SOLUTIONS TO MIDTERM 1

1. (1 point) Are you awake?

Solution: I hope you were!

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2. (6 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

You do not need to give both versions.

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3. (a) (10 points) If $f(x) = x^2 + 3$, compute the derivative $f'(4)$ *directly from the definition of the derivative* (which you are supposed to have given above). (No credit will be given for just using the differentiation rules, but see Part (b.)

Solution: We find the limit of the difference quotient, using the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$\begin{aligned} f'(4) &= \lim_{h \to 0} \frac{f(4+h) - f(4)}{h} = \lim_{h \to 0} \frac{(4+h)^2 + 3 - (4^2 + 3)}{h} \\ &= \lim_{h \to 0} \frac{16 + 8h + h^2 + 3 - 16 - 3}{h} = \lim_{h \to 0} \frac{8h + h^2}{h} \\ &= \lim_{h \to 0} (8 + h) = 8. \end{aligned}$$

(b) (1 point) Use the differentiation rules we have learned to check your answer to part (a).

Solution: $f'(x) = 2x$, so $f'(4) = 8$.

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4. (11 points) Differentiate the function $q(x) = (2x^4 - 11x + 8)\sin(x) + \dfrac{1}{15}$. (You need not do this directly from the definition.)

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Solution: The second term is a constant, so its derivative is zero. (**Don't** use the quotient rule on it!) On the first term, use the product rule:

$$\begin{aligned} q'(x) &= \frac{d}{dx}\big(2x^4 - 11x + 8\big)\sin(x) + (2x^4 - 11x + 8)\sin'(x) \\ &= (4 \cdot 2x^3 - 11)\sin(x) + (2x^4 - 11x + 8)\cos(x) = (8x^3 - 11)\sin(x) + (2x^4 - 11x + 8)\cos(x). \end{aligned}$$

Parentheses are required in the several places. For example, if if

$$\sout{\frac{d}{dx} 2x^4 - 11x + 8}, \qquad \sout{2x^4 - 11x + 8 \cos(x)},$$

or anything similar appears in your work, it is wrong.

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5. (11 points) Differentiate the function $h(x) = \cos(4x^3 + 8x)$. (You need not do this directly from the definition.)

Solution: Use the chain rule:

$$h'(x) = \cos'(4x^3 + 8x) \cdot \frac{d}{dx}\big(4x^3 + 8x\big) = -\sin(4x^3 + 8x) \cdot (4 \cdot 3x^2 + 8) = -(12x^2 + 8)\sin(4x^3 + 8x).$$

Parentheses are required in the first and second steps: if

$$\sout{\frac{d}{dx} 4x^3 + 8x}, \qquad \sout{-\sin(4x^3 + 8x) \cdot 4 \cdot 3x^2 + 8},$$

or anything similar appears in your work, it is wrong.

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6. (7 points) Find the exact value of the limit $\lim_{x \to 3} \dfrac{x^2 - 9}{2x^2 + x + 7}$, or explain why this limit does not exist.

Solution: Both the numerator and denominator are continuous at $x = 3$, and the denominator is not zero there. Therefore the limit can be evaluated by simply substituting $x = 3$. That is,

$$\lim_{x \to 3} \frac{x^2 - 9}{2x^2 + x + 7} = \frac{3^2 - 9}{2 \cdot 3^2 + 3 + 7} = \frac{0}{28} = 0.$$

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7. (16 points.) A farmer wants to build a rectangular fenced enclosure. Because of bizarre local laws, the east fence will cost 7 florins per meter, the west fence will cost 3 florins per meter, the north fence will cost 4 florins per meter, and the south fence will cost 2 florins per meter. The farmer has 6000 florins available to build the enclosure. The farmer wants to find the lengths of the south and west fences of the enclosure with the largest area that can be built.

Set up, but **do not attempt to solve**, the appropriate maximization or minimization problem. That is, give a function $f(x)$, for x a suitable quantity related to the problem (say what x actually is!), give a suitable domain, and say whether you want to maximize or minimize f on this domain. Provide justification for all steps (possibly including a picture).

Solution: Let x be the length of the south and north fences (in meters), and let y be the length of the west and east fences (in meters). Let A be the area. Thus

$$A = xy. \tag{1}$$

We want to maximize this quantity. Let C be the total cost, in florins.

We have

$$C = 4x + 3y + 2x + 7y.$$

(The terms are, in order, the costs of the north, west, south, and east fences.) We are also given $C = 6000$. Combining the two formulas for C:

$$6000 = C = 4x + 3y + 2x + 7y = 6x + 10y.$$

Solve for one of the variables, say x:

$$x = \frac{1}{6}(6000 - 10y) = 1000 - \frac{5}{3}y. \tag{2}$$

Substitute this in (1) and write as a function of y:

$$A(y) = \left(1000 - \frac{5}{3}y\right) y = 1000y - \frac{5}{3}y^2.$$

We obviously must have $y \geq 0$. Also, clearly $x \geq 0$. By (2), this means $1000 - \frac{5}{3}y \geq 0$, so $y \leq 600$. Therefore we need to *maximize* $A(y) = 1000y - \frac{5}{3}y^2$ on the interval $[0, 600]$, and in which y is the length of the west and east fences. Stop here; this is as far as you were told to go.

If you solve for y instead, you find that you need to *maximize* $A(x) = 600x - \frac{3}{5}x^2$ on the interval $[0, 1000]$, and in which x is the length of the north and south fences.

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8. (15 points) Use the methods of calculus to find the exact values of x at which the function $k(x) = x^3 - 6x^2 - 15x$ takes its absolute minimum and maximum on the interval $[-2, 2]$.

(No credit will be given for correct guesses without supporting work that is valid for general functions of the sort considered in this course.)

Solution: We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate k at all critical numbers in the interval and at the endpoints, and compare values.

To find the critical numbers, we differentiate k, solve the equation $k'(x) = 0$, and find all numbers x in our interval such that $k'(x)$ does not exist. The derivative of f is

$$k'(x) = 3x^2 - 12x - 15 = 3(x^2 - 4x - 5) = 3(x + 1)(x - 5).$$

It exists everywhere, and it is zero when $x = -1$ and $x = 5$.

Since $x = 5$ is not in $[-2, 2]$, we ignore it. (**I must see you reject** 5**.** If I don't see this, I will assume you didn't correctly solve the equation $k'(x) = 0$.)

We now have one critical number, namely -1. So we must compare the values of k at -1, and at the endpoints -2 and 2.

Since

$$k(-2) = (-2)^3 - 6(-2)^2 - 15(-2) = -8 - 6 \cdot 4 + 30 = -2,$$
$$k(-1) = (-1)^3 - 6(-1)^2 - 15(-1) = -1 - 6 \cdot 1 + 15 = 8,$$

and

$$k(2) = 2^3 - 6 \cdot 2^2 - 15 \cdot 2 = 8 - 6 \cdot 4 - 30 = -46,$$

the smallest of these is $k(2)$ and the largest is $k(-1)$. So the absolute minimum on the interval $[-2, 2]$ occurs at $x = 2$ and the absolute maximum on the interval $[-2, 2]$ occurs at $x = -1$.

Note that $x = 5$ is not correct for the minimum, even though $k(5) = -100$, because 5 is not in the interval $[-2, 2]$.

No credit will be given for any solution which does not show evidence of an attempt to find the critical numbers of f. In particular, no credit will be given for simply comparing the values of f at the integers in the interval.

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9. (10 points.) This problem is about using correct notation. Accordingly, almost all the credit is for correctness of notation.

Consider the problem of finding the exact value of $\lim_{x\to-5} \frac{x^3 + 5x^2 + 4x + 20}{x + 5}$. The method is to factor the numerator and cancel one of the factors. The factors of the numerator are $x + 5$ and $x^2 + 4$.

Write out the calculation in full, in correct notation which exhibits correctly the steps of the calculation. In particular, put "=" and "lim" everywhere they belong, and nowhere else. Start by writing $\lim_{x\to-5} \frac{x^3 + 5x^2 + 4x + 20}{x + 5}$. Show at least the following steps:

(1) After factoring but before cancellation.
(2) After cancellation but before substituting $x = -5$.
(3) After substituting $x = -5$ but before possible simplification.
(4) The simplified final result, if the result in the previous step can be simplified.

There is no need to label the steps.

Solution.

$$\begin{aligned}\lim_{x\to-5} \frac{x^3 + 5x^2 + 4x + 20}{x + 5} &= \lim_{x\to-5} \frac{(x + 5)(x^2 + 4)}{x + 5} \\ &= \lim_{x\to-5} (x^2 + 4) = (-5)^2 + 4 = 25 + 4 = 29.\end{aligned}$$

This completes the solution. □

Alternate solution. For $x \neq -5$, we have

$$\frac{x^3 + 5x^2 + 4x + 20}{x + 5} = \frac{(x + 5)(x^2 + 4)}{x + 5} = x^2 + 4.$$

Therefore

$$\lim_{x\to-5} \frac{x^3 + 5x^2 + 4x + 20}{x + 5} = \lim_{x\to-5} (x^2 + 4) = (-5)^2 + 4 = 25 + 4 = 29.$$

This completes the solution. □

Comments. In the solutions above, the symbol "=" must appear in all the places where it is shown, and may not appear anywhere else.

The symbol "$\lim_{x\to-5}$" must appear in all the places where it is shown, and may not appear anywhere else. In particular,

$$\lim_{x\to-5} (x^2 + 4) = \lim_{x\to-5} [(-5)^2 + 4]$$

is a mathematically true statement, but does not correctly show the intended step. Also, $\not{-5^2}$ is wrong: $-5^2 = -25$, but what is wanted is $(-5)^2 = 25$.

Every parenthesis shown is essential. Putting in extra parentheses is not formally wrong, but should not be done because it makes the solution harder to read.

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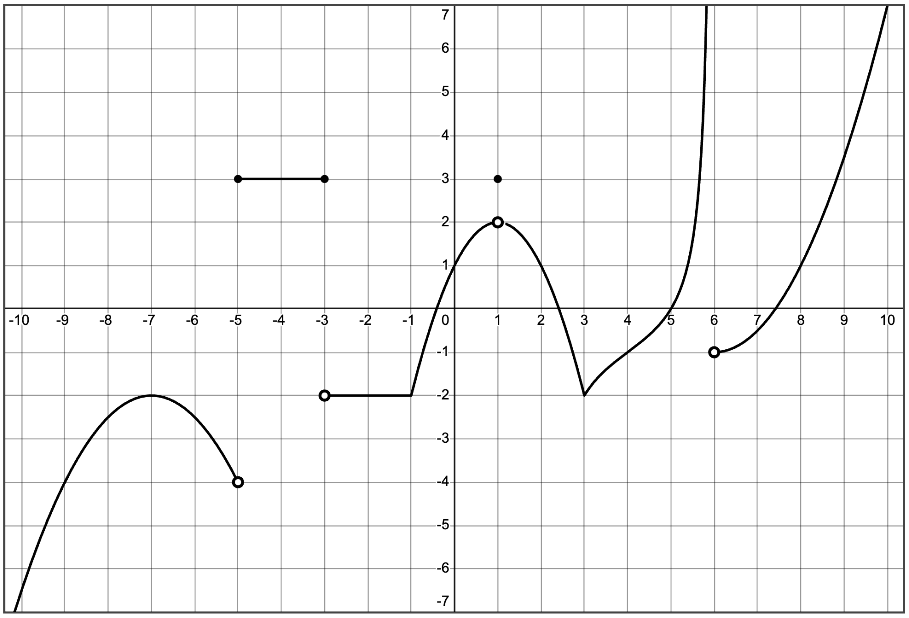
10. (6 points/part) For the function $y = L(x)$ graphed below, answer the following questions:



(a) Does $\lim_{x\to 1} L(x)$ exist? If so, what is it? If not, why not?

Solution: $\lim_{x\to 1} L(x) = 2$. You can see from the graph that when x is close enough to 1 (but $x \neq 1$), then $L(x)$ is close to 2.

The answer is not $\not{3}$. That is $L(1)$.

(b) Is L continuous at -1? Why or why not?

Solution: The function L is continuous at -1, because $L(-1)$ and $\lim_{x\to -1} L(x)$ are both equal to -2. In particular, they are equal.

Informally, the graph can be drawn near $x = -1$ without lifting the pencil off the paper.

(It is true that L is not differentiable at $x = -1$, because of the corner, but that isn't what was asked.)

Extra credit. (5 extra credit points/part; grading will be harsher than on related problems on the main exam. Do not attempt this problem until you have done and checked your answer to all the ordinary problems on this exam. It will only be counted if you get a grade of B or better on the main part of this exam.)

(a) Let $f(x) = \sin(x)$. Find the 1167th derivative $f^{(1167)}(x)$.

Solution: We have

$$f'(x) = \cos(x),$$
$$f''(x) = -\sin(x),$$
$$f'''(x) = -\cos(x),$$

and

$$f^{(4)}(x) = \sin(x) = f(x).$$

Therefore $f^{(8)}(x) = f(x)$, $f^{(12)}(x) = f(x)$, etc. In particular, since 1164 is divisible by 4, we get $f^{(1164)}(x) = f(x) = \sin(x)$. Therefore

$$f^{(1167)}(x) = f'''(x) = -\cos(x).$$

(b) Let $g(x) = x^{1167}$. Find the 1166th derivative $g^{(1166)}(x)$.

Solution: We have

$$g'(x) = 1167 \cdot x^{1166},$$
$$g''(x) = 1166 \cdot 1167 \cdot x^{1165},$$
$$g'''(x) = 1165 \cdot 1166 \cdot 1167 \cdot x^{1164},$$

etc. Continuing, we find that

$$g^{(1165)}(x) = 3 \cdot 4 \cdot 5 \cdots 1165 \cdot 1166 \cdot x^2$$

and

$$g^{(1166)}(x) = 2 \cdot 3 \cdot 4 \cdot 4 \cdots 1165 \cdot 1166 \cdot 1167 \cdot x = 1167! \cdot x.$$

(c) Let $f(x) = \cos(3x)$. Find the 1029th derivative $f^{(1029)}(x)$.

Solution: We have

$$f'(x) = -3\sin(3x),$$
$$f''(x) = -3^2\cos(3x),$$
$$f'''(x) = 3^3\sin(3x),$$

and

$$f^{(4)}(x) = 3^4\cos(3x) = 3^4 f(x).$$

Therefore

$$f^{(8)}(x) = 3^4 f^{(4)}(x) = 3^8 f(x),$$
$$f^{(12)}(x) = 3^8 f^{(4)}(x) = 3^{12} f(x),$$

etc. In particular, since 1028 is divisible by 4, we get $f^{(1028)}(x) = 3^{1028} f(x) = 3^{1028}\cos(3x)$. Therefore

$$f^{(1029)}(x) = 3^{1028} f'(x) = 3^{1028} \cdot (-3\sin(3x)) = -3^{1029}\sin(3x).$$