SOLUTIONS TO THE SAMPLE FINAL EXAM QUESTIONS, MATH 251 (PHILLIPS), SPRING 2025

Contents

1. Final Exam Information

At least 90% of the points on the real exam will be modifications of problems from Midterms 1 and 2 from the last time I taught the course, real and sample midterms and quizzes so far in this course (including Midterms 0), the problems below, homework problems (including written homework and WeBWorK), and worksheet problems. Note, though, that the exact form of the functions to be differentiated and of the limits to be computed could vary substantially, and the methods required to do them might occur in different combinations. Word problems could have rather different descriptions, but similar methods will be used.

Be sure to get the notation right! (This is a frequent source of errors.) You have seen the correct notation for limits etc. in the book, in handouts, in files posted on the course website, and on the blackboard; *use it*. The right notation will help you get the mathematics right, and incorrect notation will lose points.

The exam will be 200 points, 2 hours, and will be (in my estimation; possibly wrong) at least a little less than twice the length of a midterm. The section "Sample Final Exam" has been hastily assembled from old exams, without careful consideration of length.

I will allow one page of notes, rather than just a file card.

Exams will be available for inspection when they are graded, probably by Thursday 20 March. I keep the originals, but you can get a copy on request. Grading complaints must be submitted in writing before final grades are turned in. Extra credit for catching errors is available through 5:00 pm Thursday 20 March, possibly later (but no promises).

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2. Sample Final Exam

The problems here are intended to give a reasonable idea of how much of the final exam will come from each part (broadly interpreted) of the course. Finer details may well be different.

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1. (11 points/part.) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist or there is not enough information to evaluate them. Give justification in all cases (not just heuristic arguments). Remember to use correct notation.

(a) $\displaystyle\lim_{x \to 3} \frac{x^2 - 9}{\cos(x) - 3}$.

Solution: As always, the first thing we do is try to substitute $x = 3$. We get

$$\lim_{x \to 3} \frac{x^2 - 9}{\cos(x) - 3} = \frac{3^2 - 9}{\cos(3) - 3} = \frac{0}{\cos(3) - 3} = 0.$$

Note that L'Hospital's Rule doesn't apply, since the limit does not have an indeterminate form. If you try to use it anyway, you get

$$\lim_{x \to 3} \frac{2x}{-\sin(x)} = -\frac{6}{\sin(3)},$$

which is the wrong answer.

(b) $\displaystyle\lim_{y \to \infty} \frac{2y + 216}{13y - 5\sin(y)}$. (Be sure to show your work!)

Solution: The limit has the indeterminate form "$\frac{\infty}{\infty}$". Therefore more work is needed. We factor out y from both the numerator and denominator, and then use the limit laws:

$$\lim_{y \to \infty} \frac{2y + 216}{13y - 5\sin(y)} = \lim_{y \to \infty} \frac{2 + \frac{216}{y}}{13 - \frac{5\sin(y)}{y}} = \frac{2 + \lim_{y \to \infty} \frac{216}{y}}{13 - \lim_{y \to \infty} \frac{5\sin(y)}{y}}.$$

Certainly $\lim_{y \to \infty} \frac{216}{y} = 0$. Since $-5 \leq 5\sin(y) \leq 5$ for all y, we get $\lim_{y \to \infty} \frac{5\sin(y)}{y} = 0$ by the Squeeze Theorem. Therefore

$$\lim_{y \to \infty} \frac{2y + 216}{13y - 5\sin(y)} = \frac{2}{13}.$$

Here is a different way to arrange essentially the same calculation:

$$\begin{aligned}
\lim_{y \to \infty} \frac{2y + 216}{13y - 5\sin(y)} &= \lim_{y \to \infty} \frac{\left(\frac{1}{y}\right)(2y + 216)}{\left(\frac{1}{y}\right)(13y - 5\sin(y))} = \lim_{y \to \infty} \frac{2 + \frac{216}{y}}{13 - \frac{5\sin(y)}{y}} \\
&= \frac{2 + \lim_{y \to \infty} \frac{216}{y}}{13 - \lim_{y \to \infty} \frac{5\sin(y)}{y}} = \frac{2 + 0}{13 + 0} = \frac{2}{13}.
\end{aligned}$$

The hypotheses of L'Hospital's Rule are satisfied, but it doesn't help. Applying it once gives

$$\lim_{y \to \infty} \frac{2}{13 + 2\cos(y)}.$$

This limit does not exist: the function oscillates between $\frac{2}{15}$ and $\frac{2}{9}$.

(c) $\displaystyle\lim_{x \to 0} \frac{5x^2}{1 - \cos(7x)}$.

Solution: The limit has the indeterminate form $\dfrac{0}{0}$. Therefore we may try L'Hospital's Rule. The chain rule gives $\dfrac{d}{dx}\big(1-\cos(7x)\big) = 7\sin(7x)$, so

$$\lim_{x\to 0} \frac{5x^2}{1-\cos(7x)} = \lim_{x\to 0} \frac{10x}{7\sin(7x)}$$

if the second limit exists. The second limit also has the indeterminate form $\dfrac{0}{0}$. Therefore we may try L'Hospital's Rule again. A similar calculation shows that

$$\lim_{x\to 0} \frac{10x}{7\sin(7x)} = \lim_{x\to 0} \frac{10}{49\cos(7x)},$$

if the limit on the right exists. But the limit on the right is equal to $10/[49\cos(7\cdot 0)] = 10/49$. Therefore

$$\lim_{x\to 0} \frac{5x^2}{1-\cos(7x)} = \frac{10}{49}.$$

(d) $\displaystyle\lim_{x\to 4^+} \frac{e^{-3x}}{x-4}$. (Be sure to show your work!)

Solution: At 4, the denominator is zero but the numerator is not. Moreover, both the denominator and the numerator are continuous at 4. Therefore the function $f(x) = \frac{e^{-3x}}{x-4}$ has a vertical asymptote at $x = 4$.

For $x > 4$ but very close to 4, the denominator is positive and very close to zero. The numerator is very close to $e^{-3\cdot 4} = \frac{1}{e^{12}} > 0$, so is positive and not close to zero. Therefore the quotient is positive and very far from zero. Thus

$$\lim_{x\to 4^+} \frac{e^{-3x}}{x-4} = \infty.$$

L'Hospital's Rule doesn't apply, since the limit does not have an indeterminate form. If you try to use it anyway, you get

$$\cancel{\lim_{x\to 4^+} \frac{-3e^{-3x}}{1} = -\frac{3}{e^{12}}},$$

which is the wrong answer.

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2. (6 points.) Let $w(t)$ be the water flow at time t in a river at a particular measuring station. Assume that t is measured in days, and that $w(t)$ is measured in m^3/sec. What are the units of $w'(t)$?

Solution: m^3/sec per day, or $\text{m}^3/(\text{sec}\cdot\text{day})$.

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3. (12 points) Find the equation of tangent line to the graph of $g(x) = 2x + 4\sqrt{3x-2}$ at $x = 2$. You need not calculate the derivative directly from the definition.

Solution: We need the slope, which is $g'(2)$, and a point on the line, such as

$$(2,\ g(2)) = \big(2,\ 2\cdot 2 + 4\sqrt{3\cdot 2 - 2}\big) = \big(2,\ 4 + 4\sqrt{4}\big) = (2, 12).$$

To find $g'(2)$, we rewrite the function as $g(x) = 2x + 4(3x-2)^{1/2}$. Using the chain rule on the second term, we get

$$g'(x) = 2 + 4\left(\frac{1}{2} \cdot (3x-2)^{-1/2} \cdot 3\right) = 2 + \frac{6}{\sqrt{3x-2}},$$

so

$$g'(2) = 2 + \frac{6}{\sqrt{3 \cdot 2 - 2}} = 2 + \frac{6}{2} = 5.$$

Therefore the equation of the tangent line is $y - 12 = 5(x-2)$, which can be rearranged to give $y = 5x + 2$.

We want the slope at the *particular* value $x = 2$. Therefore we must substitute $x = 2$ in the formula for the derivative $g'(x)$ *before* using it as the slope of a line. The equation

$$y - 12 = \left(2 + \frac{6}{\sqrt{3x-2}}\right)(x-2)$$

is wrong—it is not even the equation of any line.

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4. (10 points/part) Differentiate the functions as requested.

(a) Find $f'(x)$, where $f(x) = \pi^3 + \dfrac{2x+1}{x^2+1}$.

Solution: The derivative of π^3 is zero because π^3 is a constant. On the rest, use the quotient rule,

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

This gives

$$f'(x) = \frac{2 \cdot (x^2+1) - (2x+1) \cdot (2x)}{(x^2+1)^2} = \frac{2x^2 + 2 - 4x^2 - 2x}{(x^2+1)^2} = \frac{-2x^2 - 2x + 2}{(x^2+1)^2}.$$

(The simplification is required.)

(b) Let $f(t) = e^{7t + \arcsin(t)} + \csc\left(\frac{\pi}{3}\right)$. Find $f'(t)$.

Solution: We use the chain rule. The derivative of $\csc\left(\frac{\pi}{3}\right)$ is zero because $\csc\left(\frac{\pi}{3}\right)$ is a constant. Thus

$$f'(t) = e^{7t + \arcsin(t)} \cdot \frac{d}{dt}\big(7t + \arcsin(t)\big) = e^{7t + \arcsin(t)}\left(7 + \frac{1}{\sqrt{1-t^2}}\right).$$

One can rewrite this expression as

$$f'(t) = 7e^{7t + \arcsin(t)} + \frac{e^{7t + \arcsin(t)}}{\sqrt{1-t^2}},$$

but this isn't really an improvement, and is not required.

(c) Let $q(x) = \ln(x)\cos(x^2 - cx)$, where c is a constant. Find $q'(x)$.

Solution: Use the product rule. The chain rule is needed to differentiate $\cos(x^2 - cx)$. Thus:

$$\begin{aligned} q'(x) &= \frac{d}{dx}\big(\ln(x)\big)\cos(x^2 - cx) + \ln(x)\frac{d}{dx}\big(\cos(x^2 - cx)\big) \\ &= \frac{1}{x}\cos(x^2 - cx) + \ln(x)(-\sin(x^2 - cx))(2x - c) \\ &= \frac{1}{x}\cos(x^2 - cx) - \ln(x)(2x - c)\sin(x^2 - cx). \end{aligned}$$

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5. (30 points.) A large spherical snowball was melting. (A child took it inside the house at 11:00 am, and his parents had not yet noticed.) At 11:07 am, its radius was 30 cm, and was decreasing at $\frac{1}{3}$ cm per minute. Was its surface area increasing or decreasing? At what rate? (Be sure to include the correct units in your answer.)

Note: There is no picture in this file.

Solution: Let $r(t)$ be the radius at time t, measured in centimeters and with time measured in minutes after 11:00 am, and let $S(t)$ be the surface area at time t, measured in square centimeters. Both the surface area and the radius vary with time, so must be treated as functions not constants.

The time we are interested in is $t = 7$. We are given $r(7) = 30$ and $r'(7) = -\frac{1}{3}$. The functions $S(t)$ and $r(t)$ are related by the equation

$$S(t) = 4\pi\,[r(t)]^2 .$$

(You are expected to know this formula.) Differentiate with respect to t:

$$S'(t) = 4\pi \cdot 2r(t)r'(t) = 8\pi r(t)r'(t).$$

(Don't forget the factor $r'(t)$! That will spoil the whole thing!) Evaluate this at $t = 7$, using $r(7) = 30$ and $r'(7) = -\frac{1}{3}$. This gives

$$S'(7) = 8\pi r(7)r'(7) = 8\pi \cdot 30 \cdot \left(-\frac{1}{3}\right) = -80\pi.$$

So the surface area is decreasing at the rate of 80π square centimeters per minute. (Don't forget the units!)

It is not correct to say that the the surface area is ~~decreasing at -80π~~ square centimeters per minute. That would mean it is increasing at 80π square centimeters per minute.

Here, for reference, is what the solution looks like in physicists' notation:

$$S = 4\pi r^2.$$

Differentiate with respect to t:

$$\frac{dS}{dt} = 4\pi \cdot 2r\frac{dr}{dt} = 8\pi r\frac{dr}{dt}.$$

(Don't forget the factor $\dfrac{dr}{dt}$!) Now substitute $r = 30$ and $\dfrac{dr}{dt} = -\frac{1}{3}$, getting $\dfrac{dS}{dt} = 8\pi \cdot 30 \cdot \left(-\frac{1}{3}\right) = -80\pi$.

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6. A fire-breathing monster is thrown upwards on the planet Yuggxth. Its height t seconds after it is thrown is $40t - 5t^2$ feet, until it hits the ground again.

(a) (4 points) Is the monster falling or rising 6 seconds after being thrown? How fast?

Solution: Let $y(t)$ be the height, in feet, of the monster t seconds after it is thrown. Then $y(t) = 40t - 5t^2$ for t at most the time at which it hits the ground. Since

$$y(6) = (40)(6) - (5)(6^2) = 240 - 180 > 0,$$

after 6 seconds it hasn't hit the ground yet. Therefore the vertical velocity at time t is $y'(t) = 40 - 10t$, and the vertical velocity at time 6 is $y'(6) = 40 - (10)(6) = -20$. Therefore the monster is falling at 20 feet/second.

Note: You *must* include the units in this kind of problem.

It is not correct to say that the monster is ~~falling at -20~~ feet/second. That would mean it is rising at 20 feet/second.

(b) (7 points) How long after being thrown does the monster reach its maximum height?

Solution: Let $y(t)$ be the height, in feet, of the monster t seconds after it is thrown. Then $y(t) = 40t - 5t^2$ for t at most the time at which it hits the ground. The monster reaches its greatest height when it stops rising and starts falling, that is, when the derivative $y'(t)$ changes from positive to negative. We have $y'(t) = 40 - 10t$, which changes from positive to negative when $y'(t)$ is zero, that is, at $t = 4$. So the monster reaches its greatest height 4 seconds after being thrown.

Note: You *must* include the units in this kind of problem.

The graph of position as a function of time is a parabola which opens downward. The high point is therefore at the vertex of the parabola. There are ways to find the vertex without explicitly using calculus, and these are correct solutions. However, any such solution must explicitly state that the vertex of the parabola is being found, and then use a mathematically correct way to do so; otherwise, it will receive no credit. (When I read a solution which uses a formula to find the vertex of the parabola but gives no explanation for why doing so is relevant, I see no relation to the question being asked.)

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7. (15 points) If $y^3 = \sin(7x - y) - \sqrt{2}$, find $\dfrac{dy}{dx}$ by implicit differentiation. (You must solve for $\dfrac{dy}{dx}$.)

Solution: Let's write it with y as an explicit function $y(x)$ of x:

$$y(x)^3 = \sin(7x - y(x)) - \sqrt{2}.$$

Differentiate both sides with respect to x, using the chain rule on both sides:

$$3[y(x)]^2 y'(x) = \cos(7x - y(x)) \frac{d}{dx}(7x - y(x)) = \cos(7x - y(x)) \left(7 - y'(x)\right).$$

(The derivative of $\sqrt{2}$ is zero because $\sqrt{2}$ is a constant.)

Now solve for $y'(x)$:

$$\begin{aligned} 3[y(x)]^2 y'(x) &= 7\cos(7x - y(x)) - \cos(7x - y(x))y'(x) \\ 3[y(x)]^2 y'(x) + \cos(7x - y(x))y'(x) &= 7\cos(7x - y(x)) \\ y'(x) &= \frac{7\cos(7x - y(x))}{3[y(x)]^2 + \cos(7x - y(x))}. \end{aligned}$$

This expression can't be further simplified.

For those who prefer the other notation, here it is written with $\frac{dy}{dx}$. Differentiate with respect to x, using the chain rule on both sides, just as before:

$$3y^2 \frac{dy}{dx} = \cos(7x - y) \frac{d}{dx}(7x - y) = \cos(7x - y) \left(7 - \frac{dy}{dx} \right).$$

(The derivative of $\sqrt{2}$ is zero because $\sqrt{2}$ is a constant.)

Now solve for $\frac{dy}{dx}$:

$$\begin{aligned} 3y^2 \frac{dy}{dx} &= 7\cos(7x - y) - \cos(7x - y)\frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} + \cos(7x - y)\frac{dy}{dx} &= 7\cos(7x - y) \\ \frac{dy}{dx} &= \frac{7\cos(7x - y)}{3y^2 + \cos(7x - y)}. \end{aligned}$$

As before, this expression can't be further simplified.

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8. (17 points) Suppose we know the following about the function h:

(1) h is defined and continuous on $(-\infty, \infty)$, and $h'(x)$ and $h''(x)$ exist on all of $(-\infty, \infty)$.
(2) h has only one critical number, namely 1.
(3) $h(1) \approx -2.718$.
(4) $h'(x) < 0$ for x in the interval $(-\infty, 1)$.
(5) $h'(x) > 0$ for x in the interval $(1, \infty)$.
(6) The only solution to $h''(x) = 0$ is $x = 0$.
(7) $h(0) = -2$.
(8) $h''(x) < 0$ for x in the interval $(-\infty, 0)$.
(9) $h''(x) > 0$ for x in the interval $(0, \infty)$.
(10) $\lim_{x \to -\infty} h(x) = 0$.

Find the asymptotes, intervals of increase and decrease, local minimums and maximums, intervals of concavity up and down, and inflection points. Then draw the graph of h. Make sure that the graph matches the information about concavity etc. that you found.

Solution: By (4) and (5), the function h is strictly decreasing on $(-\infty, 1)$ and strictly increasing on $(1, \infty)$. Therefore h has a local minimum at $x = 1$. This is the only possible location for a local minimum or a local maximum, because 1 is the only critical number.

By (8) and (9), the function h is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Therefore h has an inflection point at $x = 0$.

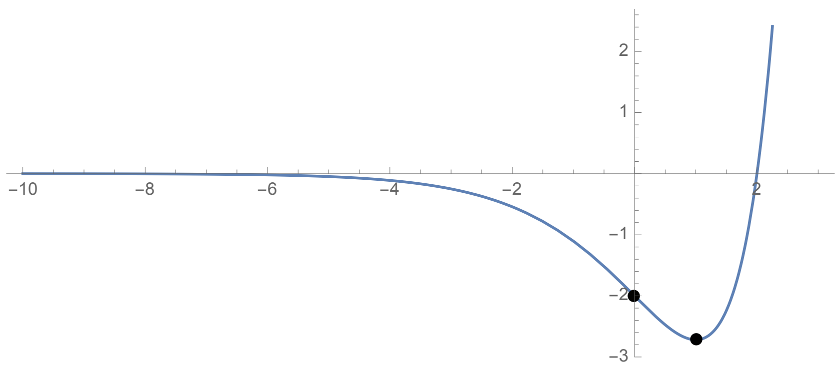
By (9), the function h has a horizontal asymptote at $y = 0$, which is approached in the negative direction.

To get the graph, it is perhaps easiest to break up the intervals and behavior at all the points listed above, as follows.

Interval	Behavior of h
$(-\infty, 0)$	Strictly decreasing and concave down
$(0, 1)$	Strictly decreasing and concave up
$(1, \infty)$	Strictly increasing and concave up

Since h is strictly increasing and concave up on $(1, \infty)$, the rate at which $h(x)$ increases must increase with x. Therefore $\lim_{x \to -\infty} h(x) = \infty$. So no horizontal asymptote can be approached in the positive direction. There are no vertical asymptotes since h is defined and continuous on the whole real line.

Here is the graph of a function satisfying these conditions. The points in (3) and (7) are shown with black dots.



Note: A correct graph must clearly show that $h(x)$ goes to $+\infty$ as $x \to \infty$.

In case anyone is interested, the function I used is $h(x) = (x - 2)e^x$. Its first two derivatives are

$$h'(x) = e^x + (x - 2)e^x = (x - 1)e^x \qquad \text{and} \qquad h''(x) = e^x + (x - 1)e^x = e^x.$$

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9. (35 points.) Consider a flat square sheet of cardboard that is 6 feet wide and 6 feet tall. You cut out a squares of equal size out of each corner of the sheet. Then you fold the remaining sides (flaps) of the cardboard up to make a rectangular box (with no top). If you want to create box with the largest volume, what is the side length of each square that is cut out?

Be sure to verify that your maximum or minimum really is what you claim it is.

Solution: There is no picture in this file.

Let x be the side length of the corner squares which are removed. The resulting box then has height x, and its base is a square of side length $6-2x$. Therefore the volume is $V(x) = x(6-2x)^2$.

Clearly $x \geq 0$. (Taking $x = 0$ gives a degenerate case, but it still makes sense.) Also, $x \leq 3$, since if $x > 3$ then the removed squares overlap. (Again, $x = 3$ is degenerate.) Therefore we must maximize the function

$$V(x) = x(6 - 2x)^2 \tag{1}$$

on the interval $[0, 3]$.

It is easiest to multiply out before differentiating:

$$V(x) = 36x - 24x^2 + 4x^3 = 4(9x - 6x^2 + x^3),$$

so

$$V'(x) = 4(9 - 12x + 3x^2) = 12(3 - 4x + x^2) = 12(x - 1)(x - 3).$$

Thus $V'(x) = 0$ exactly when $x = 1$ or $x = 3$.

We compare the values $V(0) = 0$, $V(1) = 1 \cdot 4^2 = 16$, and $V(3) = 0$. (Here $x = 3$ is both an endpoint and a critical point. For all of these, it is easiest to use the formula (1).) Obviously $V(1)$ is the largest. So the maximum volume is gotten by cutting out squares which are 1 foot by 1 foot.

The second derivative test for an absolute maximum doesn't work here. We have $V''(x) = 12(-4 + 2x)$, which is negative only on $(0, 2)$, when we need it to be negative on $(0, 3)$.

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3. Extra Sample Problems for Final Exam

These extra problems are mostly on material since the second midterm, or are slightly different problem types for earlier material. They are not representative of how much of the final exam will be on each part of the course. See the sample Midterm 1 and sample Midterm 2.

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10. (6 points.) Let $w(t)$ be the water flow at time t in a river at a particular measuring station. Assume that t is measured in days, and that $w(t)$ is measured in m^3/sec. During the beginning of the rainy season, do you expect $w'(t)$ to be positive or negative? Why?

Solution: Positive, because the amount of water flowing in the river should be increasing.

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11. (14 points.) You are told that $f(x)$ is a function defined for $x \neq -1$ whose derivative is $f'(x) = \dfrac{x^2 + 2x - 3}{(x + 1)^2}$. Find the critical points of $f(x)$ and identify each of them as a local minimum, local maximum, or neither.

Solution: We write

$$f'(x) = \frac{x^2 + 2x - 3}{(x + 1)^2} = \frac{(x + 3)(x - 1)}{(x + 1)^2}.$$

This is zero exactly when $(x + 3)(x - 1) = 0$. Therefore the critical points are at $x = -3$ and $x = 1$. (The point $x = -1$ isn't a critical point because it isn't in the domain of f.)

We use the first derivative method to determine the types of these points. The function is not defined at $x = -1$; there is a vertical asymptote there. Therefore we have **four** intervals to consider:

$$(-\infty, -3), \qquad (-3, -1), \qquad (-1, -1), \qquad \text{and} \qquad (1, \infty).$$

On all of them, $(x + 1)^2 > 0$, so we need only consider the sign of $(x + 3)(x - 1)$.

- On $(-\infty, -3)$, $x + 3 < 0$ and $x - 1 < 0$, so $(x + 3)(x - 1) > 0$. Thus $f'(x) > 0$ and f is increasing.
- On $(-3, -1)$, $x + 3 > 0$ and $x - 1 < 0$, so $(x + 3)(x - 1) < 0$. Thus $f'(x) < 0$ and f is decreasing.

- On $(-1, 1)$, $x+3 > 0$ and $x-1 < 0$, so $(x+3)(x-1) < 0$. Thus $f'(x) < 0$ and f is decreasing.
- On $(1, \infty)$, $x+3 > 0$ and $x-1 > 0$, so $(x+3)(x-1) > 0$. Thus $f'(x) > 0$ and f is increasing.

Therefore f has a local maximum at $x = -3$ and a local minimum at $x = 1$.

Since f' is continuous on each of these intervals, you can also use test points. Details are omitted, but I got

$$f'(-4) = \frac{5}{9} > 0, \qquad f'(-2) = -3 < 0, \qquad f'(0) = -3 < 0, \qquad \text{and} \qquad f'(2) = \frac{5}{9} > 0.$$

The second derivative test also works. The calculation is a bit messy, and is omitted, but the answer can be simplified to

$$f''(x) = \frac{8}{(x+1)^3}.$$

Now it is easy to see that $f''(-3) = -1 < 0$ and $f''(1) = 1 > 0$.

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12. (7 points) The derivative of the function $f(x) = (x-3)e^{-x}$ is given by $f'(x) = -(x-4)e^{-x}$, and the second derivative is given by $f''(x) = (x-5)e^{-x}$. Determine whether f has a local minimum, a local maximum, or neither at $x = 4$.

Solution: We have $f'(4) = 0$, so f does in fact have a critical point at $x = 4$..

To decide whether f has a local minimum or maximum at $x = 4$, we consider the factors of $f'(x)$:

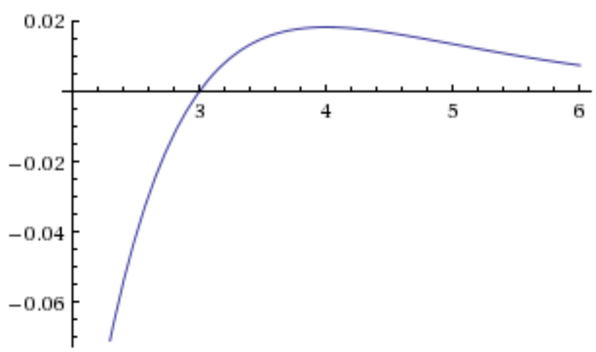
- On $(-\infty, 4)$, $-1 < 0$, $x-4 < 0$, and $e^{-x} > 0$, so $f'(x) > 0$ and f is increasing.
- On $(4, \infty)$, $-1 < 0$, $x-4 > 0$, $e^{-x} > 0$, so $f'(x) < 0$ and f is decreasing.

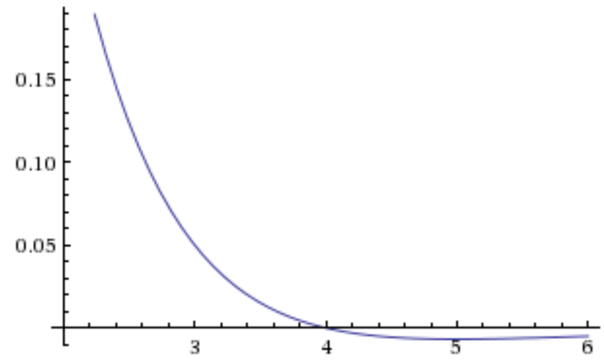
So f has a local maximum at $x = 4$.

On $(-\infty, 4)$, alternatively, since f' is continuous on this interval, we can check that, for example, $f'(0) = 4 > 0$. On $(4, \infty)$, alternatively, since f' is continuous on this interval, we can check that, for example, $f'(5) = -e^{-5} < 0$.

We can also use the second derivative test. We have $f''(4) = -e^{-4} < 0$, showing that there is a local maximum at $x = 4$.

For reference, here are graphs of f (below left) and f' (below right) on the interval $(2, 6)$. (These are not required as part of the solution.)





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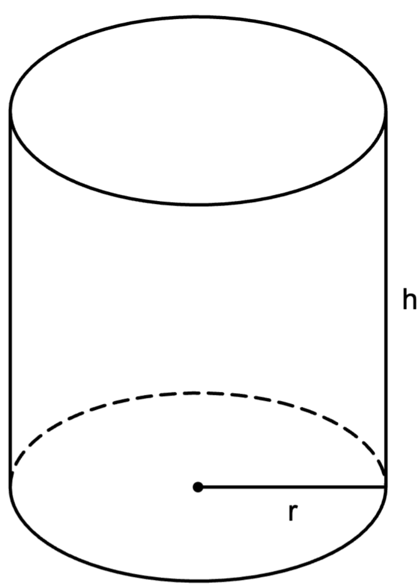
13. (18 points.) You want to construct a cylindrical water tank can, with no top. Your budget allows for the purchase of 900 square feet of material. What is the maximum possible volume?

Set up, but **do not attempt to solve**, the appropriate maximization or minimization problem. That is, give a function $f(x)$ (not necessarily calling the function f or the variable x) for x a suitable quantity related to the problem (say what it actually is!), give a suitable domain, and say whether you want to maximize or minimize your function on this domain. Provide justification for all steps (possibly including a picture).

Solution. Here is a picture, with the appropriate quantities on it labelled:



Here, r is the radius of the base of the cylinder and h is its height, both in feet. Also let V be the volume, in cubic feet, and let A be the area of the top and side combined, in square feet. The bottom has area πr^2. The side, if cut vertically and rolled out flat, is a rectangle with one side of length h and the other of length $2\pi r$ (the circumference of the circle). Therefore

$$V = \pi r^2 h \qquad \text{and} \qquad 900 = A = \pi r^2 + 2\pi r h.$$

We need to solve for one of the variables. The only reasonable choice is to solve for h: $900 - \pi r^2 = 2\pi r h$, so

$$h = \frac{900 - \pi r^2}{2\pi r} = \frac{450}{\pi r} - \frac{r}{2}. \tag{2}$$

Putting this in the formula for the volume, we get

$$V = \pi r^2 \left(\frac{450}{\pi r} - \frac{r}{2} \right) = 450r - \frac{\pi r^3}{2}.$$

Obviously $r \geq 0$. Also, $h \geq 0$. Using the first formula for h in (2), and $2\pi r > 0$ (so that multiplying by it does not reverse the inequality) this says $900 - \pi r^2 \geq 0$, so $r^2 \leq \dfrac{900}{\pi}$, whence $-\dfrac{30}{\sqrt{\pi}} \leq r \leq \dfrac{30}{\sqrt{\pi}}$. Therefore the domain for r is $\left[0, \dfrac{30}{\sqrt{\pi}}\right]$. We need to *maximize* $V(r) = 450r - \dfrac{\pi r^3}{2}$ for r in $\left[0, \dfrac{30}{\sqrt{\pi}}\right]$, and r is the radius of the bottom of the cylinder, in feet.

Stop here; this is as far as you were told to go. □

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14. (9 points) Let g be a function such that $g(2) = 3$ and whose derivative is known to be $g'(x) = -\sqrt{2x}e^{x-2}$. (You are not given a formula for g. Don't try to guess one–you won't succeed.) Use the linearization (tangent line approximation) to estimate the value of $g(1.9)$.

Solution:

$$g(1.9) \approx g(2) + g'(2)(1.9 - 2) = 3 + (-0.1)\left(-\sqrt{4}e^0\right) = 3 + (-0.1)(-2) = 3.2.$$

(It is not correct to write ~~$g(1.9) = 3.2$~~ or ~~$g(1.9) = L(1.9)$~~, because the tangent line approximation is only an *approximation*.)

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15. (8 points) The function $f(x)$ satisfies the following three properties:

$$f(7) = 4, \qquad f'(7) = 2, \qquad \text{and} \qquad f''(7) = -3.$$

Use the linearization (tangent line approximation) to estimate the value of $f(6.97)$. (You won't need to use all the information given.)

Solution:

$$f(6.97) \approx f(7) + f'(7)(6.97 - 7) = 4 + (2)(-0.03) = 3.94.$$

(It is not correct to write ~~$f(6.97) = 3.94$~~ or ~~$f(6.97) = L(6.97)$~~, because the tangent line approximation is only an *approximation*.)

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16. (10 points.) Suppose f is continuous on $[-1, 7]$, and $f'(x)$ exists and satisfies $-2 < f'(x) < 3$ for all x in $(-1, 7)$. Give the best possible estimate on $f(7) - f(-1)$. Justify your answer.

Solution: By the Mean Value Theorem, there is c in $(-1, 7)$ such that

$$\frac{f(7) - f(-1)}{7 - (-1)} = f'(c).$$

That is,

$$\frac{f(7) - f(-1)}{8} = f'(c),$$

so

$$f(7) - f(-1) = 8f'(c).$$

Since $-2 < f'(c) < 3$, we get

$$8(-2) < f(7) - f(-1) < 8 \cdot 3,$$

that is,

$$-16 < f(7) - f(-1) < 24.$$

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17. (10 points) Find the exact value of $\lim_{x \to 0} \frac{\arctan(ax)}{\tan(x)}$ (possibly including ∞ or $-\infty$), where a is a constant. Or explain why the limit does not exist or there is not enough information to evaluate it. Give justification (not just heuristic arguments).

Solution 1: Trying to substitute $x = 0$ gives the undefined expression "$\frac{0}{0}$", so more work is needed. Since "$\frac{0}{0}$" is an indeterminate form, we may use L'Hospital's Rule. Using the chain rule on the numerator, we get

$$\frac{\frac{d}{dx}(\arctan(ax))}{\frac{d}{dx}(\tan(x))} = \frac{\frac{1}{1+(ax)^2} \cdot a}{\sec^2(x)} = \frac{a}{(1 + a^2x^2)\sec^2(x)}.$$

Now

$$\lim_{x \to 0} \frac{a}{(1 + a^2x^2)\sec^2(x)} = \frac{a}{(1 + a^2 0^2)\sec^2(0)} = \frac{a}{(1 + 0) \cdot 1} = a.$$

Since this limit exists, L'Hospital's Rule tells that

$$\lim_{x \to 0} \frac{\arctan(ax)}{\tan(x)} = \lim_{x \to 0} \frac{a}{(1 + a^2x^2)\sec^2(x)} = a.$$

Solution 2: We show separately that

$$\lim_{x \to 0} \frac{\arctan(ax)}{x} = a \qquad \text{and} \qquad \lim_{x \to 0} \frac{\tan(x)}{x} = 1.$$

Then we will get

$$\lim_{x \to 0} \frac{\arctan(ax)}{\tan(x)} = \frac{\lim_{x \to 0} \frac{\arctan(ax)}{x}}{\lim_{x \to 0} \frac{\tan(x)}{x}} = \frac{a}{1} = a.$$

For the first one, let $f(x) = \arctan(ax)$. Since $f(0) = 0$, the limit is by definition the derivative $f'(0)$. Now

$$f'(x) = \frac{1}{1 + (ax)^2} \cdot a = \frac{a}{1 + a^2x^2}.$$

Therefore $f'(0) = a$. The second one is similarly the derivative at 0 of the function $\tan(x)$, which is $\sec^2(0) = 1$. Alternatively,

$$\lim_{x \to 0} \frac{\tan(x)}{x} = \lim_{x \to 0} \left(\frac{\sin(x)}{x} \cdot \frac{1}{\cos(x)} \right) = \left(\lim_{x \to 0} \frac{\sin(x)}{x} \right) \left(\lim_{x \to 0} \frac{1}{\cos(x)} \right) = 1 \cdot \frac{1}{1} = 1.$$

18. (12 points) Consider the curve given by $x^2 - 3xy + y^2 = -1$. Find the equation of the tangent line to this curve at the point $(2, 1)$.

Solution: We write

$$x^2 - 3xy(x) + y(x)^2 = -1.$$

Differentiate with respect to x, remembering to use the product rule on the second term and the chain rule on the third term:

$$2x - 3(y(x) + xy'(x)) + 2y(x)y'(x) = 0.$$

Substitute $x = 2$ and $y(x) = 1$, getting

$$4 - 3 - 6y'(2) + 2y'(2) = 0.$$

So $y'(2) = \frac{1}{4}$, and the equation of the tangent line is $y - 1 = \frac{1}{4}(x - 2)$, which simplifies to $y = \frac{1}{4}x + \frac{1}{2}$.

It is somewhat longer to solve for $y'(x)$ before substituting $x = 2$ and $y(x) = 1$, but here is what happens:

$$2x - 3y(x) - 3xy'(x)) + 2y(x)y'(x) = 0$$

$$2x - 3y(x) = 3xy'(x)) - 2y(x)y'(x) = (3x - 2y(x))y'(x)$$

$$y'(x) = \frac{2x - 3y(x)}{3x - 2y(x)}.$$

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19. (8 points) Set $h(x) = \sin(\pi + x^2)$. At $x = 0$, does h have a local minimum, local maximum, or neither?

Solution: We have $h'(x) = 2x\cos(\pi + x^2)$. Therefore $h'(0) = 0$, so at least $x = 0$ is a critical point.

The easiest test to use is the second derivative test. We have

$$h''(x) = 2\cos(\pi + x^2) + 2x \cdot (-2x\sin(\pi + x^2)) = 2\cos(\pi + x^2) - 4x^2\sin(\pi + x^2).$$

Therefore $h''(0) = 2\cos(\pi) = -2$. Since $h''(0) < 0$, it follows that h has a local maximum at $x = 0$.

It is also possible to use the first derivative test. When $0 < x < \sqrt{\pi/2}$, we have $\pi < \pi + x^2 < 3\pi/2$, so $\cos(\pi + x^2) < 0$. Since also $x > 0$, we conclude that $h'(x) = 2x\cos(\pi + x^2) < 0$. Similar reasoning shows that when $-\sqrt{\pi/2} < x < 0$, we have $h'(x) > 0$. (The difference is the sign of x.) Therefore h is increasing on $(-\sqrt{\pi/2}, 0)$ and decreasing on $(0, \sqrt{\pi/2})$. So h has a local maximum at $x = 0$.

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20. (8 points) Set $q(x) = \dfrac{x}{1 + x^2}$. At $x = 0$, does q have a local minimum, local maximum, or neither?

Solution: By the quotient rule, we have

$$q'(x) = \frac{1 \cdot (1 + x^2) - x \cdot 2x}{(1 + x^2)^2} = \frac{1 - x^2}{(1 + x^2)^2}.$$

Therefore $q'(0) = 1$. So $x = 0$ is not a critical point, and q has neither a local minimum nor local maximum at $x = 0$.
