Lecture 1: Group C*-algebras and Actions of Finite Groups on C*-Algebras

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University of Oregon

11 July 2016

The Second Summer School on Operator Algebras and Noncommutative Geometry 2016

East China Normal University, Shanghai

11–29 July 2016

- Lecture 1 (11 July 2016): Group C*-algebras and Actions of Finite Groups on C*-Algebras
- Lecture 2 (13 July 2016): Introduction to Crossed Products and More Examples of Actions.
- Lecture 3 (15 July 2016): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 4 (18 July 2016): Crossed Products by Actions with the Rokhlin Property.
- Lecture 5 (19 July 2016): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
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A rough outline of all six lectures

- The beginning: The C*-algebra of a group.
- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- More examples of actions.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
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Take $G = \mathbb{Z}$. Then $\mathbb{C}[G] \cong \mathbb{C}[x, x^{-1}]$, the ring of Laurent polynomials in one variable. The group element $k \in \mathbb{Z}$ is x^k.

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If G is not finite, one must choose a C* norm on $\mathbb{C}[G]$ and complete. This makes things harder. If G is locally compact, then one uses the convolution algebra $L^1(G)$ in place of $\mathbb{C}[G]$. (For discrete G, one can use $l^1(G)$.) Now the group elements aren't even in the algebra. The resulting C*-algebras are very important in representation theory, and this is an important direction in C*-algebras, but I will say no more here.

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$$\alpha_1 = \mathrm{id}_A, \qquad \alpha_{g_1} = \mathrm{Ad}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\alpha_{g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \text{and} \qquad \alpha_{g_1 g_2} = \mathrm{Ad}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These define an action $\alpha \colon G \to \mathrm{Aut}(A)$ such that α_g is inner for all $g \in G$, but for which there is no homomorphism $g \mapsto z_g \in U(A)$ for which $\alpha_g = \mathrm{Ad}(z_g)$ for all $g \in G$. The point is that the implementing unitaries for α_{g_1} and α_{g_2} commute up to a scalar, but can't be appropriately modified to commute exactly. Exercise: Prove this.

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