Lecture 3: Crossed Products by Actions with the Rokhlin Property

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- Lecture 1 (26 July 2014): Actions of Finite Groups on C*-Algebras and Introduction to Crossed Products.
- Lecture 2 (27 July 2014): Crossed Products by Finite Groups; the Rokhlin Property.
- Lecture 3 (28 July 2014): Crossed Products by Actions with the Rokhlin Property.
- Lecture 4 (29 July 2014): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.
- Lecture 5 (30 July 2014): Examples and Applications.

A rough outline of all five lectures

- Actions of finite groups on C*-algebras and examples.
- Crossed products by actions of finite groups: elementary theory.
- Crossed products by actions of finite groups: Some examples.
- The Rokhlin property for actions of finite groups.
- Examples of actions with the Rokhlin property.
- Crossed products of AF algebras by actions with the Rokhlin property.
- Other crossed products by actions with the Rokhlin property.
- The tracial Rokhlin property for actions of finite groups.
- Examples of actions with the tracial Rokhlin property.
- Crossed products by actions with the tracial Rokhlin property.
- Applications of the tracial Rokhlin property.

The Rokhlin property

Definition

Let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Recall the examples in the exercises from Lecture 2. Let G be a finite group. Then the following actions have Rokhlin property.

1. The action of G on G by translation gives an action of G on $C(G)$ with the Rokhlin property.
2. Let A be any unital C*-algebra. The action of G on $\bigoplus_{g \in G} A$ by translation of the summands has the Rokhlin property.
3. Let G act freely on the Cantor set X. Then the corresponding action of G on $C(X)$ has the Rokhlin property.

$\alpha\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Exercise: Let $T \subset A$ be dense. Suppose that we prove the conditions above for every finite subset $F \subset T$. Then α has the Rokhlin property.

Exercise: More generally, prove the following lemma.

Lemma

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. Let $T \subset A$ be a subset which generates A as a C*-algebra. Suppose that for every finite set $F \subset T$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Then α has the Rokhlin property.

Using a generating set

Exercise: Prove the following lemma.

Lemma

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital C*-algebra A. Let $T \subset A$ be a subset which generates A as a C*-algebra. Suppose that for every finite set $F \subset T$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Then α has the Rokhlin property.

Hint 1: The *-algebra generated by T is dense.
Hint 2: F only appears in condition (2). If, say, a and b approximately commute with e_g, then ab approximately commutes with e_g because

$$\begin{aligned}\|abe_g - e_g ab\| &= \big\|a(be_g - e_g b) + (e_g a - a e_g)b\big\| \\ &\le \|a\| \cdot \|be_g - e_g b\| + \|e_g a - a e_g\| \cdot \|b\|.\end{aligned}$$

A Rokhlin action on a simple C*-algebra

The conditions in the definition of the Rokhlin property, for $\varepsilon > 0$ and a finite set $F \subset A$:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

We want an example in which A is simple. Thus, we won't be able to satisfy condition (2) by choosing e_g to be in the center of A.

Set

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and let α be $\mathbb{Z}_2$ the product type action generated by

$$\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

We will show that this action has the Rokhlin property.

An example (continued)

We had

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The action α of $\mathbb{Z}_2$ is generated by

$$\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

Define projections $p_0, p_1 \in M_2$ by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$w p_0 w^* = p_1, \quad w p_1 w^* = p_0, \quad \text{and} \quad p_0 + p_1 = 1.$$

The action $\alpha\colon \mathbb{Z}_2 \to \mathrm{Aut}(A)$ is generated by $\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $w p_0 w^* = p_1$, $w p_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property, specialized to $G = \mathbb{Z}_2$. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_0 and e_1 such that:

1. $\|\beta(e_0) - e_1\| < \varepsilon$ and $\|\beta(e_1) - e_0\| < \varepsilon$.
2. $\|e_0 a - a e_0\| < \varepsilon$ and $\|e_1 a - a e_1\| < \varepsilon$ for all $a \in F$.
3. $e_0 + e_1 = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense in A, we can assume $F \subset A_n$ for some n. (See above.)

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that e_0 and e_1 actually commute with everything in F. (Proofs: See the next slide.) This proves the Rokhlin property.

An example (continued)

The projections e_0 and e_1 actually commute with everything in F, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of F have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \mathrm{Ad}\big(w^{\otimes n} \otimes w\big)$, so

$$\beta(e_0) = \big(w^{\otimes n} \otimes w\big)(1 \otimes p_0)\big(w^{\otimes n} \otimes w\big)^* = 1 \otimes w p_0 w^* = 1 \otimes p_1 = e_1.$$

The proof that $\beta(e_1) = e_0$ is the same.

Some other actions with the Rokhlin property

Let G be a finite group, and set $n = \mathrm{card}(G)$. Let $g \mapsto v_g$ be the left regular representation of G on $l^2(G)$, identify $L(l^2(G))$ with M_n, and let $A = \bigotimes_{n=1}^{\infty} M_n$ be the n^{∞} UHF algebra. Then the action

$$g \mapsto \alpha_g = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v_g)$$

of G on A has the Rokhlin property.

The example we just did is the case $G = \mathbb{Z}_2$, and the proof in the general case is the same.

Exercise: Write down a detailed proof that this action has the Rokhlin property.

Yet more actions with the Rokhlin property

Let G be a finite group, and set $n = \mathrm{card}(G)$.

Let $\mathcal{O}_n$ be the Cuntz algebra, but call its generators s_g for $g \in G$. The relations are thus

$$s_g^* s_g = 1$$

for all $g \in G$, and

$$\sum_{g \in G} s_g s_g^* = 1.$$

There is an action $\gamma\colon G \to \mathrm{Aut}(\mathcal{O}_n)$ such that

$$\gamma_g(s_h) = s_{gh}$$

for $g, h \in G$. This action is a special case of the quasifree actions on Cuntz algebras from Lecture 1. It turns out to have the Rokhlin property (Izumi).

Actions on Cuntz algebras (continued)

Take $G = \mathbb{Z}_2$ on the previous slide. The resulting action γ of $\mathbb{Z}_2$ on $\mathcal{O}_2$ is generated by the order 2 automorphism determined by $s_1 \mapsto s_2$ and $s_2 \mapsto s_1$.

The action of $\mathbb{Z}_2$ on $\mathcal{O}_2$ generated by $s_1 \mapsto s_1$ and $s_2 \mapsto -s_2$ is conjugate to the one gotten using $G = \mathbb{Z}_2$ above, so also has the Rokhlin property.

Exercise: Prove this conjugacy. Hint: Use an automorphism of $\mathcal{O}_2$ of the same sort as those that appeared in the definition of quasifree actions in Cuntz algebras as in Lecture 1. (It will come from a unitary operator on $\mathbb{C}^2$.)

The quasifree action of $\mathbb{Z}_2$ on $\mathcal{O}_2$ generated by $s_1 \mapsto -s_1$ and $s_2 \mapsto -s_2$ turns out to be pointwise outer but *not* to have the Rokhlin property.

Exactly permuting the projections

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Theorem (2011)

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

The difference is that in (1) we require exact equality.

Exactly permuting the projections (continued)

In the definition of the Rokhlin property, one can replace "$\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$" with "$\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$".

The proof uses methods (equivariant semiprojectivity) unrelated to those here. This result simplifies some proofs by replacing some approximate equalities by equalities, so we will assume it, but it makes no real difference.

(This simplification has not been made in the crossed product notes—proving the theorem is more complicated than doing without it.)

AF algebras

The traditional definition is that a C*-algebra A is an AF algebra if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of finite dimensional subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$. If you know about direct limits, this is the same as asking for a direct system $(B_n)_{n \in \mathbb{N}}$ of finite dimensional C*-algebras such that $\varinjlim B_n \cong A$. Examples: The UHF algebras we have already seen; $K(H)$; $C(X)$ for the Cantor set X..

We are interested in unital AF algebras. Exercise: Show that if A is a unital AF algebra, then the subalgebras A_n above can all be taken to contain 1_A. (In the direct limit definition, one can require that all the maps in the system be unital.)

Convention: When we refer to a unital subalgebra C of a unital C*-algebra A, we mean that $1_A \in C$. Thus, we can restate the unital case as: A unital C*-algebra A is AF if there is an increasing sequence $A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$ of finite dimensional *unital* subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

AF algebras (continued)

A unital C*-algebra A is AF if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of finite dimensional unital subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

AF algebras were introduced by Bratteli. They are one of the first classes of C*-algebras not of type I for which a substantial theory was developed.

Elliott proved that if AF algebras A and B have isomorphic scaled ordered K_0 groups, then $A \cong B$. (This might be discussed in the K-theory lecture series.) In retrospect, this was the beginning of the Elliott classification program.

Effros, Handelman, and Shen gave a simple description of all possible scaled ordered K_0 groups of AF algebras.

AF algebras are still a basic set of examples, and work on them continues to this day.

AF algebras (continued)

A unital C*-algebra A is AF if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of unital finite dimensional subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

The following result is the unital case of a characterization due to Bratteli.

Theorem (Bratteli)

Let A be a separable unital C*-algebra. Then

1. A is a AF algebra.
2. For every finite set $F \subset A$ and every $\varepsilon > 0$, there is a unital finite dimensional subalgebra $D \subset A$ such that $\operatorname{dist}(a, D) < \varepsilon$ for all $a \in F$.

To prove that (1) implies (2), write $A = \overline{\bigcup_{n=0}^{\infty} A_n}$ as above, and take $D = A_n$ for sufficiently large n. Exercise: Check this.

The reverse implication requires more work. Discussion: Next slide.

AF algebras (continued)

Assume that for every finite set $F \subset A$ and every $\varepsilon > 0$, there is a unital finite dimensional subalgebra $D \subset A$ such that $\operatorname{dist}(a, D) < \varepsilon$ for all $a \in F$.

We want to prove that there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of unital finite dimensional subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

One can use the hypothesis for finite subsets $F_0 \subset F_1 \subset \cdots$ with dense union and numbers $\varepsilon_n \to 0$ to construct unital finite dimensional subalgebras $A_n \subset A$ whose union is dense in A. (Exercise: Check density of the union.) but it requires work to arrange to have $A_n \subset A_{n+1}$. One can get A_{n+1} to approximately contain A_n by adding the standard matrix units for A_n to the finite set F_{n+1} and making ε_{n+1} smaller, but this isn't quite what is wanted.

The perturbation arguments required to get exact containment eventually developed into the subject of semiprojectivity. We don't have time to discuss semiprojectivity here.

Actions on AF algebras

A unital C*-algebra A is AF if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of finite dimensional unital subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

Let $\alpha \colon G \to \operatorname{Aut}(A)$ be an action of a finite group G on a unital AF algebra A. One might hope that $C^*(G, A, \alpha)$ would again be AF.

The algebraic version of this is true. That is, if a complex *-algebra A can be written as $A = \bigcup_{n=0}^{\infty} A_n$ for finite dimensional C*-algebras $A_0 \subset A_1 \subset \cdots$, and if $\alpha \colon G \to \operatorname{Aut}(A)$ is an action of a finite group G on A, then the algebraic crossed product is again an increasing union of the same type.

The idea is to replace $A_0 \subset A_1 \subset \cdots$ with finite dimensional C*-algebras $B_0 \subset B_1 \subset \cdots$ such that $\alpha_g(B_n) \subset B_n$ for all $g \in G$ and $n \in \mathbb{N}$.

Exercise: Carry it out. Hint: To start, the subalgebra generated by $\bigcup_{g \in G} \alpha_g(A_0)$ is contained in A_n for some n.

Actions on AF algebras (continued)

A unital C*-algebra A is AF if there is an increasing sequence

$$A_0 \subset A_1 \subset A_2 \subset \cdots \subset A$$

of finite dimensional unital subalgebras of A such that $\overline{\bigcup_{n=0}^{\infty} A_n} = A$.

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a unital AF algebra A. One might hope that $C^*(G, A, \alpha)$ would again be AF. If one uses algebraic direct limits, this is in fact true.

The C* version was open for some time, but turns out to be false. (The hint in the exercise on the previous slide doesn't work.) If A is AF, then $K_1(A) = 0$, $K_0(A)$ is torsion free, and A has real rank zero (definition omitted). There are examples of actions of $\mathbb{Z}_2$ on simple AF algebras such that the crossed product has nonzero K_1 (Blackadar), does not have real rank zero (Elliott), and has torsion in K_0.

Crossed products by actions with the Rokhlin property

A structure theorem for products by actions with the Rokhlin property:

Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

Crossed products by actions of finite groups with the Rokhlin property preserve many other structural properties of C*-algebras. (See below.)

The basic idea (details later): Let $e_g \in A$, for $g \in G$, be Rokhlin projections, with $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g. Then $v_{g,h} = e_g u_{gh^{-1}}$ defines a system of matrix units in $C^*(G, A, \alpha)$. (This is essentially the same formula as was used in the proof that $C^*(G, \, C(G)) \cong M_n$.) Using the homomorphism $M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ given by $v_{g,h} \otimes d \mapsto v_{g,1} d v_{1,h}$, one can approximate $C^*(G, A, \alpha)$ by matrix algebras over corners of A.

Idea of the proof

A is an AF algebra, G is a finite group, and $\alpha \colon G \to \mathrm{Aut}(A)$ has the Rokhlin property. We want to approximate a finite set $S \subset C^*(G, A, \alpha)$ by a finite dimensional subalgebra.

It turns out that it suffices to consider finite subsets of some generating set. (The argument is easier than the corresponding argument for the Rokhlin property. Exercise: Do it.) So we assume $S = F \cup \{u_g \colon g \in G\}$, with $F \subset A$ finite and $u_g \in C^*(G, A, \alpha)$ the standard unitary corresponding to $g \in G$.

In this sketch, we will find an AF algebra $D \subset C^*(G, A, \alpha)$ which approximately contains S. It is not hard to see that this is enough (exercise: check this!), but the actual proof will be organized a bit differently.

Preliminary exercise: Let B be a C*-algebra and let $q \in B$ be a projection. Show that qBq is a C*-algebra, with identity q.

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha \colon G \to \mathrm{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g \colon g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We will approximate S by an AF algebra.

Apply the Rokhlin property to the finite set F. Use the version in which the projections are exactly permuted by the group. Thus, we get projections $e_g \in A$ for $g \in G$ such that:

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\| e_g a - a e_g \| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Informally: $e_g a \approx a e_g$ for all $g \in G$ and all $a \in F$.

In particular, for $g \neq h$ and $a \in F$, $e_g a e_h \approx a e_g e_h = 0$. Therefore, if $a \in F$,

$$a = \sum_{g,h \in G} e_g a e_h \approx \sum_{g \in G} e_g a e_g .$$

That is, a is approximately in $D_0 = \sum_{g \in G} e_g A e_g \subset A$.

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g : g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We will approximate S by an AF algebra. We chose Rokhlin projections $e_g \in A$ for $g \in G$.

We have found that F is approximately contained in the unital subalgebra (justification for subalgbra and direct sum below)

$$D_0 = \sum_{g \in G} e_g A e_g = \bigoplus_{g \in G} e_g A e_g \subset A.$$

The sum is direct because the projections e_g are orthogonal, and D_0 is unital because $\sum_{g \in G} e_g = 1$. Exercise: Prove that if B is a C*-algebra and $p_1, p_2, \ldots, p_n \in B$ are mutually orthogonal projections, then $\sum_{k=1}^n p_k B p_k$ is a C* subalgebra of B isomorphic to $\bigoplus_{k=1}^n p_k B p_k$.

Recall that we are assuming that $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$. Exercise: Use this to prove that D_0 is G-invariant.

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g : g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We will approximate S by an AF algebra. We chose Rokhlin projections $e_g \in A$ for $g \in G$, and we found that $D_0 = \bigoplus_{g \in G} e_g A e_g$ is a unital G-invariant subalgebra of A which approximately contains F.

The action of G permutes the summands. Exercise: Prove that D_0 is equivariantly isomorphic to $C(G, \, e_1 A e_1)$ with the action $\beta_g(b)(h) = b(h^{-1}g)$ for $g, h \in G$ and $b \in C(G, \, e_1 A e_1)$.

Set $n = \mathrm{card}(G)$. We showed before that $C^*(G, \, C(G)) \cong M_n$. Exercise: Use the same method to prove that if B is any unital C*-algebra, then $C^*(G, \, C(G, B)) \cong M_n(B)$.

Set $D = C^*(G, D_0, \alpha) \subset C^*(G, A, \alpha)$. Thus $D \cong M_n(e_1 A e_1)$.

Idea of the proof (continued)

A is an AF algebra, G is a finite group, and $\alpha\colon G \to \mathrm{Aut}(A)$ has the Rokhlin property. Our finite set is $S = F \cup \{u_g : g \in G\} \subset C^*(G, A, \alpha)$, with $F \subset A$ finite. We got $D \subset C^*(G, A, \alpha)$ as $D = C^*(G, D_0, \alpha)$, in which $D_0 \subset A$ is a unital subalgebra which approximately contains F.

Since D_0 is unital, $u_g \in D$ for all $g \in G$. Therefore D approximately contains $S = F \cup \{u_g : g \in G\}$, as wanted.

All that remains is to show that D is AF. Recall that $D \cong M_n(e_1 A e_1)$.

It is a general fact that if C is an AF algebra and $q \in C$ is a projection, then qCq is an AF algebra. (Suppose $C = \overline{\bigcup_{n=0}^\infty C_n}$ for an increasing sequence of finite dimensional C*-algebras $(C_n)_{n \in \mathbb{N}}$. Using methods from the K-theory lectures, show that q is unitarily equivalent to a projection in one of the C_n.) Since $e_1 A e_1$ is AF, so is $D \cong M_n(e_1 A e_1)$.

A future modification of the argument

We want to approximate elements of $C^*(G, A, \alpha)$ using unital homomorphisms from $M_n \otimes e_1 A e_1$ to $C^*(G, A, \alpha)$.

In Lecture 4, we are going to need the same argument again, but under slightly weaker conditions. We will still assume that the projections e_g are orthogonal, are exactly permuted by the group action, and can be chosen to approximately commute with a given finite subset of A. However, the sum $e = \sum_{g \in G} e_g$ will no longer necessarily be equal to 1.

We can nevertheless carry out the same argument; we get unital homomorphisms from $M_n \otimes e_1 A e_1$ to $e C^*(G, A, \alpha) e$, and we just get the weaker conclusion that we can approximate a finite set in $e C^*(G, A, \alpha) e$, rather than one in $C^*(G, A, \alpha)$, by a matrix algebra over a corner of A.

Crossed products by actions with the Rokhlin property (continued)

Recall the conclusion of the theorem: $C^*(G, A, \alpha)$ is AF.

To prove the theorem, we prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g. Thus, a general element has the form $\sum_{g \in G} c_g u_g$, with $c_g \in A$ for $g \in G$. It suffices to consider a finite set of the form $S = F \cup \{u_g : g \in G\}$, where F is a finite subset of A. So let $F \subset A$ be a finite subset and let $\varepsilon > 0$.

Set

$$n = \operatorname{card}(G) \quad \text{and} \quad \delta = \frac{\varepsilon}{n(n-1)}.$$

Crossed products by actions with the Rokhlin property (continued)

We had: $S = F \cup \{u_g : g \in G\}$, with F a finite subset of A.

Apply the Rokhlin property to α with F as given and with δ in place of ε, obtaining projections $e_g \in A$ for $g \in G$ such that $\alpha_g(e_h) = e_{gh}$ for $g, h \in G$, $\|e_g a - a e_g\| < \delta$ for $g \in G$ and $a \in F$, and $\sum_{g \in G} e_g = 1$.

Define $v_{g,h} = e_g u_{gh^{-1}}$ for $g, h \in G$. In particular, $v_{g,g} = e_g$, so the $v_{g,g}$ are orthogonal projections which add up to 1.

We claim that the $v_{g,h}$ form a system of $n \times n$ matrix units in $C^*(G, A, \alpha)$. Recall for comparison: when proving that $C^*(G, C(G)) \cong M_n$, we used the matrix units $v_{g,h} = \chi_{\{g\}} u_{gh^{-1}}$. The computation here is exactly the same as there, so we don't repeat it.

Crossed products by actions with the Rokhlin property (continued)

We had: $(v_{g,h})_{g,h \in G}$ is an $n \times n$ system of matrix units in $C^*(G, A, \alpha)$.

Let $(w_{g,h})_{g,h \in G}$ be a system of matrix units for M_n. There is a unital homomorphism $\varphi_0 \colon M_n \to C^*(G, A, \alpha)$ such that $\varphi_0(w_{g,h}) = v_{g,h}$ for all $g, h \in G$. In particular, $\varphi_0(w_{g,g}) = e_g$ for all $g \in G$.

Now define a unital homomorphism $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ by $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$ for $g, h \in G$ and $d \in e_1 A e_1$.

Corners of AF algebras are AF, and φ is injective, so $D = \varphi(M_n \otimes e_1 A e_1)$ is an AF subalgebra of $C^*(G, A, \alpha)$. We complete the proof by showing that every element of S is within ε of an element of D. Recall that $S = F \cup \{u_g : g \in G\}$, and F is a finite subset of A.

Crossed products by actions with the Rokhlin property (continued)

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$ by elements of $D = \varphi(M_n \otimes e_1 A e_1)$.

We first consider u_g with $g \in G$. In fact, for u_g no approximation is necessary. Recall that $v_{g,h} = e_g u_{gh^{-1}}$. We have

$$\varphi\left(\sum_{h \in G} w_{h, g^{-1}h}\right) = \varphi_0\left(\sum_{h \in G} w_{h, g^{-1}h}\right) = \sum_{h \in G} v_{h, g^{-1}h} = \sum_{h \in G} e_h u_g = u_g.$$

Crossed products by actions with the Rokhlin property (continued)

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$, with $F \subset A$ finite, by elements of $D = \varphi(M_n \otimes e_1 A e_1)$. Recall that $\varphi\colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ is defined by $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$ for $g, h \in G$ and $d \in e_1 A e_1$. We already took care of u_g.

Let $a \in F$. The obvious first step in approximating a is to use

$$\sum_{g \in G} e_g a e_g.$$

In fact, one needs to (implicitly) use this approximation in the form

$$\sum_{g \in G} \alpha_g \big(e_1 \alpha_g^{-1}(a) e_1 \big).$$

This happens because the definition of φ sends $w_{g,h} \otimes d$, for $d \in e_1 A e_1$, to an element obtained by using the action of the group elements g and h.

Set

$$b = \sum_{g \in G} w_{g,g} \otimes e_1 \alpha_g^{-1}(a) e_1 \in M_n \otimes e_1 A e_1.$$

For $g \neq h$, we have

$$\|e_g a e_h\| \leq \|e_g a - a e_g\| + \|a e_g e_h\| = \|e_g a - a e_g\| < \delta.$$

We then get

$$\Big\| a - \sum\nolimits_{g \in G} e_g a e_g \Big\| \leq \sum\nolimits_{g \neq h} \|e_g a e_h\| < n(n-1)\delta.$$

We use this, and the relations

$$v_{g,1} e_1 = e_g u_g e_1 = e_g u_g \quad \text{and} \quad e_1 \alpha_g^{-1}(a) e_1 = \alpha_g^{-1}(e_g a e_g),$$

to get

$$\begin{aligned} \|a - \varphi(b)\| &= \Big\| a - \sum\nolimits_{g \in G} v_{g,1} e_1 \alpha_g^{-1}(a) e_1 e_{1,g} \Big\| \\ &= \Big\| a - \sum\nolimits_{g \in G} e_g u_g \alpha_g^{-1}(a) u_g^* e_g \Big\| = \Big\| a - \sum\nolimits_{g \in G} e_g a e_g \Big\| \\ &< n(n-1)\delta = \varepsilon. \end{aligned}$$

This completes the proof of the theorem.

Other structural consequences of the Rokhlin property

Crossed products by actions of finite groups with the Rokhlin property preserve various other classes of C*-algebras. In many cases, the proofs are similar to what we did for AF algebras. Some examples of such classes:

1. Simple unital C*-algebras.
2. Various classes of unital but not necessarily simple countable direct limit C*-algebras using semiprojective building blocks. (With Osaka.)
3. Simple unital AH algebras with slow dimension growth and real rank zero. (With Osaka.)
4. D-absorbing separable unital C*-algebras for a strongly self-absorbing C*-algebra D. (Hirshberg-Winter.)
5. Separable nuclear unital C*-algebras whose quotients all satisfy the Universal Coefficient Theorem. (With Osaka.)
6. Separable unital approximately divisible C*-algebras. (Hirshberg-Winter.)
7. Unital C*-algebras with the ideal property and unital C*-algebras with the projection property. (With Pasnicu.)