Lecture 2: Introduction to Crossed Products and More Examples of Actions

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University of Oregon

13 July 2016

The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
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A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
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We saw some examples coming from actions on compact Hausdorff spaces. We also saw the inner action: if $g \mapsto z_g$ is a (continuous) homomorphism from G to the unitary group $U(A)$ of a unital C*-algebra A, then $\alpha_g(a) = z_g a z_g^*$ defines an action of G on A. (We write $\alpha_g = \mathrm{Ad}(z_g)$.)

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by $\varphi_n(a) = a \otimes 1$. Let A be the (completed) direct limit $\varinjlim_n A_n$. (This is just the 2^∞ UHF algebra.)

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Infinite tensor product action example (continued)

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Recall: If G is finite and $w\colon G \to U(H)$ is a unitary representation, then $\pi_w\colon \mathbb{C}[G] \to L(H)$ is $\pi_w \left(\sum_{g \in G} a_g \cdot u_g \right) = \sum_{g \in G} a_g \cdot w_g$. Moreover, $w \mapsto \pi_w$ is a bijection from unitary representations to unital *-homomorphisms.

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The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).

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- A_θ is the crossed product of the action of $\mathbb{Z}$ on S^1 generated by rotation by $e^{2\pi i \theta}$.
- There is a "natural" continuous field over S^1 whose fiber over $e^{2\pi i \theta}$ is A_θ. (Obviously it isn't locally trivial.)

The algebra A_θ is often considered to be a noncommutative analog of the torus $S^1 \times S^1$ (more accurately, a noncommutative analog of $C(S^1 \times S^1)$).

The rotation algebras

Let $\theta \in \mathbb{R}$. Recall the irrational rotation algebra A_θ, the universal C*-algebra generated by two unitaries u and v satisfying the commutation relation $vu = e^{2\pi i\theta}uv$. If $\theta_1 - \theta_2 \in \mathbb{Z}$, then $A_{\theta_1} = A_{\theta_2}$. (The commutation relation is the same.) Some standard facts, presented without proof:

- If $\theta \notin \mathbb{Q}$, then A_θ is simple. In particular, *any* two unitaries u and v in any C*-algebra satisfying $vu = e^{2\pi i\theta}uv$ generate a copy of A_θ.
- If $\theta \in \mathbb{Q}$, then A_θ is Type I. In fact, if $\theta = \frac{m}{n}$ in lowest terms, with $n > 0$, then A_θ is isomorphic to the section algebra of a locally trivial continuous field over $S^1 \times S^1$ with fiber M_n.
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