SOLUTIONS TO MIDTERM 1 REVIEW SESSION WORKSHEET FOR SPRING 2025

It actually has 103 points.

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1. (5 points) State carefully the definition of the derivative of a function.

Solution: Let f be a function defined on an open interval containing a. Then the derivative of f at a is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

if this limit exists.

The last phrase is an essential part of the answer.

An alternate formulation is: Then the derivative of f at a is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

if this limit exists.

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2. (a) (9 points) If $f(x) = 7 - x^2$, compute the derivative $f'(2)$ *directly from the definition.* (No credit will be given for just using the differentiation rules, but see Part (b.)

Solution: We find the limit of the difference quotient, using the technique of cancelling common factors in the numerator and denominator to handle the resulting expression:

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \to 0} \frac{7 - (2+h)^2 - (7 - 2^2)}{h} = \lim_{h \to 0} \frac{7 - (4 + 4h + h^2) - (7 - 4)}{h}$$

$$= \lim_{h \to 0} \frac{7 - 4 - 4h - h^2 - 7 + 4}{h} = \lim_{h \to 0} \frac{-4h - h^2}{h} = \lim_{h \to 0}(-4 - h) = -4.$$

(b) (1 point) Use the differentiation rules we have learned to check your answer to part (a).

$f'(x) = -2x$, so $f'(2) = -4$.

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3. (9 points.) Differentiate the function $w(t) = t^2 \cos(t) - \dfrac{5}{\sqrt{t}} + \dfrac{7}{71}$.

Solution: We rewrite the second term of the function to make it easy to differentiate:

$$w(t) = t^2 \cos(t) - 5t^{-1/2} + \frac{7}{71}.$$

Use the product rule on the first term, use the power rule on the second term, and note that $\frac{7}{71}$ is a constant so that its derivative is zero. This gives

$$g'(t) = 2t\cos(t) + t^2(-\sin(t)) - 5 \cdot \left(-\tfrac{1}{2}\right) t^{-1/2-1} = 2t\cos(t) - t^2 \sin(t) + \tfrac{5}{2} t^{-3/2}.$$

(I have seen people waste lots of time using the quotient rule to differentiate expressions like $\frac{7}{71}$. Moreover, people often make mistakes when doing so, thus ending up with the wrong answer.)

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4. (9 points.) Differentiate the function $h(q) = \sqrt[3]{12 + q^4 + \sin(q)}$.

Solution: We rewrite the function to make it easy to differentiate:

$$h(q) = (12 + q^4 + \sin(q))^{1/3}.$$

Now use the chain rule:

$$\begin{aligned} h'(q) &= \frac{1}{3}\big(12 + q^4 + \sin(q)\big)^{1/3-1} \cdot \frac{d}{dq}\big(12 + q^4 + \sin(q)\big) \\ &= \frac{1}{3}\big(12 + q^4 + \sin(q)\big)^{-2/3}(4q^3 + \cos(q)). \end{aligned}$$

If you like, you can rewrite this as

$$h'(q) = \frac{4q^3 + \cos(q)}{3\sqrt[3]{[12 + q^4 + \sin(q)]^2}},$$

but this is not necessary.

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5. (20 points) The Wang Container Corporation plans to manufacture wooden boxes with square bases and hinged lids. The wood for the bottom and sides costs \$3 per square foot, and the wood for the lid costs \$1 per square foot. [Evidently the lid will be rather flimsy.] Furthermore, each box requires hinges and a latch costing a total of \$6. If the total cost of the materials is only allowed to be \$54, what are the dimensions of the largest volume box that can be manufactured?

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

Solution: Note: There is no picture in this file. A picture may be provided separately.

Let the linear dimensions of the box be x, y, and z, measured in feet, with z being the height. Since the base is square,

$$x = y.$$

We want to maximize the volume

$$V = xyz = x^2 z.$$

The constraint on the total cost allows the elimination of one more variable (necessary if we are to have a single variable problem). Let C be the total cost, measured in dollars. The bottom costs $3x^2$, the four sides cost $3xz$ each, the top costs x^2, and the hinges and latch cost 6. So the total cost is

$$C = 3x^2 + 4 \cdot 3xz + x^2 + 6 = 4x^2 + 12xz + 6.$$

There is clearly no benefit to using less than \$54 worth of material, so we set

$$54 = C = 4x^2 + 12xz + 6.$$

It is easier to solve for z than for x, so we get:

$$48 - 4x^2 = 12xz$$

$$\frac{48 - 4x^2}{12x} = z.$$

Since z is determined by x, we can write the volume V as a function $V(x)$ of x. That is,

$$V(x) = x^2 z = x^2 \left(\frac{48 - 4x^2}{12x} \right) = \frac{48x - 4x^3}{12} = 4x - \tfrac{1}{3}x^3.$$

Note that we want to write $V(x)$ in the simplest possible form, to make further work easy.

We now determine the constraints. One is obvious: $x \geq 0$. (Allowing the "degenerate" case $x = 0$ will allow us to find the maximum over a closed bounded interval, which simplifies later steps.) The other constraint is $z \geq 0$, which is the same as $48 - 4x^2 \geq 0$. Rewriting this, we get $x^2 \leq 12$, which means $-\sqrt{12} \leq x \leq \sqrt{12}$. Since we already know $x \geq 0$, our final constraint is:

$$0 \leq x \leq \sqrt{12}.$$

(Note that $x = \sqrt{12}$ is also a degenerate case.)

We now must maximize $V(x) = 4x - \frac{1}{3}x^3$ for x in $[0, \sqrt{12}]$. Differentiate:

$$V'(x) = 4 - x^2.$$

Set the derivative equal to zero and solve:

$$0 = V'(x) = 4 - x^2$$

$$x = \pm 2.$$

We ignore the solution $x = -2$, since -2 is not in the interval $[0, \sqrt{12}]$. Since we are maximizing over a closed bounded interval, we need only compare the numbers $V(0)$, $V(2)$, and $V(\sqrt{12})$. These are

$$V(0) = 4 \cdot 0 - \tfrac{1}{3} \cdot 0^3 = 0, \quad V(2) = 4 \cdot 2 - \tfrac{1}{3} \cdot 2^3 = 8 - \tfrac{8}{3} = \tfrac{16}{3},$$

and

$$V(\sqrt{12}) = 4 \cdot \sqrt{12} - \tfrac{1}{3} \left(\sqrt{12}\right)^3 = 4 \cdot \sqrt{12} - \tfrac{12}{3} \cdot \sqrt{12} = 0.$$

Clearly the largest value is at $x = 2$. Therefore $y = 2$ and

$$z = \frac{48 - 4x^2}{12x} = \frac{48 - 16}{24} = \frac{4}{3}.$$

So the dimensions are 2 feet $\times$ 2 feet $\times$ $\frac{4}{3}$ feet. (Be sure to include the units!)

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6. (15 points) Use the methods of calculus to find the exact values of x at which the function $f(x) = \dfrac{x}{x^2 + 1}$ takes its absolute minimum and maximum on the interval $[0, 7]$.

(No credit will be given for correct guesses without supporting work that is valid for general functions of the sort considered in this course.)

Solution: We apply the procedure for continuous functions on closed finite intervals. That is, we evaluate f at all critical numbers and at the endpoints, and compare values.

To find the critical numbers, we differentiate f, solve the equation $f'(x) = 0$, and find all numbers x in our interval such that $f'(x)$ does not exist. The derivative of f is

$$f'(x) = \frac{(x^2 + 1) - x(2x)}{(x^2 + 1)^2} = -\frac{x^2 - 1}{(x^2 + 1)^2} = -\frac{(x + 1)(x - 1)}{(x^2 + 1)^2}.$$

It exists everywhere, and it is zero when $x = -1$ and $x = 1$.

Since $x = -1$ is not in $[0, 7]$, we ignore it. (**I must see you reject -1.** If I don't see this, I will assume you didn't correctly solve the equation $f'(x) = 0$.)

We now have one critical number, namely 1. So we must compare the values of f at 1, and at the endpoints 0 and 7.

We have $f(0) = 0$, $f(1) = \frac{1}{2}$, and $f(7) = \frac{7}{50}$. The smallest of these is $f(0)$ and the largest is $f(1)$. So the absolute minimum on the interval $[0, 7]$ occurs at $x = 0$ and the absolute maximum on the interval $[0, 7]$ occurs at $x = 1$.

Note that $x = -1$ is not correct for the minimum, even though $f(-1) = -\frac{1}{2}$, because -1 is not in the interval $[0, 7]$.

No credit will be given for any solution which does not show evidence of an attempt to find the critical numbers of f. In particular, no credit will be given for simply comparing the values of f at the integers in the interval.

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7. (9 points.) Find the exact value of $\displaystyle\lim_{x \to -2} \frac{x+2}{x^2 - 3x - 10}$, or explain why it does not exist:

Solution: This limit has the indeterminate form "$\frac{0}{0}$", so work is needed. We factor the denominator and cancel common factors:

$$\lim_{x \to -2} \frac{x+2}{x^2 - 3x - 10} = \lim_{x \to -2} \frac{x+2}{(x+2)(x-5)} = \lim_{x \to -2} \frac{1}{x-5} = \frac{1}{-2-5} = -\frac{1}{7}.$$

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8. (9 points.) Find the exact value of $\displaystyle\lim_{x \to -1} \frac{x^2 - 6x - 7}{x - 2}$, or explain why it does not exist:

Solution: Both the numerator and denominator are continuous at -1, and the denominator is not zero there. Therefore the limit can be evaluated by simply substituting $x = -1$. That is,

$$\lim_{x \to -1} \frac{x^2 - 6x - 7}{x - 2} = \frac{(-1)^2 - 6(-1) - 7}{(-1) - 2} = \frac{0}{-3} = 0.$$

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9. (9 points.) This problem is about using correct notation. Accordingly, almost all the credit is for correctness of notation.

Consider the problem of finding the exact value of $\lim\limits_{x \to 2} \dfrac{x^4 - 2x^3 - x\sin(x) + 2\sin(x)}{x - 2}$. The method is to factor the numerator (the factors are $x - 2$ and $x^3 - \sin(x)$) and cancel the factor $x - 2$, getting a limit which can be found by direct substitution.

Write out the calculation in full, in correct notation which exhibits correctly the steps of the calculation. In particular, put "=" and "lim" everywhere they belong, and nowhere else. Start by writing $\lim\limits_{x \to 2} \dfrac{x^4 - 2x^3 - x\sin(x) + 2\sin(x)}{x - 2}$. Show at least the following steps:

(1) After factoring but before cancellation.
(2) After cancellation but before substituting $x = 2$.
(3) After substituting $x = 2$ but before possible simplification.
(4) The simplified final result, if the result in the previous step can be simplified. (Don't give a decimal approximation to $\sin(2)$.)

There is no need to label the steps.

Solution.

$$\begin{aligned}\lim_{x \to 2} \frac{x^4 - 2x^3 - x\sin(x) + 2\sin(x)}{x - 2} &= \lim_{x \to 2} \frac{(x-2)\big(x^3 - \sin(x)\big)}{x - 2} \\ &= \lim_{x \to 2} \big(x^3 - \sin(x)\big) = 2^3 - \sin(2) = 8 - \sin(2).\end{aligned}$$

This completes the solution. □

Alternate solution. For $x \neq 2$, we have

$$\frac{x^4 - 2x^3 - x\sin(x) + 2\sin(x)}{x - 2} = \frac{(x-2)\big(x^3 - \sin(x)\big)}{x - 2} = x^3 - \sin(x).$$

Therefore

$$\lim_{x \to 2} \frac{x^4 - 2x^3 - x\sin(x) + 2\sin(x)}{x - 2} = \lim_{x \to 2} \big(x^3 - \sin(x)\big) = 2^3 - \sin(2) = 8 - \sin(2).$$

This completes the solution. □

Comments. In the solutions above, the symbol "=" must appear in all the places where it is shown, and may not appear anywhere else.

The symbol "$\lim\limits_{x \to 2}$" must appear in all the places where it is shown, and may not appear anywhere else. In particular,

$$\lim_{x \to 2} \big(x^3 - \sin(x)\big) = \lim_{x \to 2} \big(2^3 - \sin(2)\big)$$

is a mathematically true statement, but does not correctly show the intended step.

Every parenthesis shown is essential, except for the possibility of writing "$\sin x$" and "$\sin 2$" in place of "$\sin(x)$" and "$\sin(2)$", an unfortunately standard bad habit. Putting in extra parentheses is not formally wrong, but should not be done because it makes the solution harder to read.

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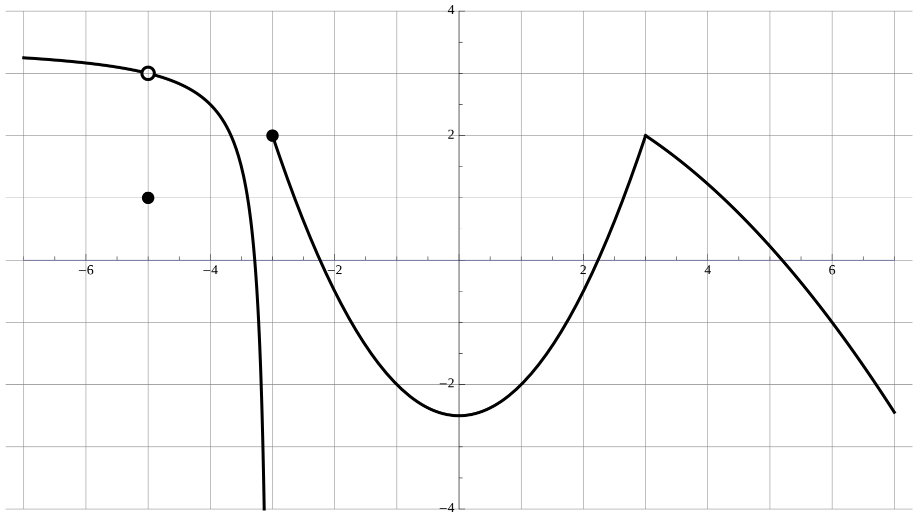
10. (4 points/part) For the function $y = k(x)$ graphed below, answer the following questions:



(a) List all numbers a in $(-7, 7)$ such that k is not differentiable at a. Give reasons.

Solution: The answer is $a = -5$, $a = -3$, and $a = 3$. The function h is not differentiable at -5 and at -3, because k is not continuous at these places. It is not differentiable at 3 because there is a corner in the graph, so that there is no tangent line at 3.

(b) List all numbers a in $(-7, 7)$ such that $\lim_{x\to a} k(x)$ does not exist.

Solution: The answer is only $a = -3$. It is clear from the graph that $\lim_{x\to -3^-} k(x) = -\infty$ and $\lim_{x\to -3^+} k(x) = 2$. So the one sided limits disagree.

Note: $\lim_{x\to -5} k(x)$ *does* exist: it is equal to 2.