Lecture 6: Applications and Problems

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- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 6 (20 July 2016): Applications and Problems.

A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.
- Some open problems.

Recall: The tracial Rokhlin property

Definition

Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that, with $e = \sum_{g \in G} e_g$:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $1 - e$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$.
4. $\|exe\| > 1 - \varepsilon$.

Recall the simplifications:

1. If A is finite, the last condition can be omitted.
2. We need only consider finite subsets F of a fixed generating set.
3. We can require $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
4. In good cases, replace (3), (4) by $\tau(1 - e) < \varepsilon$ for all tracial states τ.

The tracial Rokhlin property is common

We saw some examples of actions with the tracial Rokhlin property, but mostly without the Rokhlin property or even finite Rokhlin dimension with commuting towers (its nearly as good generalization):

- The action of $\mathbb{Z}_2$ on the 3^∞ UHF algebra generated by

$$\gamma = \bigotimes_{n=1}^{\infty} \mathrm{Ad} \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right).$$

- Actions on A_θ from finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$.
- The action of $\mathbb{Z}_n$ on an irrational rotation algebra generated by $u \mapsto e^{2\pi i/n} u$ and $v \mapsto v$. (It does have a higher dimensional Rokhlin property with commuting towers.)
- The tensor flip on any UHF algebra.
- Pointwise outer actions of a finite group on a unital Kirchberg algebra.

Recall: tracial rank zero

Recall the definition of tracial rank zero:

Definition

Let A be a simple separable unital C*-algebra. Then A has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there exists a nonzero projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.
2. $\mathrm{dist}(pap, D) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$.

In both definitions, the strong version (the Rokhlin property, or local approximation by finite dimensional C*-algebras) is supposed to hold only after cutting down by a "large" projection which approximately commutes with all elements of the given finite set.

Crossed products and the tracial Rokhlin property

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

The idea of the proof is essentially the same as for crossed products of AF algebras by actions with the Rokhlin property. The definitions of both tracial rank zero and the tracial Rokhlin property allow a "small" (in trace) error projection. One must show that the sum of two "small" error projections is again "small". One must do a little work here, but these lectures won't address this point.

There is one additional difficulty. The hypotheses give an error which is "small" relative to A. One must prove that it is also "small" relative to $C^*(G, A, \alpha)$. This uses a theorem of Jeong and Osaka. We give the ideas of a proof here.

"Small" in $C^*(G, A, \alpha)$ vs. "small" in A

The following result implies that one can make the error projection "small" relative to $C^*(G, A, \alpha)$ by requiring that it be "small" relative to A.

Theorem

Let A be an infinite dimensional simple separable unital C*-algebra which has Property (SP) (every nonzero hereditary subalgebra contains a nonzero projection), and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group which has the tracial Rokhlin property. Then for every nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$, there is a nonzero projection $p \in D$ which is Murray-von Neumann equivalent to a projection $q \in A$.

So, to ensure the error projection $1 - e$ is Murray-von Neumann equivalent to a projection in D, it is enough to require that $1 - e \precsim q$ in A.

Much weaker conditions suffice: provided one uses $C^*_{\mathrm{r}}(G, A, \alpha)$, one can allow any pointwise outer action of a discrete group (Jeong-Osaka).

Tracial rank zero is known to imply Property (SP).

Kishimoto's condition

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A.

We use what we call Kishimoto's condition (from his paper on simplicity of reduced crossed products). Here is the version for finite groups:

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. We say that α satisfies *Kishimoto's condition* if for every $x \in A_+$ with $\|x\| = 1$, for every finite set $S \subset A$, and for every $\varepsilon > 0$, there is $a \in A_+$ with $\|a\| = 1$ such that:

1. $\|axa\| > 1 - \varepsilon$.
2. $\|ab\alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$ and $b \in S$.

We will get (2) by arranging that:

1. $\|ab - ba\|$ is small.
2. a is approximately orthogonal to $\alpha_g(a)$ for $g \neq 1$.

Kishimoto's condition (continued)

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. We say that α satisfies *Kishimoto's condition* if for every $x \in A_+$ with $\|x\| = 1$, for every finite set $S \subset A$, and for every $\varepsilon > 0$, there is $a \in A_+$ with $\|a\| = 1$ such that:

1. $\|axa\| > 1 - \varepsilon$.
2. $\|ab\alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$ and $b \in S$.

For general discrete groups, one uses finite subsets of $G \setminus \{1\}$.

Kishimoto shows that this condition holds for pointwise outer actions on simple C*-algebras (in fact, under weaker assumptions). We sketch the (easier) proof that Kishimoto's condition follows from the tracial Rokhlin property. and show how to use it.

We will get (2) by arranging that $\|ab - ba\|$ is small, and that a is approximately orthogonal to $\alpha_g(a)$ for $g \neq 1$.

Getting Kishimoto's condition

Given $x \in A_+$ with $\|x\| = 1$, and a finite set $S \subset A$, we want:

1. $\|axa\| > 1 - \varepsilon$.
2. $\|ab\alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$ and $b \in S$.

For $F \subset A$ finite and $\delta > 0$, the tracial Rokhlin property gives mutually orthogonal projections e_g such that (omitting a condition we won't need):

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\|e_g b - b e_g\| < \delta$ for all $g \in G$ and all $b \in F$.
3. With $e = \sum_{g \in G} e_g$, we have $\|exe\| > 1 - \varepsilon$.

Apply the tracial Rokhlin property with $F = S \cup \{x\}$. Then $exe \approx \sum_{h \in G} e_h x e_h$ (exercise: check this!) so, provided δ is small enough, orthogonality of the sum implies that there is some $h \in G$ such that $\|e_h x e_h\| > 1 - \delta$. We will take $a = e_h$. Now $g \neq 1$ implies

$$ab\alpha_g(a) = e_h b \alpha_g(e_h) \approx b e_h \alpha_g(e_h) = b e_h e_{gh} = 0.$$

This proves Kishimoto's condition. Exercise: Write out the details.

Using Kishimoto's condition

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A. We assume α satisfies Kishimoto's condition.

Step 1: Choose a nonzero positive element $c = \sum_{g \in G} c_g u_g \in D$. We claim $c_1 \geq 0$ and we can arrange to have $\|c_1\| = 1$.

To see this, write $c = yy^*$ with $y = \sum_{g \in G} y_g u_g$. Multiply out, getting $c_1 = \sum_{g \in G} y_g u_g u_g^* y_g^* = \sum_{g \in G} y_g y_g^* \geq 0$. (Exercise: Check this!) If $c_1 = 0$, then $y_g = 0$ for all g, so $y = 0$, so $c = 0$.

Now multiply c by a suitable scalar.

Step 2: Apply Kishimoto's condition, with suitable $\varepsilon > 0$, with $x = c_1$, and using the finite set of coefficients c_g for $g \in G$, getting $a \in A_+$ with $\|a\| = 1$ such that:

1. $\|ac_1a\| > 1 - \varepsilon$.
2. $\|ac_g\alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$.

Using Kishimoto's condition (continued)

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A.

We have $c = \sum_{g \in G} c_g u_g \in D_+$ and $a \in A_+$ with $\|c_1\| = \|a\| = 1$ and

1. $\|a c_1 a\| > 1 - \varepsilon$.
2. $\|a c_g \alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$.

Step 3: Using $u_g a u_g^* = \alpha_g(a)$ at the second step and (2) at the third step:

$$aca = \sum_{g \in G} a c_g u_g a = \sum_{g \in G} a c_g \alpha_g(a) u_g \approx a c_1 a u_1 = a c_1 a.$$

Step 4: Choose $f \colon [0,1] \to [0,1]$ such that $f = 0$ on $[0, 1 - 2\varepsilon]$ and $f(1 - \varepsilon) = 1$. Then $f(a c_1 a) \neq 0$ by (1). By Property (SP), there is a nonzero projection p in the hereditary subalgebra of A generated by $f(a c_1 a)$. One can show that $p a c_1 a p \approx p$. Exercise: Do this, giving precise estimates. (Nothing special to crossed products is needed.)

Using Kishimoto's condition (continued)

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A.

We have $c \in D_+$, $a \in A_+$, and a nonzero projection $p \in A$, all satisfying

$$aca \approx a c_1 a \quad \text{and} \quad p a c_1 a p \approx p.$$

Step 5: So

$$pacap \approx p.$$

Define

$$s_0 = c^{1/2} a p.$$

Then $s_0^* s_0 \approx p$ and is in $p C^*(G, A, \alpha) p$. So we can form $s = s_0 (s_0^* s_0)^{-1/2}$ (functional calculus in $p C^*(G, A, \alpha) p$), getting

$$s^* s = p \quad \text{and} \quad s s^* = c^{1/2} a p \left((s_0^* s_0)^{-1/2} \right)^2 p a c^{1/2} \in \overline{c C^*(G, A, \alpha) c} \subset D.$$

Thus $s s^*$ is a projection in D equivalent to the nonzero projection $p \in A$. This is what we want. End of the sketch of the proof.

An application: Crossed products of rotation algebras by finite groups

The action of $\mathrm{SL}_2(\mathbb{Z})$ on the rotation algebra A_θ (generated by unitaries u and v with $vu = e^{2\pi i \theta} uv$) sends $n = \left(\begin{smallmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{smallmatrix} \right)$ to the automorphism

$$\alpha_n(u) = \exp(\pi i n_{1,1} n_{2,1} \theta) u^{n_{1,1}} v^{n_{2,1}}, \quad \alpha_n(v) = \exp(\pi i n_{1,2} n_{2,2} \theta) u^{n_{1,2}} v^{n_{2,2}}.$$

The finite cyclic subgroups of orders 2, 3, 4, and 6, are generated by

$$\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix} \right), \quad \left(\begin{smallmatrix} -1 & -1 \\ 1 & 0 \end{smallmatrix} \right), \quad \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix} \right), \quad \text{and} \quad \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix} \right).$$

In terms of generators of A_θ, and omitting the scalar factors (not needed when one restricts to these subgroups), the action of $\mathbb{Z}_2$ is generated by $u \mapsto u^*$ and $v \mapsto v^*$, and the action of $\mathbb{Z}_4$ is generated by $u \mapsto v$ and $v \mapsto u^*$. (An old exercise: Find the analogous formulas for $\mathbb{Z}_3$ and $\mathbb{Z}_6$.)

Theorem (Joint with Echterhoff, Lück, and Walters; known previously for the order 2 case)

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\alpha \colon G \to \mathrm{Aut}(A_\theta)$ be the action on A_θ of one of the finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$. Then $C^*(G, A_\theta, \alpha)$ is an AF algebra.

Crossed products of rotation algebras by finite groups

Theorem (Joint with Echterhoff, Lück, and Walters)

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\alpha \colon G \to \mathrm{Aut}(A_\theta)$ be the action on A_θ of one of the finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (of order 2, 3, 4, or 6). Then $C^*(G, A_\theta, \alpha)$ is an AF algebra.

This solved a problem open for some years. The result is initially unexpected, since A_θ itself is not AF. It was suggested by K-theory computations done for rational θ.

The proof uses the tracial Rokhlin property for the action to show that $C^*(G, A_\theta, \alpha)$ has tracial rank zero. One then applies classification theory (specifically, Lin's classification theorem), but one must compute the K-theory of $C^*(G, A_\theta, \alpha)$ and show that it satisfies the Universal Coefficient Theorem. This is done using known cases of the Baum-Connes conjecture.

Direct proof?

$C^*(G, A_\theta, \alpha)$ is AF for finite subgroups $G \subset \mathrm{SL}_2(\mathbb{Z})$. For $G = \mathbb{Z}_2$, much more direct methods are known. For the other cases, our proof, using several different pieces of heavy machinery (the Elliott classification program and the Baum-Connes conjecture), is the only one known.

Problem

Prove that $C^*(G, A_\theta, \alpha)$ is AF for $G \subset \mathrm{SL}_2(\mathbb{Z})$ of order 3, 4, and 6, by explicitly writing down a direct system of finite dimensional C*-algebras and proving directly that its direct limit is $C^*(G, A_\theta, \alpha)$.

An application: Higher dimensional noncommutative tori

Theorem

Every simple higher dimensional noncommutative torus is an AT algebra.

A higher dimensional noncommutative torus is a version of the rotation algebra using more unitaries as generators. An AT algebra is a direct limit of finite direct sums of C*-algebras of the form $C(S^1, M_n)$.

Elliott and Evans proved that A_θ is an AT algebra for θ irrational. A general simple higher dimensional noncommutative torus can be obtained from some A_θ by taking repeated crossed products by $\mathbb{Z}$. If all the intermediate crossed products are simple, the theorem follows from a result of Kishimoto. Using classification theory and the tracial Rokhlin property for actions generalizing the one on A_θ generated by

$$u \mapsto e^{2\pi i/n} u \qquad \text{and} \qquad v \mapsto v,$$

one can reduce the general case to the case proved by Kishimoto.

Some directions for further work: Summary

- Infinite discrete groups. This is a vast area, with little known beyond elementary amenable groups. (I will say no more here.)
- Actions of finite dimensional quantum groups (recently started by Kodaka, Osaka, Teruya).
- What if there are few or no projections? Some things are known, but there is still work to do.
- The nonunital case.
- The nonsimple case: There are many open questions, including analogs of facts which in the simple case are easier than the results we have presented.

For many open problems for actions of finite groups, see the survey article (now somewhat out of date):

N. C. Phillips, *Freeness of actions of finite groups on C*-algebras*, pages 217–257 in: *Operator structures and dynamical systems*, M. de Jeu, S. Silvestrov, C. Skau, and J. Tomiyama (eds.), Contemporary Mathematics vol. 503, Amer. Math. Soc., Providence RI, 2009.

Finite dimensional quantum groups

There are versions of the Rokhlin and tracial Rokhlin properties for actions (coactions?) of finite dimensional quantum groups on (simple) C*-algebras (Kodaka, Osaka, Teruya), and some theorems. Some of it even generalizes to inclusions of C*-algebras of “index-finite type”. It is not clear the the definitions used so far are the right ones. (From a conversation with Osaka.) In particular, if G is a finite group, the product type action given as the infinite tensor product of copies of conjugation by the regular representation has the Rokhlin property. With current definitions, analogous statement can fail for finite dimensional quantum groups.

There is a great shortage of examples of such actions which don't come from groups (regardless of whether they have the Rokhlin property or the tracial Rokhlin property).

Few or no projections

What if there are few or no projections? There are several competing suggested conditions generalizing the tracial Rokhlin property, and some theorems, but there is no known analog of the statement that crossed products by tracial Rokhlin actions preserve tracial rank zero. (There isn't a known suitable generalization of tracial rank zero. Recent classification work might suggest some ideas involving tensoring with UHF algebras.)

The higher dimensional Rokhlin properties (both with and without "commuting towers") of Hirshberg-Winter-Zacharias do not need projections, and there are examples of such actions on simple C*-algebras with few projections. The "right" property (which gives results close to what one gets with the Rokhlin property) is the one *with* commuting towers, but there are no actions of nontrivial finite groups on the Jiang-Su algebra with this property (joint with Hirshberg).

The nonunital case

What should be done in the nonunital case? There is a definition for the Rokhlin property (and the higher dimensional Rokhlin properties), and several theorems, but still work to be done. There is at least one suggested answer for the tracial Rokhlin property, but as far as I know almost no theorems.

Problem

Find a suitable definition of the tracial Rokhlin property for actions on nonunital simple C*-algebras, so that crossed products of nonunital simple C*-algebras with tracial rank zero by tracially Rokhlin actions again have tracial rank zero.

The tracial Rokhlin property in the nonsimple case

The result on crossed products of AF algebras by Rokhlin actions did not need simplicity. Lin has a definition of tracial rank zero for unital nonsimple algebras, but some things go wrong: it no longer implies either real rank zero or stable rank one. (There are theorems relating tracial rank zero to tracial quasidiagonality of extensions, which suggest that, nevertheless, this definition is the "right" one.)

Problem

Find a suitable definition of the tracial Rokhlin property for actions on nonsimple unital C*-algebras, and possibly a different definition of tracial rank zero, so that crossed products of C*-algebras with tracial rank zero by tracially Rokhlin actions again have tracial rank zero.

One can extend the definition of the tracial Rokhlin property to the nonsimple case by imitating Lin. It looks likely that then crossed products of C*-algebras with tracial rank zero by tracially Rokhlin actions again have tracial rank zero. Some partial results are known (using strong hypotheses on the ideal structure of A), but this has not been proved in general.

Open problems with weaker conditions on the action

The rest of the open problems discussed here are not related to the Rokhlin property or the tracial Rokhlin property. They assume weaker conditions on the action. Some of them have no condition on the action.

An old problem on stable rank:

Problem

Let A be a simple C*-algebra with stable rank one (in the unital case: the invertible elements are dense), and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group on A. Does it follow that $C^*(G, A, \alpha)$ has stable rank one?

Without simplicity, there is a counterexample (Blackadar).

Preservation of structure in the nonsimple case: pure infiniteness

Definition (Kirchberg-Rørdam)

A not necessarily simple C*-algebra A is *purely infinite* if there is no nonzero homomorphism from A to $\mathbb{C}$, and for every $a, b \in A$ such that $a \in \overline{AbA}$, we have $a \precsim b$ (Cuntz subequivalence; it means that there exists a sequence $(v_n)_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} v_n^* b v_n = a$).

Direct sums of purely infinite simple C*-algebras are purely infinite. $C([0,1], \mathcal{O}_d)$ is purely infinite.

The following is a corollary of a result of Jeong and Osaka.

Proposition

Let A be a purely infinite simple C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be any action. Then $C^*(G, A, \alpha)$ is purely infinite (even if it isn't simple).

Nonsimple pure infiniteness (continued)

Proposition (Jeong-Osaka)

Let A be a purely infinite simple C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be any action. Then $C^*(G, A, \alpha)$ is purely infinite (even if it isn't simple).

Is simplicity of A really needed?

Problem

Let A be a purely infinite C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be any action. Does it follow that $C^*(G, A, \alpha)$ is purely infinite?

A later paper of Kirchberg and Rørdam gives several other versions of pure infiniteness for nonsimple C*-algebras. It isn't known whether they are all equivalent, and the problem above is open for all of them as well.

Nonsimple pure infiniteness (continued)

Problem

Let A be a purely infinite C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be any action. Does it follow that $C^*(G, A, \alpha)$ is purely infinite?

Something is known for a slightly different condition.

Definition

Let A be a C*-algebra. We say that A is *hereditarily infinite* if for every nonzero hereditary subalgebra $B \subset A$, there is an infinite positive element $a \in B$ in the sense of Kirchberg-Rørdam, that is, there is $b \in A_+ \setminus \{0\}$ such that $a \oplus b \precsim a$. We say that A is *residually hereditarily infinite* if A/I is hereditarily infinite for every ideal I in A.

Kirchberg and Rørdam asked whether residual hereditary infiniteness implies pure infiniteness. This is still open.

Nonsimple pure infiniteness (continued)

A C*-algebra A is residually hereditarily infinite if for every ideal $I \subset A$ and nonzero hereditary subalgebra $B \subset A/I$, B has an infinite positive element.

Theorem (with Pasnicu)

Let A be a residually hereditarily infinite C*-algebra, let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action. If α is strongly pointwise outer (defined below), or if G is a finite abelian 2-group and α is arbitrary, then $C^*(G, A, \alpha)$ is residually hereditarily infinite.

This is for the wrong condition, and one should get the result for arbitrary actions of arbitrary finite groups.

Definition

Let A be a C*-algebra and let G be a group. An action $\alpha \colon G \to \mathrm{Aut}(A)$ is said to be *strongly pointwise outer* if, for every $g \in G \setminus \{1\}$ and any two α_g-invariant ideals $I \subset J \subset A$ with $I \neq J$, the automorphism of J/I induced by α_g is outer, that is, not of the form $a \mapsto \mathrm{Ad}(u)(a) = uau^*$ for any unitary u in the multiplier algebra $M(J/I)$.

Preservation of structure in the nonsimple case: the ideal property

Definition

A C*-algebra A has the *ideal property* if every ideal in A is generated, as an ideal, by the projections it contains.

So all C*-algebras with real rank zero, and all simple unital C*-algebras, have the ideal property. $C([0,1], \mathcal{O}_d)$ does not have the ideal property.

Problem

Let A have the ideal property, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be any action. Does it follow that $C^*(G, A, \alpha)$ has the ideal property?

There is some progress. It is true (joint with Pasnicu) if α is strongly pointwise outer. It can probably be proved easily using known results if A is simple and unital. In general, we don't know.

Preservation of structure in the nonsimple case: pure infiniteness *and* the ideal property

Theorem (with Pasnicu)

Let A be a purely infinite C*-algebra which also has the ideal property, let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. If α is strongly pointwise outer, or if G is a finite abelian 2-group and α is arbitrary, then $C^*(G, A, \alpha)$ is purely infinite and has the ideal property.

Again, it should be true for arbitrary actions of arbitrary finite groups.

The base case for arbitrary actions is $G = \mathbb{Z}_2$. We get finite abelian 2-groups by a bootstrap argument. We don't know how to deal with $\mathbb{Z}_3$.

(The proof, and the difficulty with $\mathbb{Z}_3$, are the same as for the earlier result on preservation of residual hereditary infiniteness.)

Kishimoto's result for the nonsimple case (infinite groups)

Recall that Kishimoto showed that if A is simple and separable, G is discrete, and $\alpha\colon G \to \mathrm{Aut}(A)$ is pointwise outer, then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple. In the nonsimple case, every invariant ideal of A gives an ideal in the crossed product, so one should ask for the following property.

Definition

Let A be a C*-algebra and let G be a group. An action $\alpha\colon G \to \mathrm{Aut}(A)$ is said to *have only crossed product ideals* if every ideal in $C^*_{\mathrm{r}}(G, A, \alpha)$ has the form $C^*_{\mathrm{r}}(G, J, \alpha)$ for some G-invariant ideal $J \subset A$.

By abuse of terminology, we say $C^*_{\mathrm{r}}(G, A, \alpha)$ has only crossed product ideals, with the particular way this algebra is thought of as being a crossed product left implicit.

Problem

Is there a suitable version of pointwise outerness of an action $\alpha\colon G \to \mathrm{Aut}(A)$ of a discrete group G which guarantees that $C^*_{\mathrm{r}}(G, A, \alpha)$ has only crossed product ideals?

Kishimoto without simplicity (continued)

Problem

Is there a suitable version of pointwise outerness of an action $\alpha\colon G \to \mathrm{Aut}(A)$ which guarantees that $C^*_{\mathrm{r}}(G, A, \alpha)$ has only crossed product ideals?

For G abelian, there is a necessary and sufficient condition in terms of the strong Connes spectrum, but this is hard to compute. There also is an analog of the strong Connes spectrum for actions of finite groups. But no related construction is known for actions of general locally compact groups, not even for general discrete groups.

If instead we ask for something with hypotheses more like pointwise outerness of α (as in Kishimoto's theorem), one might try strong pointwise outerness. Recall that an action $\alpha\colon G \to \mathrm{Aut}(A)$ is strongly pointwise outer if, for every $g \in G \setminus \{1\}$ and any two α_g-invariant ideals $I \subset J \subset A$ with $I \neq J$, the automorphism of J/I induced by α_g is outer.

Kishimoto without simplicity (continued)

$\alpha\colon G \to \mathrm{Aut}(A)$ is strongly pointwise outer if, for every $g \in G \setminus \{1\}$ and any two α_g-invariant ideals $I \subset J \subset A$ with $I \neq J$, the automorphism of J/I induced by α_g is outer.

In order to prove that $C^*_{\mathrm{r}}(G, A, \alpha)$ has only crossed product ideals, one needs at least to know that for every subgroup $H \subset G$ and every H-invariant subquotient J/I of A, the induced action of H on J/I is pointwise outer. (There is a finite dimensional counterexample if one doesn't consider subgroups.)

If G is finite, then strong pointwise outerness is sufficient. If G is exact and discrete, a condition we call "spectral freeness" seems appropriate and works. If G is finite, they are equivalent. Exactness of the group (or at least of the action) is necessary in any case.

Question

If G is exact and discrete, and $\alpha\colon G \to \mathrm{Aut}(A)$ is strongly pointwise outer, does it follow that $C^*_{\mathrm{r}}(G, A, \alpha)$ has only crossed product ideals?