MATH 251: SUMMARY OF TOPICS

1. Summary of the main topics

See the separate page about arithmetic you are expected to be able to do without a calculator. Also, remember that notation counts! See the separate notation page.

Here is a list of most of the topics of the course (not necessarily complete).

(1) Understanding the meaning of the limit, both one sided and two sided, at finite values and at $\pm\infty$, and which have finite values, are $\pm\infty$, or do not exist and are not even $\pm\infty$. Understanding includes recognizing them from graphs and numerical interpretation, for example, how the values of $f(2.1)$, $f(2.01)$, $f(2.001)$, etc. and $f(1.9)$, $f(1.99)$, $f(1.999)$ etc. relate to $\lim_{x\to 2} f(x)$.

(2) Limit laws, and finding limits of all the kinds above. We have seen many methods for doing this. See Section 2 for more.

(3) Continuity, in particular, its relation to limits and recognizing continuity from the graph.

(4) Definition, meaning, and interpretation of the derivative:
   - (a) $f'(a) = \lim_{h\to 0} \dfrac{f(a+h) - f(a)}{h}$, if the limit exists.
   - (b) Limit of the slopes of secant lines to the graph.
   - (c) Slope of the tangent line to the graph.
   - (d) Recognizing differentiability (existence of a tangent line) from the graph.
   - (e) Instantaneous rate of change. The most common example is velocity, but there are many other rates of change, some of which don't involve time. (Acceleration is included here: it is the rate of change of velocity.)
   - (f) Linear approximation.

(5) Applications of derivatives. Some examples, not already in (4):
   - (a) Shape of a graph:
     - (i) Critical points.
     - (ii) Local minimums and maximums.
     - (iii) Relation between the sign of the derivative and whether the function is increasing or decreasing.
     - (iv) Relations between the second derivative and concavity.
     - (v) Inflection points.

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(vi) Second derivative test for a local minimum or maximum.
(b) Finding the maximum or minimum of a function on a closed bounded interval.
(c) Applied maximization and minimization problems. (See the cartoon at the end.)
(d) Related rates.
(e) Using the Mean Value Theorem to get information about the change in a function from information about its derivative.

2. Additional comments on limits

Warning: using L'Hopital's Rule when its hypotheses don't apply will result in getting the entire problem wrong. This could cost **half or more of a letter grade** on the final exam.

Warning: you will be expected to know that $\sin(0) = 0$, $\cos(0) = 1$, $\ln(e) = 1$, $a^0 = 1$ when $a \neq 0$, etc. **without using a calculator.** Getting these wrong may result in a misidentification of an indeterminate form, and thus getting an entire limit problem wrong, costing **half or more of a letter grade** on the final exam.

We have seen at least seven methods for calculating limits at finite values of the variable.

(1) Direct substitution: if f is continuous at a, then $\lim_{x \to a} f(x) = f(a)$. **Always** check first if this case applies!
(2) Cancellation of common factors in a fraction.
(3) Rationalization in a ratio containing roots.
(4) Identifying vertical asymptotes, for example in a limit of the form $\lim_{x \to a} \frac{f(x)}{g(x)}$ in which $g(a) = 0$ but $f(a) \neq 0$. (Vertical asymptotes also occur elsewhere. For example, $\lim_{x \to 0^+} \ln(x) = -\infty$.)
(5) L'Hopital's Rule, but **only** for indeterminate forms!
(6) Squeeze Theorem.
(7) The limit does not exist because the function oscillates too much. For example, $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist. (But $\lim_{x \to 0} x \sin\left(\frac{1}{x}\right) = 0$ by the Squeeze Theorem.)

As an example, specifically for fractions, consider $\lim_{x \to a} \frac{f(x)}{g(x)}$. For the purposes of illustration, assume that f and g are both continuous at a. Then there are three possibilities.

(1) $g(a) \neq 0$. Then the function $q(x) = \dfrac{f(x)}{g(x)}$ is continuous at a, so
$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} q(x) = q(a) = \frac{f(a)}{g(a)}.$$
This is true even if $f(a) = 0$.

(2) $g(a) = 0$ but $f(a) \neq 0$. Then the function $q(x) = \dfrac{f(x)}{g(x)}$ has a vertical asymptote at $x = a$. (The limit $\lim_{x\to\pi/2} \tan(x)$ is of this type.) This is **not** an indeterminate form, since no finite limit is possible. You expect the one sided limits $\lim_{x\to a^+} \dfrac{f(x)}{g(x)}$ and $\lim_{x\to a^-} \dfrac{f(x)}{g(x)}$ to be ∞ or $-\infty$. If they aren't both the same, then $\lim_{x\to a} \dfrac{f(x)}{g(x)}$ does not exist, even as ∞ or $-\infty$.

(3) $g(a) = 0$ and $f(a) = 0$. This is an indeterminate form, and methods like cancelling common factors, rationalization, or L'Hopital's Rule might be useful. You can certainly try L'Hopital's Rule, but **only** in this case..

It may be that instead $\lim_{x\to a} f(x) = \pm\infty$, $\lim_{x\to a} g(x) = \pm\infty$, or both. Among these cases, only $\lim_{x\to a} f(x) = \pm\infty$ **and** $\lim_{x\to a} g(x) = \pm\infty$ is an indeterminate form, namely "$\dfrac{\pm\infty}{\pm\infty}$". But some other combinations, for example $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = 0$, that is, the form "$\dfrac{\pm\infty}{0}$", give vertical asymptotes.

We have seen a similar collection of methods for calculating limits at $\pm\infty$. First, here are examples of limits at infinity which we implicitly already knew before this course, for example just by looking at the graphs:
$$\lim_{x\to\infty} x = \lim_{x\to\infty} x^2 = \lim_{x\to\infty} x^3 = \lim_{x\to\infty} \sqrt{x} = \infty, \qquad \text{etc.}$$
$$\lim_{x\to\infty} x = \lim_{x\to\infty} x^3 = \lim_{x\to\infty} x^5 = -\infty, \qquad \text{etc.}$$
$$\lim_{x\to\infty} x^2 = \lim_{x\to\infty} x^4 = \lim_{x\to\infty} x^6 = \infty, \qquad \text{etc.}$$
$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to\infty} \ln(x) = \infty, \quad \lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}, \quad \text{etc.}$$
$$\lim_{x\to\infty} \frac{1}{x} = \lim_{x\to\infty} \frac{1}{x^2} = \lim_{x\to\infty} \frac{1}{x^3} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0, \qquad \text{etc.}$$
$$\lim_{x\to-\infty} \frac{1}{x} = \lim_{x\to-\infty} \frac{1}{x^2} = \lim_{x\to-\infty} \frac{1}{x^3} = 0, \qquad \text{etc.}$$

Here are some of the methods we have seen.

(1) Limits of combinations of the functions appearing above gotten by using the limit laws and reasoning with $\pm\infty$ in ways analogous to the limit laws. (These are the analogs of both direct substitution in

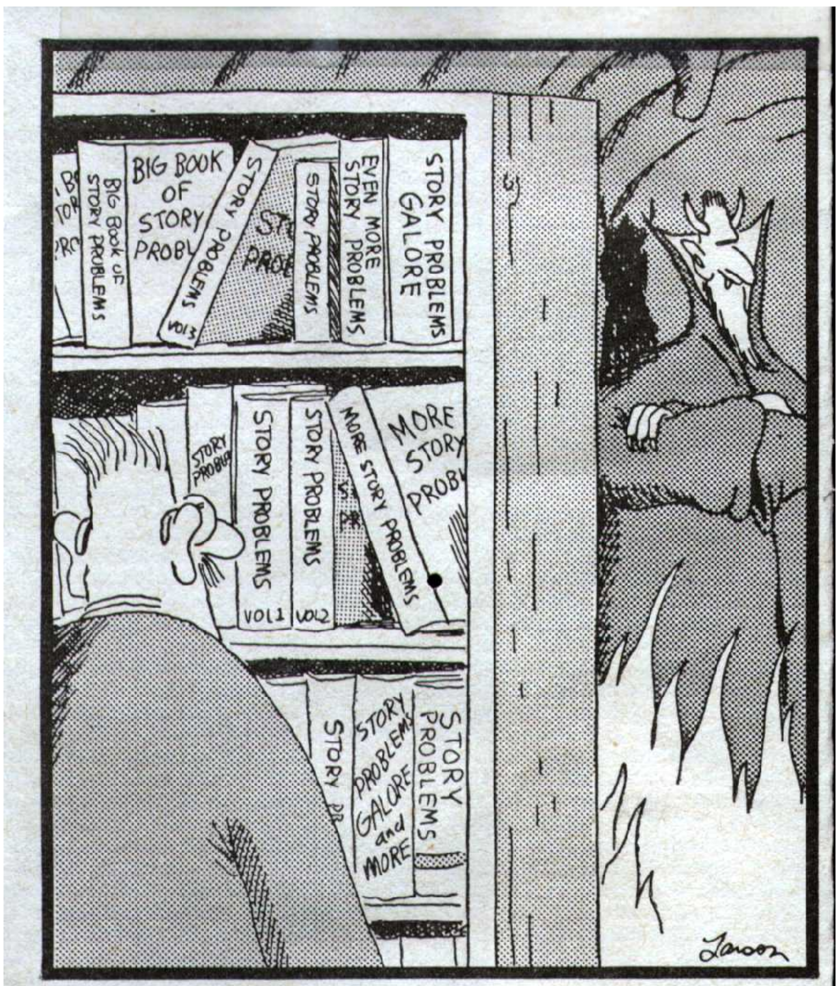
continuous functions and some cases of identifying vertical asymptotes.)

(2) Multiplying both numerator and denominator of a fraction by $\frac{1}{x^n}$, or similar things (for example, e^{-2x}). This is the analog of cancellation of common factors in a fraction.

(3) L'Hopital's Rule, but **only** for indeterminate forms!

(4) Squeeze Theorem.

(5) The limit does not exist because the function oscillates too much. For example, $\lim_{x\to\infty} \sin(x)$ does not exist. (But $\lim_{x\to 0} \frac{\sin(x)}{x} = 0$ by the Squeeze Theorem.)



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