Lecture 4: Crossed Products by Actions with the Rokhlin Property

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University of Oregon

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The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
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- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
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$$\begin{aligned}\|abe_g - e_g ab\| &= \big\|a(be_g - e_g b) + (e_g a - a e_g)b\big\| \\ &\leq \|a\| \cdot \|be_g - e_g b\| + \|e_g a - a e_g\| \cdot \|b\|.\end{aligned}$$

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Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. Let $T \subset A$ generate A as a C*-algebra. Suppose that for every finite set $F \subset T$ and every $\varepsilon > 0$, there are projections $e_g \in A$ for $g \in G$ such that:

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3. $\sum_{g \in G} e_g = 1$.

Then α has the Rokhlin property.

Hint 1: The *-algebra generated by T is dense.

Hint 2: F only appears in condition (2). If, say, a and b approximately commute with e_g, then ab approximately commutes with e_g because

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The proof that $\beta(e_1) = e_0$ is the same. We are done.

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Exactly permuting the projections

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\| \alpha_g (e_h) - e_{gh} \| < \varepsilon$ for all $g, h \in G$.
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Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

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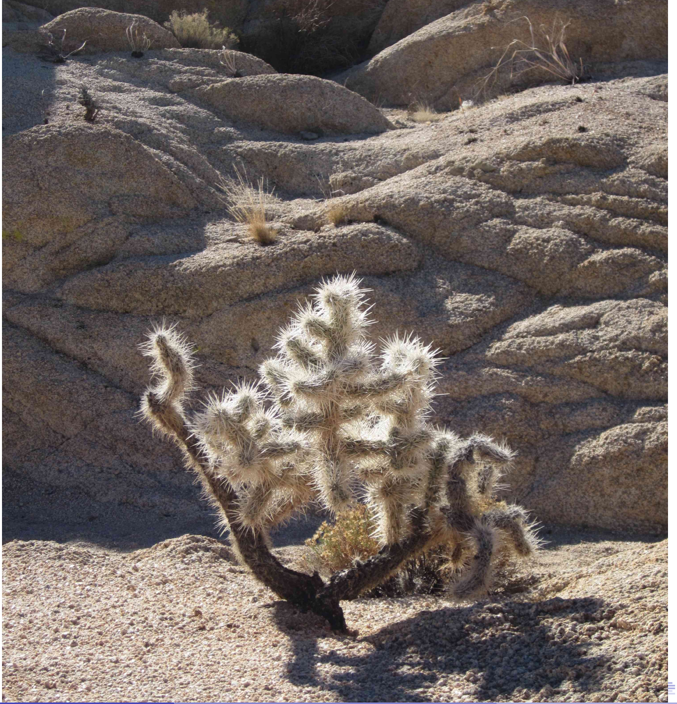

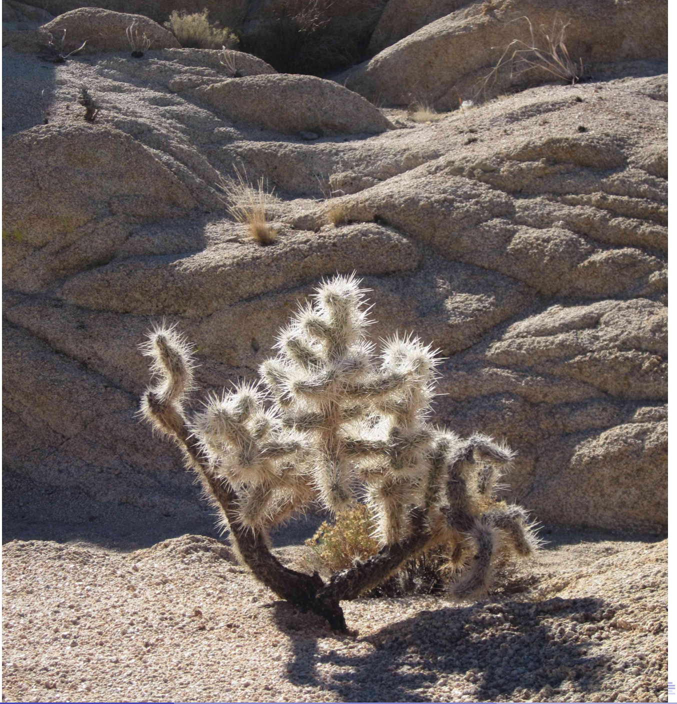

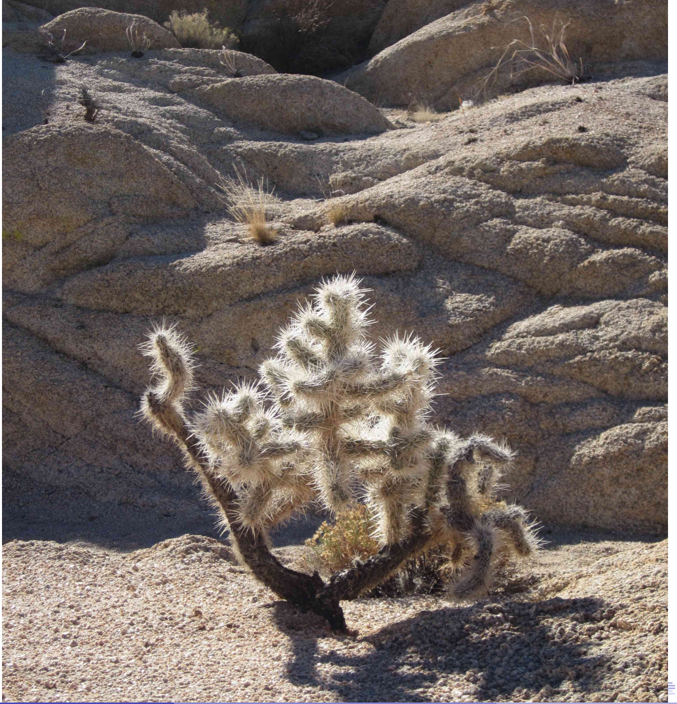

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In particular, for $g \neq h$ and $a \in F$, $e_g a e_h \approx a e_g e_h = 0$.

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Informally: $e_g a \approx a e_g$ for all $g \in G$ and all $a \in F$.

In particular, for $g \neq h$ and $a \in F$, $e_g a e_h \approx a e_g e_h = 0$. Therefore, if $a \in F$,

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Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha\colon G \to \text{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g : g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We will approximate S by an AF algebra.

Apply the Rokhlin property to the finite set F. Use the version in which the projections are exactly permuted by the group. Thus, we get projections $e_g \in A$ for $g \in G$ such that:

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The action of G permutes the summands. Exercise: Prove that D_0 is equivariantly isomorphic to $C(G, \, e_1 A e_1)$ with the action $\beta_g(b)(h) = b(g^{-1}h)$ for $g, h \in G$ and $b \in C(G, \, e_1 A e_1)$.

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Crossed products by actions with the Rokhlin property (continued)

Recall the conclusion of the theorem: $C^*(G, A, \alpha)$ is AF.

To prove the theorem, we prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D.

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Corners of AF algebras are AF, and φ is injective, so $D = \varphi(M_n \otimes e_1 A e_1)$ is an AF subalgebra of $C^*(G, A, \alpha)$.

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Exercise: Prove that φ is a *- homomorphism.

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Now (justifications given afterwards):

$$\begin{aligned} \varphi(b) &= \sum_{g \in G} v_{g,1} e_1 \alpha_g^{-1}(a) e_1 v_{g,1}^* = \sum_{g \in G} e_g u_g e_1 \alpha_g^{-1}(a) e_1 u_g^* e_g \\ &= \sum_{g \in G} e_g \alpha_g \big(e_1 \alpha_g^{-1}(a) e_1 \big) e_g = \sum_{g \in G} e_g a e_g . \end{aligned}$$

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