MATH 251, FALL 2010: NOTATION AND OTHER ERRORS OFTEN SEEN IN WRITTEN HOMEWORK AND ON EXAMS

Contents

This sheet describes some notation errors and procedural mistakes made in written homework and on exams in my Math 251 classes. Almost all of them are seen every time I teach this class.

The first section has a large collection of short comments on notation, terminology, and other common mistakes. The next sections are on notation and terminology. I will assume everyone in the class has seen these sections and knows that the things described in it are in fact not correct notation. The last few sections are on procedural issues, like not reading the instructions in a problem and doing intermediate steps in such a way as to make later steps harder.

1. Miscellaneous

Remember that additional incorrect statements added to correct solutions will result in not getting full credit.

Symbols in equations must be *clearly written*, in particular, so that there is no reasonable way to mistake what is intended. Once, in a problem about differentiating the function $g(x) = 2x^k - ax^{-1} - bx^{1/2} + \frac{3}{7} - \pi^2$, several people changed a to 9 by the end of the problem. I can't tell whether they misread their own handwriting or usually write a so that it can't be distinguished from 9. Regardless, those answers were wrong.

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Never use $\times$ for multiplication. (In handwritten work, it is often indistinguishable from x.)

Many calculator and computer expressions are not acceptable in written work, because we already have better notation. For example,

$$\cancel{2^{\wedge -x}},$$

written for 2^{-x}, is wrong.

Don't use any of

$$\emptyset, \qquad \varnothing, \qquad \text{or} \qquad \phi$$

for zero. Each of these symbols has another meaning already.

You are expected to know the formulas for the area and circumference of a circle, the volume and surface area of a sphere, cylinder, cone, and box, the area of a triangle and rectangle, and various other similar things (including some easier ones not mentioned). If you don't know these, write them on your note card. (I have seen people use the wrong formula for the area of a circle.)

Remember that final answers must *always* be simplified. You *must* simplify $-\sqrt{\frac{4}{9}}$ to $-\frac{2}{3}$.

When you use the Squeeze Theorem, the Intermediate Value Theorem, or some other theorem, *cite it* by name.

Don't use methods that have not yet appeared in the course. For example, any use of L'Hospital's Rule before we cover it in class will result in getting no credit on the problem.

2. Units

In all problems in which units are given in the problem (for example, the height of a thrown rock on the planet Yuggxth at time t in seconds is $6t - t^2$ feet), the answers must always include correct units. For example, the height at time 2 in seconds is 8 *feet*. If the units are left out, or wrong (such as ~~8 seconds~~ or ~~8 feet / second~~), the answer is wrong. (Beware on `WeBWorK`: different problems use different methods for putting in the units.)

Many choices of units are obviously wrong. In the situation above, the answer ~~8 meters~~ is wrong, but at lease a meter is a unit of distance. The answers ~~8 seconds~~ and ~~8 feet / second~~ are *obviously* wring, because neither seconds nor feet / second are units of distance. Similarly, ~~8 square feet~~, ~~8 cubic meters~~, and ~~8 kg~~ are *obviously* wring. As another example, an area can have units of square inches (or square kilometers, square feet, square meters, square miles, etc.), but *never* inches, kilometers, feet, meters, miles, etc.

Using the same formula as above, at time 4 seconds, the height was 8 feet, and at time 5 seconds, the height was 3 feet. This means the height *decreased* by 3 feet, which is the same as saying the height *increased* by -3 feet. It is *wrong* to say that the height ~~*decreased* by -3 feet~~; since this means that the height *increased* by 3 feet.

In calculus, we **always** measure angles in **radians**. The derivatives we have learned for trigonometric functions and their inverses are **only** valid if those functions are interpreted as involving the radian measure of angles. In particular, if in a related rates problem, you find that the derivative of an angle is $\frac{1}{4}$, then it is increasing at the rate of $\frac{1}{4}$ radians per hour (which is $45/\pi$ degrees per hour), **not** at ~~$\frac{1}{4}$ degrees~~ per hour.

Feet are not units for measuring angles. It does not make sense to say that an angle is decreasing at $\frac{1}{4}$ ~~feet~~ per hour.

3. Fraction notation

Never use mixed fractions in mathematics. (There are places where they are acceptable, even standard, but, as explained on the sheet of instructions for written homework, mathematics is not one of those places.) In particular, the expression $3\frac{1}{3}$ will be read as $3\left(\frac{1}{3}\right) = 1$. Write $\frac{10}{3}$ instead.

To avoid ambiguity, all fraction lines must be exactly horizontal and cover the entire numerator and denominator, unless enough parentheses are used.

If the answer is supposed to be $16/(x+7)$, the following are all fine:

$$16/(x+7) \qquad 16/[x+7] \qquad \frac{16}{x+7} \qquad {}^{16}\!\big/_{(x+7)} \qquad {}^{16}\!\big/_{(x+7)}$$

The following are all wrong, since they are $[16/x]+7$ or are ambiguous and could be interpreted as $[16/x]+7$. (For the last one, I don't want to be trying to make judgement calls on how close to horizontal a nonhorizontal line needs to be to count as horizontal.)

$$\cancel{16/x+7} \qquad \cancel{\frac{16}{x}+7} \qquad \cancel{{}^{16}\!\big/_{x}+7} \qquad \cancel{{}^{16}\!\big/_{x}+7}$$

Similarly, if the answer is supposed to be $(x+7)/16$, the following are all fine:

$$(x+7)/16 \qquad [x+7]/16 \qquad \frac{x+7}{16} \qquad {}^{(x+7)}\!\big/_{16} \qquad {}^{(x+7)}\!\big/_{16}$$

The following are all wrong, since they are $x+[7/16]$, or are ambiguous and could be interpreted as $x+[7/16]$. (As above, for the last one, I don't want to be trying to make judgement calls on how close to horizontal a nonhorizontal line needs to be to count as horizontal.)

$$\cancel{x+7/16} \qquad \cancel{x+\frac{7}{16}} \qquad \cancel{x+{}^{7}\!\big/_{16}} \qquad \cancel{x+{}^{7}\!\big/_{16}}$$

Multiplication in the denominator causes the same problem, but worse. The expression $1/2x$ is ambiguous: physicists seem to read it one way, but computer scientists read it the other way, and I have seen both intended meanings in Math 251 homework. Unfortunately, it even appears in textbooks.
textbfDon't use it! If the correct answer is $1/(2x)$, the following are all fine:

$$1/(2x) \qquad 1/[2x] \qquad \frac{1}{2x} \qquad \frac{1}{2x} \qquad {}^{1}\!\big/_{(2x)} \qquad {}^{1}\!\big/_{(2x)} \qquad \frac{1}{2}x^{-1} \qquad (2x)^{-1}$$

But the following are all wrong:

$$\cancel{1/2x} \qquad \cancel{\frac{1}{2}\,x} \qquad \cancel{{}^{1}\!\big/_{2}x} \qquad \cancel{{}^{1}\!\big/_{2}x}$$

If the correct answer is instead $x/2$, the following are all fine. (The third one looks a bit strange, and the parentheses in the last one are essential.)

$$x/2 \qquad \frac{x}{2} \qquad \frac{x}{\;\;2} \qquad {}^{x}\!\diagup_{2} \qquad \frac{x}{\;2} \qquad \frac{1}{2}x \qquad 2^{-1}x \qquad (1/2)x$$

But the following are all wrong. (Some of them are the same as for $1/(2x)$.)

$$\cancel{1/2x} \qquad \cancel{1/2\ x} \qquad \cancel{\frac{1}{2}\ x} \qquad \cancel{{}^{1}\!\diagup_{2x}} \qquad \cancel{\frac{1}{\;2x}}$$

4. Never write two operation signs next to each other

Never write two operation signs next to each other. Always use parentheses. Thus,

$$4 + (-x), \qquad \textbf{not} \qquad \cancel{4 + -x},$$

and

$$4 \cdot (-x), \qquad \textbf{not} \qquad \cancel{4 \cdot -x}.$$

The second one is particularly bad, because it is easily read as just $4 - x$. Here are three more examples, taken from actual student work:

$$\left[3\left(\sqrt{19}+\cos(x)\right)^2+\pi^2\right]^{237}\cdot(-\sin(x)), \quad \textbf{not} \quad \cancel{\left[3\left(\sqrt{19}+\cos(x)\right)^2+\pi^2\right]^{237}\cdot -\sin(x)},$$

$$-5 + (-2)(-0.03), \qquad \textbf{not} \qquad \cancel{4 + -x},$$

and

$$\pi \cdot 20 \cdot (-3), \qquad \textbf{not} \qquad \cancel{\pi \cdot 20 \cdot -3}.$$

Again, the last example is particularly bad, because it is likely to be read as $\pi \cdot 20 - 3$.

5. Missing parentheses

Leaving out required parentheses is a very common error. Example:

$$\frac{6}{c-4} - \frac{1}{c-2} = \frac{6(c-2) - \cancel{c-4}}{(c-4)(c-2)} = \cancel{\frac{6c-12-c-4}{(c-4)(c-2)}} = \cancel{\frac{5c-16}{(c-4)(c-2)}}.$$

The correct calculation, getting the correct answer, is:

$$\frac{6}{c-4} - \frac{1}{c-2} = \frac{6(c-2) - (c-4)}{(c-4)(c-2)} = \frac{6c-12-c+4}{(c-4)(c-2)} = \frac{5c-8}{(c-4)(c-2)}.$$

Another example (which a student actually did): the expression $-x\left(e^{-x^2}+\sqrt{3}\right)$ is **not** equal to

$$\cancel{-xe^{-x^2}+\sqrt{3}}.$$

If you multiply out $-x\left(e^{-x^2}+\sqrt{3}\right)$, you get

$$-xe^{-x^2} - x\sqrt{3}.$$

Mistakes like this made further from the end of a problem can ruin any chance of even being able to do the rest of the problem.

6. Use of "="

Use the symbol "=" when you want to say two things are equal. For example,

$$\frac{x^2+5x-14}{x-2} = \frac{(x-2)(x+7)}{x-2} = x+7.$$

The following are all wrong:

$$\cancel{\frac{x^2+5x-14}{x-2} \qquad \frac{(x-2)(x+7)}{x-2} \qquad x+7,}$$

$$\cancel{\frac{x^2+5x-14}{x-2} \longrightarrow \frac{(x-2)(x+7)}{x-2} \longrightarrow x+7,}$$

and

$$\cancel{\frac{x^2+5x-14}{x-2} \Longrightarrow \frac{(x-2)(x+7)}{x-2} \Longrightarrow x+7,}$$

as is the use of any other symbol in place of "=". The first version is just a list of unrelated mathematical expressions. It doesn't count as showing work, because it doesn't say how the expressions are supposed to be related. The symbols "$\longrightarrow$" and "$\Longrightarrow$" both have meanings, but mean something different from "=".

Do not use "=" when tee two expressions are not equal, or for anything else that is not equality. This is an actual student solution (in a precalculus class) to the problem of finding $\log_2(1/8)$:

$$\cancel{\log_2(1/8) = 2^x = 1/8 = x = -3.}$$

As shown work, this gets very little credit. Among other things, it assets that $\cancel{\log_2(1/8) = 1/8}$ (because both are claimed equal to 2^x) and $\cancel{1/8 = -3}$ (because both are claimed equal to x).

Because "=" means you are claiming the expressions on each side of it are equal, those expressions must be mathematically meaningful. Thus,

$$\cancel{\lim_{x\to 0} \frac{\sin(x)}{x} = \frac{0}{0}}$$

is a false statement. (The correct statement is $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x} = 1$. One may say that the limit $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$ "has the form $\dfrac{0}{0}$" as a way of explaining why one must use methods other than substituting $x = 0$ to calculate it.) Because they are not meaningful, nothing may ever be *equal to* any of following expressions; in fact, they may never appear in any mathematical equation:

$$\cancel{\frac{0}{0}}, \qquad \cancel{\frac{\infty}{\infty}}, \qquad \cancel{0\cdot\infty}, \qquad \cancel{\infty-\infty}, \qquad \cancel{\frac{2}{0}},$$

or even

$$\cancel{\infty+\infty}, \qquad \cancel{\infty\cdot\infty}, \qquad \cancel{\frac{1}{\infty}}, \qquad \cancel{\frac{0}{\infty}}, \qquad \cancel{e^{\infty}}, \qquad \cancel{e^{-\infty}}. \qquad \cancel{\infty^2}.$$

(Neither list is complete.) The expressions in the first list have no reasonable definitions in the context of this course. Those in the second line could be given reasonable definitions, but we have not done so. Without having definitions, they may not be used as mathematical expressions.

For similar reasons,

$$\cancel{h(6) = \text{negative}}$$

is not correct notation; it must be "$h(6)$ **is** negative", since nothing can be equal to "negative". Similarly

$$\text{“}\cancel{f = \text{incr}}\text{”}, \qquad \text{“}\cancel{f(x) = \text{incr}}\text{”}, \qquad \text{and} \qquad \text{“}\cancel{\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = \text{DNE}}\text{”}$$

do not make sense.

The following is yet another abuse of "=" when something else was meant:

$$\cancel{f'(x) > 0 = \text{incr}}.$$

The statement "$\cancel{0 = \text{incr}}$" does not make sense. Write

$$f'(x) > 0 \text{ on } (-\ln(2), \infty), \text{ so } f \text{ is strictly increasing on } (-\ln(2), \infty).$$

In a chain of steps written in a column, an operation symbol always relates to the immediately preceding item. Thus, the calculation

$$\begin{aligned} e^{x^2} - \frac{1}{2} &> e^{x^2} - 1 \\ &\geq \left(1 + x^2 + \frac{x^4}{2}\right) - 1 \\ &= x^2 + \frac{x^4}{2} \\ &\geq x^2 \end{aligned}$$

is an abbreviation for

$$e^{x^2} - \frac{1}{2} > e^{x^2} - 1, \qquad e^{x^2} - 1 \geq \left(1 + x^2 + \frac{x^4}{2}\right) - 1,$$

$$\left(1 + x^2 + \frac{x^4}{2}\right) - 1 = x^2 + \frac{x^4}{2}, \qquad \text{and} \qquad x^2 + \frac{x^4}{2} \geq x^2,$$

all of which is correct as written. The final conclusion is $e^{x^2} - \frac{1}{2} > x^2$.

Similarly, if $f(4) = -5$ and $f'(4) = -2$, so that the linear approximation to $f(x)$ near 4 is $L(x) = 5 - 2(x - 4)$, the calculation

$$\begin{aligned} f(4.1) &\approx L(4.1) \\ &= 5 - 2(4.1 - 4) \\ &= 5 - 0.2 \\ &= 4.8 \end{aligned}$$

is an abbreviation for

$$f(4.1) \approx L(4.1), \qquad L(4.1) = 5 - 2(4.1 - 4),$$

$$5 - 2(4.1 - 4) = 5 - 0.2, \qquad \text{and} \qquad 5 - 0.2 = 4.8,$$

all of which is correct as written. The final conclusion is $f(4.1) \approx 4.8$ (not $\cancel{f(4.1) = 4.8}$, which is usually false). It is wrong to write

$$\begin{aligned} f(4.1) &\approx L(4.1) \\ &\cancel{\approx 5 - 2(4.1 - 4)}, \end{aligned}$$

because you intend to say that $L(4.1) = 5 - 2(4.1 - 4)$, not $\cancel{L(4.1) \approx 5 - 2(4.1 - 4)}$.

7. Solution sets, intervals, and using the right variable

Always use the correct variable. Suppose you are asked to solve the equation $2a - 14 = 0$. The correct statement of the solution is

$$a = 7.$$

The following are both wrong:

$$\cancel{x = 7} \qquad \text{and} \qquad \cancel{A = 7}.$$

These supposed solutions don't answer the problem, since they say nothing about a. In particular, a and A are *different* variables. Mathematics is "case sensitive": like (probably) most of your passwords, lowercase and uppercase letters are considered different.

As another example, consider the problem:

> Find all real numbers a such that $(2a, -a)$ is in the second quadrant (and not on any of the coordinate axes).

The answer "$\cancel{x < 0}$" isn't correct, since the problem asks about a. Since mathematics is case sensitive, "$\cancel{A < 0}$" is just as incorrect an answer as "$x < 0$".

Suppose you are asked to solve the inequality $5 \leq a + 9 \leq 8$. Here are several correct ways to write the solution. (If you haven't seen the symbol "$\in$" before, ignore those answers which contain it; some people have seen it but have never been told how to use it correctly.)

$$-4 \leq a \leq -1.$$

$$\text{all real numbers } a \text{ in } [-4, -1].$$

$$\text{all real numbers } a \text{ such that } -4 \leq a \leq -1.$$

$$a \in [-4, -1].$$

The following answers are wrong because they say the solution set is empty:

$$\cancel{-1 \leq a \leq -4} \qquad \text{and} \qquad \cancel{a \in [-1, -4]}.$$

There is no real number a satisfying both $a \geq -1$ and $a \leq -4$. Similarly, $[-1, -4]$ is the set of all real numbers a satisfying both $a \geq -1$ and $a \leq -4$, but there are no such numbers.

The answers

$$\cancel{-4 \leq x \leq -1}, \qquad \cancel{-4 \leq A \leq -1}, \qquad \cancel{x \in [-4, -1]}, \qquad \text{and} \qquad \cancel{A \in [-4, -1]}$$

are all wrong, because none of them says anything about a.

The answers

$$\cancel{\text{all } \mathbb{R} \in [-4, -1]} \qquad \text{and} \qquad \cancel{a \in \mathbb{R}\{-4 \leq x \leq -1\}}$$

(both of which I have seen) don't mean anything as written, so are also wrong. The answer

$$\cancel{a = [-4, -1]}$$

is also wrong: it asserts that the interval $[-4, -1]$ is a solution to the inequality, when what is true is the the numbers **in** $[-4, -1]$ are solutions to the inequality.

Points are *in* intervals, not on them. Thus, π is **in** $[0, 7]$.

8. Limit notation

The expression "$\lim_{x\to a}$" (or other limit) must be present when the limit remains to be taken, and must not be present after the limit has been taken. Thus, both

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\lim_{x\to 2}\frac{(x+7)(x-2)}{x-2}=\lim_{x\to 2}(x+7)=\cancel{\lim_{x\to 2}}(2+7)=9$$

and

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\cancel{\frac{(x+7)(x-2)}{x-2}}=\cancel{x+7}=2+7=9$$

are wrong—with errors at the crossed out expressions. The only correct placement of "$\lim_{x\to 2}$" in this calculation is as shown here:

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\lim_{x\to 2}\frac{(x+7)(x-2)}{x-2}=\lim_{x\to 2}(x+7)=2+7=9.$$

Putting "$\lim_{x\to a}$" where it belongs, and not where it doesn't belong, is part of understanding what you are doing in this course.

No expression containing the combination

$$\cancel{\lim_{x\to 0} =},$$

or anything analogous, is ever correct. One may never have "=" (or any operation or relation symbol) directly after a limit symbol. One must always take the limit *of something*.

Parentheses are needed when taking the limit of a sum. (You have already seen some rules about the order of operations, such as multiplication before addition. Here are more: multiplication and division before limits, and limits before addition and subtraction.)

For example, it is not correct to write "$\cancel{\lim_{x\to 2} x+7}$" instead of "$\lim_{x\to 2}(x+7)$" in the computation

$$\lim_{x\to 2}\frac{x^2+5x-14}{x-2}=\lim_{x\to 2}\frac{(x+7)(x-2)}{x-2}=\lim_{x\to 2}(x+7).$$

Similarly, the notation

$$\cancel{\lim_{h\to 0} 1-2x-h}$$

is not correct. It means

$$\left(\lim_{h\to 0} 1\right)-2x-h,$$

which does not make sense. Write

$$\lim_{h\to 0}\big(1-2x-h\big).$$

You need parentheses even in

$$\lim_{h\to 0}\big(-h\big).$$

It is not correct to take the limit in one part of an expression but not another. For example, if $\lim\limits_{x\to 9} f(x)=-4$, then it is *not correct* to write without justification

$$\lim_{x\to 9^+}\frac{f(x)}{x-9}=\cancel{\lim_{x\to 9^+}\frac{-4}{x-9}}.$$

This statement does happen to be true, but it *needs justification*, and it does *not* follow from any limit laws we have seen. Example:

$$\lim_{x\to 3}(x^2-9)=0 \qquad \text{but} \qquad \lim_{x\to 3}\frac{x^2-9}{x-3} \neq \lim_{x\to 3}\frac{0}{x-3}.$$

The limit laws (limit of a sum is the sum of the limits, etc.) only apply when the limits exist. For this purpose, if a limit is ∞ or $-\infty$, then it does not exist. *Sometimes*, correctly interpreted, a statement is true anyway. For example, if $\lim_{x\to 7} f(x)=\infty$ and $\lim_{x\to 7} g(x)=\infty$, then it is true that $\lim_{x\to 7}[f(x)+g(x)]=\infty$. But in many combinations the laws *don't* hold, or even make sense. Thus, the equations

$$\cancel{\lim_{x\to 3}\frac{x^2-9}{x-3}=\frac{0}{0}} \qquad \text{and} \qquad \cancel{\lim_{x\to \infty}\frac{x}{x^3-1}=\frac{\infty}{\infty}}$$

do not make sense—nothing can *equal* $\frac{0}{0}$ or $\frac{\infty}{\infty}$. (For example, the correct value of the first expression is 6, and $\frac{0}{0}$ is certainly not equal to 6.) See the discussion of undefined symbols in Section 6.

Relatedly,

$$\lim_{x\to 0}\frac{4}{x^2}=\infty$$

is, by our conventions, a correct statement, even though ∞ is not a number. However,

$$\cancel{\lim_{x\to 0}\frac{4}{x^2}=\frac{4}{0}}$$

is not a correct statement: since $\frac{4}{0}$ is not a meaningful expression, it may not ever appear in an equation or algebraic expression. Also,

$$\lim_{x\to \infty}\frac{4}{x^2}=0$$

is correct, but

$$\cancel{\lim_{x\to \infty}\frac{4}{x^2}=\frac{4}{\infty}=0}$$

is not, because $\frac{4}{0}$ is meaningless and may never appear in an equation or algebraic expression. It is correct (by our conventions) to write

$$\lim_{x\to \infty} x^2=\infty; \qquad \text{therefore} \qquad \lim_{x\to \infty}\frac{4}{x^2}=0.$$

Heuristic argument instead of mathematically correct calculations, for example in problems like $\displaystyle\lim_{x\to -\infty}\frac{x^2+6070x+193}{6x^2-9x+21}$, get very little credit.

9. Functions might be continuous or differentiable, but points, equations, etc. never are

Generally, when one says a function f has certain behavior (is continuous, is not differentiable, has a local minimum, is strictly increasing, etc.), it is **always** at numbers (points) in the **domain** of f.

For example, only a *function* can be continuous or discontinuous, and can moreover only be continuous or discontinuous *at an element of its domain*. The following statements are nonsense:

$$\cancel{\text{The equation } -x^7-x+1=0 \text{ is continuous.}}$$

You have to say that the *function* $f(x) = -x^7 - x + 1$ is continuous.

$$\cancel{a \text{ is continuous.}}$$

You must say that the *function* f (or whatever) is continuous at a.

$$\cancel{\text{The point } -2 \text{ is discontinuous.}}$$

You must say that the *function* f (or whatever) is discontinuous at -2.

$$\cancel{\text{The function } f \text{ is discontinuous at } (4, 1).}$$

Unless f is a function of two variables (which we don't see in Math 251–253), the point $(4, 1)$ isn't in the domain of f, so the statement is nonsense.

Just as for continuity, only a *function* can be differentiable or not differentiable, and can moreover only be differentiable or not differentiable *at an element of its domain*. Thus, if a in the domain of f, and $f'(a)$ dos not exist, we say that f is not differentiable at a. We do **not** say that f is not differentiable at $\cancel{f(a)}$ or at $\cancel{f'(a)}$.

The same principle applies to minimums, inflection points (in some books, inflection numbers), etc. The function $h(x) = 3x^3 - 4x$ has a global minimum on the interval $[-1, 0]$ at 0 (or "at $x = 0$"), *not* at $\cancel{(0, 0)}$. The function $f(x) = e^{2x} - e^x$ has a global minimum at $-\ln(2)$ (or "at $x = -\ln(2)$"), *not* at $\cancel{\left(-\ln(2), -\frac{1}{4}\right)}$, and *certainly* not at $\cancel{(-\ln(2), 0)}$ (which is not even a point on the graph of f). Similarly, it has an inflection point at $-\ln(4)$, *not* at $\cancel{\left(-\ln(4), -\frac{3}{16}\right)}$.

If a function f is not defined at a, then we do **not** say any of: f is continuous at a, f is discontinuous at a, or f is not continuous at a. Similarly, we do **not** say either that f is differentiable at a or that f is not differentiable at a.

Only a curve or graph can have a slope. The expressions

$$\cancel{\text{the slope of } f(3)} \qquad \text{and} \qquad \cancel{\text{the slope of } f'(3)}$$

are both meaningless—numbers don't have slopes. (Presumably what was meant is the slope of the graph of $y = f(x)$ at $x = 3$.)

The logic is often badly wrong in problems using the Intermediate Value Theorem. For example, in a problem asking you to show that $-x^7 - x + 1 = 0$ has a solution in $[-1, 1]$, the function $f(x) = -x^7 - x + 1$ is continuous *because it is a polynomial and we have a theorem saying that polynomials are continuous*, not because of something about $f(1)$ and $f(-1)$.

10. Notation for derivatives and differentiation

The notation $\dfrac{d}{dx}$ means that the expression it is applied to is being differentiated with respect to x.

- It requires parentheses whenever there is an algebraic operation in the expression to be differentiated (except possibly for a horizontal fraction bar). Thus:
$$\frac{dy}{dx}, \qquad \frac{d}{dx}y, \qquad \frac{d}{dx}(xy), \qquad \text{but not} \qquad \cancel{\frac{d}{dx}xy}.$$
- The expression
$$\frac{dy}{dx}\sin(x)$$

is the product of $\dfrac{dy}{dx}$ and $\sin(x)$. It has **nothing to do** with the derivative of $\sin(x)$.

- The notation $\dfrac{d}{dx}$ means to differentiate with respect to x. It you are differentiating with respect to t, use $\dfrac{d}{dt}$.
- The notation $\dfrac{d}{dx}$ means to differentiate *everything* inside. Thus,

$$\frac{d}{dx}(\sin(x^2)) = \cos(x^2) \cdot 2x,$$

not $\cancel{\cos(x^2)}$. The only good notation we have to show that step in using the chain rule is $\sin'(x^2)$.

Expressions like $\ln(2\pi)$ and $\sin(6)$ are clearly constants, so their derivatives are obviously zero. Using the chain rule to differentiate them is a waste of time, even if done correctly, and is likely to lead to errors. The same applies to the quotient rule and $\frac{1}{6}$. When differentiating $\dfrac{3}{x^2}$, do **not** use the quotient rule. For differentiation purposes, $3x^{-2}$ is simpler: it is now clear that the derivative is $-6x^{-3}$.

The following is not correct:

$$\begin{aligned} g(x) &= 2x^k - ax^{-1} - bx^{1/2} + \frac{3}{7} - \pi^2 \\ &\cancel{= 2kx^{k-1} + ax^{-2} - \tfrac{1}{2}bx^{-1/2}}. \end{aligned}$$

It asserts that the function g is equal to its derivative, which is (almost) never true. It thus violates the meaning of "="; see Section 6.

It is not correct to differentiate some parts on an expression but not others. For example, if

$$g(x) = 2x^k - ax^{-1} - bx^{1/2} + \frac{3}{7} - \pi^2,$$

then

$$\cancel{2kx^{k-1} - ax^{-1} - \tfrac{1}{2}bx^{-1/2} + \frac{3}{7} - \pi^2}$$

can't be part of any correct calculation—it is neither $g(x)$ nor $g'(x)$.

If $A(t) = \pi r(t)^2$ for all t, and you differentiate both sides with respect to t, you get $A'(t) = 2\pi r(t)r'(t)$. The derivative of $A(t)$ is *not* $\cancel{A(t)A'(t)}$.

Let

$$f(x) = \begin{cases} (e^x - 1)/x & x \neq 0 \\ 1 & x = 0. \end{cases}$$

If you want to compute $f'(0)$, none of the differentiation formulas is of any use. Instead, you must go back to the definition:

$$f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}.$$

You must now evaluate f in the expression on the right hand side. To evaluate $f(0)$, you use the second line in the definition of f, which says that if $x = 0$ then $f(x) = 1$. To evaluate $f(h)$, since you are trying to find a limit as $h \to 0$ of something, you are not allowed to take $h = 0$. Therefore you *must* use the *first* line in the definition of f, which says that if $x \neq 0$ then $(e^x - 1)/x$. Thus, $f(h) = (e^h - 1)/h$.

11. Linear approximation

I will explain very clearly in class the difference between "=" and "≈". If $f(4) = -5$ and $f'(4) = -2$, so that the linear approximation to $f(x)$ near 4 is $L(x) = 5 - 2(x - 4)$, then generally

$$\cancel{f(x) = -5 - 2(x-4)} \qquad \text{and} \qquad \cancel{f(x) = L(x)}$$

are **false**. Therefore it is wrong to write those statements. Usually the best that can be said is that they are *approximately* equal, that is,

$$f(x) \approx 5 - 2(x-4) \qquad \text{and} \qquad f(x) \approx L(x).$$

In particular,

$$f(3.97) \approx -4.96, \qquad \text{not} \qquad \cancel{f(3.97) = -4.96}.$$

On the other hand, we do have

$$L(3.97) = -4.96,$$

so that it is inappropriate to write

$$\cancel{L(3.97) \approx -4.96}.$$

When you are trying to say two things are equal, you *must* write "=".

When doing linear approximations, don't multiply out the expression involving $x - a$. If, say, $L(x) = -5 - 2(x - 4)$, for further computations it is better to leave it as is, since $x - 4$ will be close to zero. It is a waste of effort, and also makes later calculations harder, to rewrite it as $L(x) = 3 - 2x$.

It is correct to write

$$\ln\left(\tfrac{1}{2}\right) \approx -0.693.$$

It is wrong to write

$$\cancel{\ln\left(\tfrac{1}{2}\right) = -0.693},$$

because the right hand side is only a decimal approximation. This violates the meaning of "="; see Section 6.

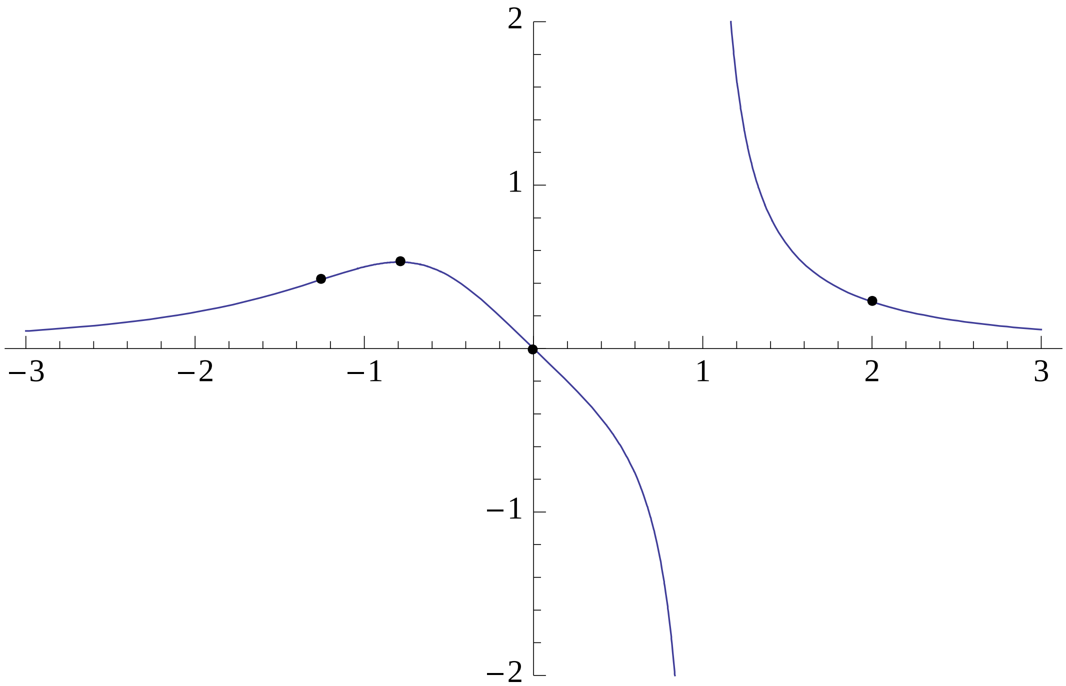
12. Pay attention in graphing problems

All graphs must show a scale on both axes. It needn't be elaborate, but something must be there. The axes themselves must be labelled.

Many graphs given in problems asking for the graph of a function do not match the information presented about the graph. They are not strictly increasing where they are claimed to be, or not concave up where they are claimed to be, or do not even have the intercepts they are claimed to have. If your graph doesn't match the information you derive about the function from calculus, you have clearly done something wrong.

The following is one common mistake. Here is the graph of the function

$$f(x) = \frac{x}{x^3 - 1}:$$



It is clearly not true that f is strictly decreasing on the interval $\left(-1/\sqrt[3]{2},\, \infty\right)$. For example, $f(2) > f(0)$. What is true is that f is strictly decreasing on the intervals $\left(-1/\sqrt[3]{2},\, 1\right)$ and $(1, \infty)$. (**Caution:** Some WeBWorK problems were written by people who didn't understand this.)

13. Read the problems

Read the problems! There will be problems giving the graph of the **derivative** of a function, not of the function itself. A number of people answer questions in such a problem as if the graph were the graph of the original function; obviously, the answers will be wrong.

If the problem asks for reasons, and you don't give any, obviously you didn't do the problem.

If the problem asks about points in $(-10, 8)$, this excludes -10 and 8.

In a problem asking specifically about what happens at $x = 6$, $x = 3.35$, and $x = -6$, but nowhere else, what happens at, for example, $x = -7.5$ is not relevant.

14. Simplify, but that does not necessarily mean to multiply out

Leave expressions in factored form unless there is good reason not to. In particular, "simplify" does not mean "always multiply out". For example, the following answers need have nothing further done to them:

$$\frac{t^{-1/2}}{2(t+1)} \qquad \text{and} \qquad \frac{5t}{(t-9)(2t-3)}.$$

Here are three examples, which are steps in actual homework problems in Math 251.

First, suppose that f is the function defined by $f(x) = \frac{1}{6-x}$, and you are supposed to simplify the expression

$$\frac{f(x) - f(2)}{x - 2}.$$

Here is the right way to carry this out:

$$\frac{f(x) - f(2)}{x - 2} = \frac{\frac{1}{6-x} - \frac{1}{6-2}}{x - 2} = \frac{\frac{1}{6-x} - \frac{1}{4}}{x - 2} = \frac{\left(\frac{4-(6-x)}{4(6-x)}\right)}{x - 2} = \frac{\left(\frac{x-2}{4(6-x)}\right)}{x - 2} = \frac{1}{4(6 - x)}.$$

If you get to the third expression and proceed as follows:

$$\frac{\left(\frac{4-(6-x)}{4(6-x)}\right)}{x - 2} = \frac{4 - (6 - x)}{4(6 - x)(x - 2)} = \frac{4 - (6 - x)}{-4x^2 + 32x - 48} = \frac{x - 2}{-4x^2 + 32x - 48},$$

then you have made the cancellation, which was obvious before, hard to find. (This calculation is part of the problem of finding $f'(2)$ directly from the definition, a standard type of problem in Math 251.)

Similarly, suppose that in the expression

$$\frac{x - 5}{3\left(\sqrt{5x} - 5\right)},$$

you are supposed to rationalize the denominator and then simplify as much as possible. Do it this way:

$$\begin{aligned} \frac{x - 5}{3\left(\sqrt{5x} - 5\right)} &= \frac{(x - 5)\left(\sqrt{5x} + 5\right)}{3\left(\sqrt{5x} - 5\right)\left(\sqrt{5x} + 5\right)} = \frac{(x - 5)\left(\sqrt{5x} + 5\right)}{3(5x - 25)} \\ &= \frac{(x - 5)\left(\sqrt{5x} + 5\right)}{3 \cdot 5(x - 5)} = \frac{\sqrt{5x} + 5}{3 \cdot 5}. \end{aligned}$$

At the third expression, you should do no further multiplication in either the numerator or denominator. It is far harder to find the correct cancellation from the second expression below than from the first, even though they are equal:

$$\frac{(x - 5)\left(\sqrt{5x} + 5\right)}{3(5x - 25)} = \frac{x\sqrt{5x} - 5\sqrt{5x} + 5x - 25}{15x - 75}.$$

(This calculation is part of finding $\displaystyle\lim_{x \to 5} \frac{x - 5}{3\left(\sqrt{5x} - 5\right)}$, also a standard type of problem in Math 251.)

However, to solve for D in the equation

$$y + xD = -\sin(x + y)\,(1 + D)\,,$$

you *must* multiply out the right hand side. Otherwise, you will be unable to isolate D. (This calculation is part of finding $\dfrac{dy}{dx}$, with $D = \dfrac{dy}{dx}$, by implicit differentiation of the implicit formula $xy = \cos(x + y)$, another standard type of problem in Math 251.)

Simplify things intended for use in further calculations. Simplify appropriately as you go along,, not just at the end, to keep down the complexity. For the function $h(x) = 3x^3 - 4x$, here is the right way to find $h\left(-\sqrt{\frac{4}{9}}\right)$:

$$h\big(-\tfrac{2}{3}\big) = 3\big(-\tfrac{2}{3}\big)^3 - 4\big(-\tfrac{2}{3}\big) = -\frac{3 \cdot 2^3}{3^3} + \frac{4 \cdot 2}{3} = -\frac{2^3}{3^2} + \frac{8}{3} = -\frac{8}{9} + \frac{24}{9} = \frac{16}{9}.$$

Cancel a factor of 3 in the first term *before* multiplying anything out, so that the common denominator is 9, not 27 (or, worse, 81).

15. Do all the steps in optimization problems

On applied maximization/minimization problems:

First, you must explicitly calculate the domain over which you are trying to minimize your function.

Second, calling your function f, you must find **all** solutions to $f'(x) = 0$. You must explicitly reject those that are not in the domain. Otherwise, I will assume you did not correctly solve the equation $f'(x) = 0$.

Third, you must explicitly check that you found a global minimum (or maximum) on your domain. (Remember the stealth bomber blunder!) It isn't enough to check that you found a *local* minimum (or maximum). The problem asks for a *global* minimum (or maximum), and you must show me that you really found one.

If you omit all of the things described above, even if everything else is done perfectly, you will get not much more than half credit for the problem, both on homework and on exams.