Math 251 Final exam worksheet NAME: SOLUTIONS

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Not all types of problems that will appear on the final exam are here. In particular, see the separate sample final exam for other possibilities, but that is not the only place you need to look. For example, a problem on the real final exam like Problem 1 here might not give you the derivatives of the function, and a problem like Problem 3 here might only ask you to set up the problem.

The point values are rough.

Solutions have not been properly proofread!

1. (15 points) Let $g(x) = 128x - 8x^3 + x^4$. Its first two derivatives are

$$g'(x) = 128 - 24x^2 + 4x^3 = 4(x-4)^2(x+2) \qquad \text{and} \qquad g''(x) = -48x + 12x^2 = 12x(x-4).$$

(You do **not** need to check these.)

Find the open intervals of increase and decrease, values of x at which critical points occur, values of x at which local minimums occur, values of x at which local maximums occur, open intervals of concavity up and down, and values of x at which inflection points occur.

Solution: The factorization shows that $g'(x) = 0$ exactly when $x = 4$ and $x = -2$. Therefore the critical points of g are -2 and 4.

We therefore need to consider the open intervals $(-\infty, -2)$, $(-2, 4)$, and $(4, \infty)$. By inspecting the factorizations, $g'(x) < 0$ for x in the interval $(-\infty, -2)$, $g'(x) > 0$ for x in the interval $(-2, 4)$, and $g'(x) > 0$ for x in the interval $(4, \infty)$.

Alternate method using test points, valid since g' is continuous:

$$g'(-3) = 4(-3-4)^2(-3+2) = 4 \cdot (-7)^2 \cdot (-1) = -4 \cdot 7^2 < 0,$$

$$g'(0) = 128 > 0, \qquad \text{and} \qquad g'(5) = 4(5-4)^2(5+2) = 4 \cdot 7 > 0.$$

We conclude that g is decreasing on $(-\infty, -2)$ and increasing on $(-2, 4)$ and on $(4, \infty)$. Therefore g has a local minimum at $x = -2$, but neither a local minimum nor a local maximum at $x = 4$. (You can also see that there is a local minimum at $x = -2$ by calculating $g''(-2) = 12(-2)(-2-4) = 12 \cdot 2 \cdot 2 > 0$, but this method doesn't help at $x = 4$.) Because g is continuous at 4 and increasing on $(-2, 4)$ and on $(4, \infty)$, it follows that g is increasing on $(-2, \infty)$.

The factorization shows that $g''(x) = 0$ exactly when $x = 0$ and $x = 4$. For concavity we therefore need to consider the open intervals $(-\infty, 0)$, $(0, 4)$, and $(4, \infty)$. By inspecting the factorizations, $g''(x) > 0$ for x in the interval $(-\infty, 0)$, $g''(x) < 0$ for x in the interval $(0, 4)$, and $g''(x) > 0$ for x in the interval $(4, \infty)$.

Alternate method using test points, valid since g'' is continuous:

$$g''(-1) = 12(-1)(-1-4) = 12 \cdot 5 > 0,$$

$$g'(1) = 12 \cdot 1 \cdot (1-4) = 12 \cdot (-3) < 0, \qquad \text{and} \qquad g''(5) = 12 \cdot 5 \cdot (5-4) = 12 \cdot 5 > 0.$$

It follows that g is concave up on $(-\infty, 0)$, concave down on $(0, 4)$, and concave up on $(4, \infty)$. Therefore g has inflection points at $x = 0$ and at $x = 4$.

2. (11 points/part.) Find the exact values of the following limits (possibly including ∞ or $-\infty$), or explain why they do not exist (not even being ∞ or $-\infty$) or there is not enough information to evaluate them. Give justification in all cases (not just heuristic arguments).

(a) $\displaystyle\lim_{x \to 1} \frac{x^2 - e^x - 6}{x - 2}$.

Solution: Both the numerator and denominator are continuous at 1, and the denominator is not zero there. Therefore the limit can be evaluated by simply substituting $x = 1$. That is,

$$\lim_{x \to 1} \frac{x^2 - e^x - 6}{x - 2} = \frac{1^2 - e - 6}{1 - 2} = e + 5.$$

L'Hospital's Rule doesn't apply, since the limit does not have an indeterminate form. If you try to use it anyway, you get

$$\require{cancel}\cancel{\lim_{x \to 1} \frac{2x - e^x}{1} = 2 - e},$$

which is the wrong answer.

(b) $\displaystyle\lim_{\varphi \to 0} \frac{\cos(2\varphi)}{\varphi}$. (Be sure to show your work!)

Solution: Since $\varphi \mapsto \cos(2\varphi)$ is continuous and takes the value $\cos(0) = 1$ when $\varphi = 0$, it follows that $\cos(2\varphi)$ is close to 1 for φ close to zero. For φ close to zero and positive, we are dividing by something close to zero and positive, giving a very large positive number. For φ close to zero and negative, we are dividing by something close to zero and negative, giving a negative number very far from zero. Therefore

$$\lim_{\varphi \to 0^+} \frac{\cos(2\varphi)}{\varphi} = \infty \qquad \text{and} \qquad \lim_{\varphi \to 0^-} \frac{\cos(2\varphi)}{\varphi} = -\infty.$$

So $\displaystyle\lim_{\varphi \to 0} \frac{\cos(2\varphi)}{\varphi}$ does not exist.

L'Hospital's Rule doesn't apply, since the limit does not have an indeterminate form. If you try to use it anyway, you get

$$\cancel{\lim_{\varphi \to 0} \frac{\sin(2\varphi) \cdot 2}{1}} = 0,$$

which is the wrong answer.

(c) $\displaystyle\lim_{x \to 0} \frac{e^{k \sin(x)} - 1}{\sin(x)}$, where k is a nonzero constant.

Solution: Trying to substitute $x = 0$ gives the undefined expression "$\frac{0}{0}$", so more work is needed. In this case, L'Hospital's Rule applies. Remember to use the chain rule when differentiating the numerator:

$$\lim_{x \to 0} \frac{e^{k \sin(x)} - 1}{\sin(x)} = \lim_{x \to 0} \frac{e^{k \sin(x)} \cdot k \cos(x)}{\cos(x)},$$

provided the last limit exists. The last limit can be evaluated by substituting $x = 0$. That is,

$$\lim_{x \to 0} \frac{e^{k \sin(x)} - 1}{\sin(x)} = \lim_{x \to 0} \frac{e^{k \sin(x)} \cdot k \cos(x)}{\cos(x)} = \frac{e^{k \sin(0)} \cdot k \cos(0)}{\cos(0)} = \frac{e^0 \cdot k \cdot 1}{1} = k.$$

3. (25 points.) An open rectangular box (no top) is to have a base that is twice as long as it is wide. 96 square feet of material are available to make the box. Find the dimensions of the box which maximizes the volume.

Include units, and be sure to verify that your maximum or minimum really is what you claim it is.

Solution: Note: No picture is included in this file. A picture may be provided separately.

We want to maximize the volume of the box; call it V. Let the dimensions of the box be l feet long, w feet wide, and h feet high. Then $V = lwh$. There are too many variables here; we must eliminate two of them.

One relation comes from the assumption that the base is twice as long as it is wide. This says $l = 2w$. We need another relation, and it will come from the requirement that the surface area be 96 square feet. (We may assume that the surface area is not less than 96 square feet, since clearly there is no benefit from not using all the material.) The front and back have area lh each, the two ends have area wh each, and the bottom has area lw. Since there is no top, this is everything, and the total surface area is $2lh + 2wh + lw$. Thus $2lh + 2wh + lw = 96$.

The easiest thing to do first is to substitute $2w$ for l everywhere. This gives

$$V = 2w^2 h \qquad \text{and} \qquad 96 = 4wh + 2wh + 2w^2 = 6wh + 2w^2.$$

Next, we solve the second equation for one of the variables and substitute in the first equation. Clearly it is easier to solve for h:

$$h = \frac{96 - 2w^2}{6w} = \frac{48 - w^2}{3w}.$$

Substitute, and use function notation since we have one variable and want to differentiate:

$$V(w) = 2w^2 h = 2w^2 \cdot \frac{48 - w^2}{3w} = 32w - \tfrac{2}{3}w^3.$$

(Simplify before differentiating!)

What are the restrictions? Clearly $w \geq 0$. (It is helpful to allow the degenerate case $w = 0$, since we will then be able to use the shortcut for maximizing a function on a closed bounded interval.) Also $h \geq 0$. Since $h = \frac{48-w^2}{3w}$ and $3w \geq 0$, this means $48 - w^2 \geq 0$. Thus $w \leq \sqrt{48}$. (Also, this tells us that $w \geq -\sqrt{48}$, but we already know $w \geq 0$.) We therefore must maximize $V(w) = 32w - \frac{2}{3}w^3$ subject to the condition $0 \leq w\sqrt{48}$.

We look for critical numbers: $V'(w) = 32 - 2w^2$, and this is zero when $w^2 = 16$, that is, $w = \pm 4$. We reject $w = -4$ because it is not in $[0, \sqrt{48}]$. (I must see you do this!) But $w = 4$ is in $[0, \sqrt{48}]$.

We are working over a closed bounded interval, so we can compare values at the critical numbers and the endpoints. We check:

$$V(0) = 0, \qquad V(4) = 32 \cdot 4 - \tfrac{2}{3} \cdot 4^3 = \tfrac{256}{3}, \qquad \text{and} \qquad V(\sqrt{48}) = 0.$$

Obviously $V(4)$ is the largest. Thus

$$w = 4, \qquad l = 2w = 8, \qquad \text{and} \qquad h = \frac{48 - w^2}{3w} = \tfrac{48-16}{12} = \tfrac{8}{3}.$$

So the dimensions are 8 feet $\times$ 4 feet $\times$ $\frac{8}{3}$ feet. (Be sure to include the units!)

Both the first and second derivative tests also work for identifying the maximum. For them, we do not need to include the degenerate cases $w = 0$ and $w = \sqrt{48}$, so we assume $0 < w < \sqrt{48}$.

Second derivative test: Calculate $V''(w) = -4w$. This is negative for all w with $0 < w < \sqrt{48}$. Therefore the function is concave down on the entire allowed interval, and any critical number (here, $w = 4$) must be an absolute maximum on this interval.

First derivative test: The derivative $V'(w) = 32 - 2w^2 = 2(16 - w^2)$ is easily seen to be positive for $0 < w < 4$ and negative for $w > 4$. Therefore $w = 4$ must represent the absolute maximum on the interval $(0, \sqrt{48})$.

4. (15 points) Find $\dfrac{dy}{dx}$ if $xy = \tan(y - x) + \ln(2)$. (Use implicit differentiation. You must solve for $\dfrac{dy}{dx}$.)

Solution: Differentiate both sides with respect to x, using the product rule on the left and the chain rule on the right:

$$1 \cdot y + x\frac{dy}{dx} = \sec^2(y - x)\frac{d}{dx}(y - x) = \sec^2(y - x)\left(\frac{dy}{dx} - 1\right).$$

(The derivative of $\ln(2)$ is immediately seen to be zero because $\ln(2)$ is a constant.) Now solve for $\dfrac{dy}{dx}$:

$$\begin{aligned} y + x\frac{dy}{dx} &= \sec^2(y - x)\frac{dy}{dx} - \sec^2(y - x) \\ x\frac{dy}{dx} - \sec^2(y - x)\frac{dy}{dx} &= -\sec^2(y - x) - y \\ [x - \sec^2(y - x)]\frac{dy}{dx} &= -\sec^2(y - x) - y \\ \frac{dy}{dx} &= \frac{-\sec^2(y - x) - y}{x - \sec^2(y - x)}. \end{aligned}$$

For those who prefer the other notation, here it is written with y as an explicit function $y(x)$ of x. Start with

$$xy(x) = \tan(y(x) - x) + \ln(2).$$

Then differentiate with respect to x, just as before:

$$1 \cdot y(x) + xy'(x) = \sec^2(y(x) - x)\frac{d}{dx}(y(x) - x) = \sec^2(y(x) - x)\left(y'(x) - 1\right).$$

Now solve for $y'(x)$:

$$\begin{aligned} y(x) + xy'(x) &= \sec^2(y(x) - x)y'(x) - \sec^2(y(x) - x) \\ xy'(x) - \sec^2(y(x) - x)y'(x) &= -\sec^2(y(x) - x) - y(x) \end{aligned}$$

$$[x - \sec^2(y(x) - x)]y'(x) = -\sec^2(y(x) - x) - y(x)$$
$$y'(x) = \frac{-\sec^2(y(x) - x) - y(x)}{x - \sec^2(y(x) - x)}.$$

5. (10 points/part) Differentiate the following functions.

(a) $w(t) = \dfrac{\cos(t)}{3t} - \dfrac{5}{\sqrt{t}} + e^2.$

Solution: We rewrite the second term of the function to make it easy to differentiate:

$$w(t) = \frac{\cos(t)}{3t} - 5t^{-1/2} + e^2.$$

Use the quotient rule on the first term, use the power rule on the second term, and note that e^2 is a constant so that its derivative is zero. This gives

$$g'(t) = \frac{-\sin(t) \cdot 3t - \cos(t) \cdot 3}{(3t)^2} - 5 \cdot \left(-\tfrac{1}{2}\right) t^{-1/2-1} = -\frac{3t\sin(t) + 3\cos(t)}{3 \cdot 3t^2} + \tfrac{5}{2}t^{-3/2} = -\frac{t\sin(t) + \cos(t)}{3t^2} + \tfrac{5}{2}t^{-3/2}.$$

The simplification is required.

Alternate (easier) solution: Rewrite both the first and second terms to make them easier to differentiate:

$$w(t) = \tfrac{1}{3}t^{-1}\cos(t) - 5t^{-1/2} + e^2.$$

Use the product rule on the first term, use the power rule on the second term, and note that e^2 is a constant so that its derivative is zero. This gives

$$g'(t) = \tfrac{1}{3}(-1)t^{-2}\cos(t) + \tfrac{1}{3}t^{-1}(-\sin(t)) - 5 \cdot \left(-\tfrac{1}{2}\right) t^{-1/2-1} = -\tfrac{1}{3}t^{-2}\cos(t) - \tfrac{1}{3}t^{-1}\sin(t) + \tfrac{5}{2}t^{-3/2}.$$

(One can check that this answer is the same as in the first solution.)

(b) $h(t) = \pi^2 - 3t^2\cos(t).$

Solution: Use the product rule on the second term, getting

$$h'(t) = \frac{d}{dt}(\pi^2) - \frac{d}{dt}\big(3t^2\cos(t)\big) = 0 - \left[\frac{d}{dt}(3t^2)\cos(t) + 3t^2\frac{d}{dt}\big(\cos(t)\big)\right]$$
$$= -\left[\frac{d}{dt}(3t^2)\cos(t) + 3t^2\cos'(t)\right] = -\big(6t\cos(t) - 3t^2\sin(t)\big) = -6t\cos(t) + 3t^2\sin(t).$$

(c) $w(t) = \arctan\left(\sqrt{t}\right) - \ln(2\pi).$

Solution: Use the chain rule on the first term, and rewrite $\sqrt{t} = t^{1/2}$. The derivative of the second term is zero because $\ln(2\pi)$ is a constant. This gives the answer

$$w'(t) = \frac{1}{1 + \left(\sqrt{t}\right)^2} \cdot \frac{d}{dt}\left(\sqrt{t}\right) = \frac{1}{1+t} \cdot \left(\tfrac{1}{2}t^{-1/2}\right) = \frac{t^{-1/2}}{2(t+1)}.$$

6. (22 points) A certain section of the San Andreas Fault runs straight north-south. On 1 January 1997, the west side was moving north (relative to the east side) at 3 cm/year (0.03 meters/year). At the same time, the town of Hicksville was 2 km (2000 meters) west of the fault, and the town of Gorman was 1 km (1000 meters) east of the fault and 4 km (4000 meters) farther south than Hicksville. Were these two towns getting closer together or farther apart at this time? At what rate?

Solution: Note: There is no picture in this file. A picture may be provided separately.

The route from Gorman to Hicksville at the present time, as described in the problem, is a zigzag line, first going west 1000 meters to the fault, then north 4000 meters along the fault, then west 2000 meters from the fault to Gorman. Both the east-west distances are constant, but the north-south distance along the fault is actually changing. So let's call it $y(t)$, and say that $t = 0$ is the present time. Thus $y(0) = 4000$ (measuring all distances in meters). A more appropriate description of the route from Gorman to Hicksville, valid at an arbitrary time, is therefore a zigzag line which first goes west 1000 meters to the fault, then north $y(t)$ meters along the fault, then west 2000 meters from the fault to Hicksville.

To find the distance, we are better off considering the route which starts at Gorman, goes 3000 meters west, and then goes $y(t)$ meters north to Hicksville. These are two sides of a right triangle, so the distance $l(t)$ in meters from Hicksville to Gorman at time t is

$$l(t) = \sqrt{3000^2 + y(t)^2} = \left(3000^2 + y(t)^2\right)^{1/2}.$$

We want to find $l'(0)$. Differentiating, we get

$$l'(t) = \tfrac{1}{2}\left(3000^2 + y(t)^2\right)^{-1/2} \cdot 2y(t)y'(t) = y(t)y'(t)\left(3000^2 + y(t)^2\right)^{-1/2}.$$

(Don't forget to use the chain rule!) Put $t = 0$ and substitute values. (Note that this can only be done *after* differentiating!) We know $y(0) = 4000$. We need $y'(0)$, which we can see from the statement is 0.03. (It is positive because $y(t)$ is increasing.) So

$$l'(0) = y(0)y'(0)\left(3000^2 + y(0)^2\right)^{-1/2} = 4000 \cdot (0.03) \cdot \tfrac{1}{5000} = 0.024.$$

Therefore Hicksville and Gorman are getting farther apart at 0.024 meters per year, or 2.4 cm/year. (The units are necessary!)

Some people in Gorman consider this to be good news. (So do some people in Hicksville.)

Here are descriptions of some alternatives. First, you could differentiate the equation $l(t)^2 = 3000^2 + y(t)^2$, getting

$$2l(t)l'(t) = 2y(t)y'(t),$$

so that

$$l'(t) = \frac{y(t)y'(t)}{l(t)}.$$

Now put $t = 0$, and substitute $y(0) = 4000$, $y'(0) = 0.03$, and $l(0) = 5000$. (You still need to calculate $l(0)$ from the Pythagorean Theorem.)

You could also do everything in physicists' notation. I will only show the first version. The equation for l is

$$l = \sqrt{3000^2 + y^2} = \left(3000^2 + y^2\right)^{1/2}.$$

Differentiating (using the chain rule, because everything is a function of t!), we get

$$\frac{dl}{dt} = \tfrac{1}{2}\left(3000^2 + y^2\right)^{-1/2} \cdot 2y\frac{dy}{dt} = y\frac{dy}{dt}\left(3000^2 + y^2\right)^{-1/2}.$$

Substituting values (implicitly putting $t = 0$, and using $l = 5000$ at $t = 0$, as above):

$$\frac{dl}{dt} = y\frac{dy}{dt}\left(3000^2 + y^2\right)^{-1/2} = 4000 \cdot (0.03) \cdot \tfrac{1}{5000} = 0.024,$$

as before.

7. (10 points) You have just removed a pizza from a hot oven. Let $T(t)$ be the temperature of the pizza in degrees Fahrenheit at time t minutes after removing it from the oven. If $T(5) = 400$ and $T'(5) = -6$, use the linear approximation (tangent line to $y = T(t)$ at $t = 5$) to estimate $T(10)$.

Solution: Let $y = L(t)$ be the equation of the tangent line. Then

$$L(t) = T(5) + T'(5)(t - 5) = 400 - 6(t - 5).$$

So

$$T(10) \approx L(10) = 400 - 6(10 - 5) = 370.$$

(It is not correct to write ~~$T(10) = L(10)$~~ or ~~$T(10) = 370$~~, because the tangent line approximation is only an *approximation*.)

8. (15 points; point values of parts as shown) A small spacecraft takes off from the surface of a planet, reaches a maximum height, and then crashes. Its position at time t is given by $y(t) = 9t^2 - 4t^3$, where $y(t)$ is measured in kilometers (km) above the surface and t is measured in minutes (min). Answer the following questions, being careful to give correct units when called for.

(a) (5 points) Find the upwards velocity of the spacecraft at time $t = 1$.

Solution: Calculate $y'(t) = 18t - 12t^2$, so the upwards velocity is $y'(1) = 18 - 12 = 6$. Thus, the answer is 6km/min. (The units are *required*.)

(b) (5 points) Find the average upwards velocity of the spacecraft between time $t = 0$ and time $t = 2$.

Solution: The average velocity is the how far it went up by how long it took to go up that far, which here is

$$\frac{y(2) - y(0)}{2 - 0} = \frac{9 \cdot 2^2 - 4 \cdot 2^3 - (9 \cdot 0^2 - 4 \cdot 0^3)}{2} = \frac{36 - 32 - (0 - 0)}{2} = 2.$$

So the average upwards velocity is 2 kilometers/minute.

Note: You *must* include the units in this kind of problem.

It is not correct to average the velocities at the endpoints of the interval. That is, do not use

$$\frac{y'(2) + y'(0)}{2} = -6.$$

(c) (5 points) How long will it take for the spacecraft to reach its maximum height?

Solution: The maximum height occurs at a time t for which when $y'(t) = 0$. We have $y'(t) = 18t - 12t^2$, so solve:

$$18t - 12t^2 = 0$$
$$6t(3 - 2t) = 0$$
$$t = 0 \qquad \text{or} \qquad t = \frac{3}{2}.$$

We are told that the spacecraft starts out by rising, reaches a maximum height, and then crashes, to the answer is after $\frac{3}{2}$min. (This can also be checked by applying the methods we have learned to the function $y(t) = 9t^2 - 4t^3$.)

9. (15 points.) Let $g(x) = x^3 - 15x^2 + 16$. Use the methods of calculus to find the exact values of x at which has its maximum and minimum values on the interval $[-1, 1]$.

(No credit will be given for correct guesses without supporting work that is valid for general functions of the sort considered in this course.)

Solution: We apply the procedure for continuous functions on closed bounded intervals. That is, we evaluate g at all critical numbers and at the endpoints, and compare values.

To find the critical numbers, we differentiate g, solve the equation $g'(x) = 0$, and find all numbers x in $[-1, 1]$ such that $g'(x)$ does not exist. The derivative of g is

$$g'(x) = 3x^2 - 30x = 3x(x - 10).$$

It exists everywhere, and it is zero when $x = 0$ and $x = 10$.

Since $x = 10$ is not in $[-1, 1]$, we ignore it. (**I must see you reject** 10**.** If I don't see this, I will assume you didn't correctly solve the equation $g'(x) = 0$.)

We now have one critical number, namely 0. So we must compare the values of g at 0, and at the endpoints -1 and 1.

Since $g(0) = 16$, $g(-1) = 0$, and $g(1) = 2$, the smallest of these is $g(-1)$ and the largest is $g(0)$. So the absolute minimum on the interval $[-1, 1]$ occurs at $x = -1$ and the absolute maximum on the interval $[-1, 1]$ occurs at $x = 0$.

Note that $x = 10$ is not correct for the minimum, even though $g(-1) = -484$, because -1 is not in the interval $[-1, 1]$.

No credit will be given for any solution which does not show evidence of an attempt to find the critical numbers of g. In particular, no credit will be given for simply comparing the values of g at the integers in the interval.