v(x) = x^3 + 48x^{-1}$ on the interval $(0, \infty)$.

Differentiate: $v'(x) = 3x^2 - 48x^{-2}$. So we solve:

$$3x^2 - 48x^{-2} = 0$$

$$3x^4 - 48 = 0$$

$$x^4 = 16$$

$$x = 2 \qquad \text{or} \qquad x = -2.$$

We reject $x = -2$ because it is not in $(0, \infty)$. (I must see you do this! Otherwise, I don't know that you correctly solved the equation $v'(x) = 0$.) So 2 is the only critical number in the interval $(0, \infty)$.

Is it a maximum or minimum? The easiest test to use here is the second derivative test. One easily checks that

$$v''(x) = 6x + 96x^{-3},$$

which is positive for all x in the interval $(0, \infty)$. Therefore v is concave up everywhere on the interval $(0, \infty)$, and any critical number must be an absolute minimum.

The problem asked for the dimensions giving the minimum three dimensional volume. The base is 2 feet by 2 feet by 2 feet. Since $y = 8x^{-3}$, the height is 1 foot. (Don't forget the units!) □

For those who tried it instead of the second derivative test as done above, here is how the first derivative test works. We factor the derivative as follows:

$$v'(x) = 3x^{-2}(x^4 - 16).$$

All factors except $x^4 - 16$ are always positive. For $0 < x < 2$, the factor $x^4 - 16$ is negative. Thus $v'(x) < 0$, and v is decreasing on the whole interval $(0, 2)$. For $x > 2$, the factor $x^4 - 16$ is positive. Thus $v'(x) > 0$, and v is increasing on the whole interval $(2, \infty)$. Clearly, then, there is an absolute minimum at $x = 2$.

Other tests are also possible, but the details are not given here.