Lecture 3: Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property

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- Lecture 1 (7 Dec. 2011): Crossed Products of C*-Algebras by Finite Groups.
- Lecture 2 (8 Dec. 2011): Crossed Products of AF Algebras by Actions with the Rokhlin Property.
- Lecture 3 (9 Dec. 2011): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.

Recall the Rokhlin property

Definition

Let A be a unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Crossed products by actions with the Rokhlin property are very well behaved.

But the Rokhlin property is rare. There are no Rokhlin actions of any nontrivial finite group on any irrational rotation algebra or on $\mathcal{O}_\infty$. Also, if $\mathrm{card}(G)$ is not a power of 2 there is no Rokhlin action of G on the 2^∞ UHF algebra. Therefore we look for a weaker condition.

Pointwise outer actions

For simplicity of the crossed product, a much weaker condition suffices.

Definition

An action $\alpha \colon G \to \mathrm{Aut}(A)$ is said to be *pointwise outer* if, for $g \in G \setminus \{1\}$, the automorphism α_g is outer, that is, not of the form $a \mapsto \mathrm{Ad}(u)(a) = uau^*$ for some unitary u in the multiplier algebra $M(A)$ of A.

Theorem (Kishimoto)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a simple C*-algebra A. Suppose that α is pointwise outer. Then $C^*_{\mathrm{r}}(G, A, \alpha)$ is simple.

Pointwise outerness is not enough

The product type action

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(\mathrm{diag}(-1, 1, 1, \ldots, 1)) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

is pointwise outer. However, its crossed product has "too many" tracial states. (A has a unique tracial state, but the crossed product has two extreme tracial states.)

Worse, Elliott has constructed an example of a pointwise outer action α of $\mathbb{Z}_2$ on a simple unital AF algebra A such that $C^*(\mathbb{Z}_2, A, \alpha)$ does not have real rank zero.

Pointwise outerness thus seems not to be enough for proving classifiability of crossed products.

The tracial Rokhlin property

Definition

Let A be an infinite dimensional simple separable unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *tracial Rokhlin property* if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

The Rokhlin property corresponds to $e = 1$.

If A is finite, the last condition can be omitted.

Exactly permuting the projections

The conditions in the definition:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.
4. With e as in (3), we have $\|exe\| > 1 - \varepsilon$.

As for the Rokhlin property, a very recent theorem implies that we can replace

$$\text{“}\|\alpha_g(e_h) - e_{gh}\| < \varepsilon \text{ for all } g, h \in G\text{”}$$

(in (1)) with

$$\text{“}\alpha_g(e_h) = e_{gh} \text{ for all } g, h \in G\text{”}.$$

We do this in these lectures.

The tracial Rokhlin property (continued)

The conditions in the definition for the finite case:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. With $e = \sum_{g \in G} e_g$, the projection $1 - e$ is Murray-von Neumann equivalent to a projection in the hereditary subalgebra of A generated by x.

The first two conditions are the same as for the Rokhlin property.

If the algebra "has enough tracial states" (for example, for irrational rotation algebras and simple AF algebras), the element x can be omitted, and the third condition replaced by:

- With $e = \sum_{g \in G} e_g$, the projection $1 - e$ satisfies $\tau(1 - e) < \varepsilon$ for all tracial states τ on A.

Comparison: tracial rank zero

The tracial Rokhlin property was motivated by the definition of tracial rank zero (originally called "tracially AF"):

Definition

Let A be a simple unital C*-algebra. Then A has tracial rank zero if for every finite subset $F \subset A$, every $\varepsilon > 0$, and every nonzero positive element $x \in A$, there exists a projection $p \in A$ and a unital finite dimensional subalgebra $D \subset pAp$ such that:

1. $\|[a, p]\| < \varepsilon$ for all $a \in F$.
2. $\operatorname{dist}(pap, D) < \varepsilon$ for all $a \in F$.
3. $1 - p$ is Murray-von Neumann equivalent to a projection in $\overline{xAx}$.

In both definitions, the strong version (the Rokhlin property, or local approximation by finite dimensional C*-algebras) is supposed to hold only after cutting down by a "large" projection.

An example for the tracial Rokhlin property

Here is an example for which it is fairly easy to see that the tracial Rokhlin property holds but the Rokhlin property doesn't hold.

Take $v_k \in M_{3^k}$ to be the unitary

$$v_k = \operatorname{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1),$$

in which the diagonal entry 1 occurs $\frac{1}{2}(3^k + 1)$ times and the diagonal entry -1 occurs $\frac{1}{2}(3^k - 1)$ times. Let α be the order 2 automorphism

$$\alpha = \bigotimes_{k=1}^{\infty} \operatorname{Ad}(v_k) \quad \text{of} \quad A = \bigotimes_{k=1}^{\infty} M_{3^k}.$$

It generates an action of $\mathbb{Z}_2$ (also called α). Note that A is just the 3^∞ UHF algebra.

We consider the tracial Rokhlin property below, but we can see right away that α does not have the Rokhlin property: there is no action at all of $\mathbb{Z}_2$ on this algebra which has the Rokhlin property.

An example for the tracial Rokhlin property (continued)

$$v_k = \operatorname{diag}(1, 1, \ldots, 1, -1, -1, \ldots, -1),$$

with -1 occurring $r(k) = \frac{1}{2}(3^k - 1)$ times and 1 occurring $r(k) + 1 = \frac{1}{2}(3^k + 1)$ times, and α is the $\mathbb{Z}_2$ action on the 3^∞ UHF algebra generated by $\bigotimes_{n=1}^{\infty} \operatorname{Ad}(v_k)$.

v_k is unitarily equivalent to the block unitary (in which subscripts indicate matrix sizes)

$$w_k = \begin{pmatrix} 0 & 1_{r(k)} & 0 \\ 1_{r(k)} & 0 & 0 \\ 0 & 0 & 1_1 \end{pmatrix} \in M_{3^k}.$$

So α is conjugate to $\beta = \bigotimes_{n=1}^{\infty} \operatorname{Ad}(w_k)$.

$$w_k = \begin{pmatrix} 0 & 1_{r(k)} & 0 \\ 1_{r(k)} & 0 & 0 \\ 0 & 0 & 1_1 \end{pmatrix} \in M_{3^k} \quad \text{and} \quad \beta = \bigotimes_{n=1}^{\infty} \operatorname{Ad}(w_k).$$

For $F \subset A$ finite, we have to find two projections e_0 and e_1 in A such that:

1. The action approximately exchanges e_0 and e_1.
2. e_0 and e_1 approximately commute with all elements of F.
3. $1 - e_0 - e_1$ is "small", here, the (unique) tracial state τ on A gives $\tau(1 - e_0 - e_1) < \varepsilon$.

We assume $F \subset A_n = \bigotimes_{k=1}^{n} M_{3^k}$. We can increase n, so also assume $3^{-n-1} < \varepsilon$. Set

$$p_0 = \begin{pmatrix} 1_{r(n+1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n+1}} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{r(n+1)} & 0 \\ 0 & 0 & 0_1 \end{pmatrix} \in M_{3^{n+1}}.$$

Then $w_{n+1} p_0 w_{n+1}^* = p_1$, $w_{n+1} p_1 w_{n+1}^* = p_0$, and the normalized trace of $1 - p_0 - p_1$ is $\dfrac{1}{2r(n+1)+1} = 3^{-(n+1)} < \varepsilon$.

An example for the tracial Rokhlin property (continued)

We had, in $M_{3^{n+1}}$,

$$w_{n+1} p_0 w_{n+1}^* = p_1 \quad \text{and} \quad w_{n+1} p_1 w_{n+1}^* = p_0,$$

and the normalized trace of $1 - p_0 - p_1$ is less than ε.

Now we take

$$e_0 = 1_{A_n} \otimes p_0, \;\; e_1 = 1_{A_n} \otimes p_1 \in A_n \otimes M_{3^{n+1}} = A_{n+1}.$$

Since all elements of F have the form $a \otimes 1_{M_{3^{n+1}}}$ with $a \in A_n$, these projections exactly commute with the elements of F.

Also, $\tau(1 - e_0 - e_1) < \varepsilon$ because $\tau(1 - e_0 - e_1)$ is the normalized trace of $1 - p_0 - p_1$.

Finally, on $A_n \otimes M_{3^{n+1}}$, the automorphism β has the form $\mathrm{Ad}\big(w^{(n)} \otimes w_{n+1}\big)$, so

$$\beta(e_0) = e_1 \quad \text{and} \quad \beta(e_1) = e_0.$$

This proves the tracial Rokhlin property.

An example for the tracial Rokhlin property (continued)

$$v_k = \mathrm{diag}\big(1, 1, \ldots, 1, -1, -1, \ldots, -1\big),$$

and α is the $\mathbb{Z}_2$-action on the 3^∞ UHF algebra generated by $\bigotimes_{n=1}^{\infty} \mathrm{Ad}(v_k)$.

The matrix size is very large, and the numbers of 1's and -1's are very close. This looks special.

Recall from Lecture 1 the action of $\mathbb{Z}_2$ on $\bigotimes_{n=1}^{\infty} M_3$ generated by $\gamma = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v)$, with

$$v = \mathrm{diag}(1, 1, -1).$$

Rewrite:

$$\gamma = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v^{\otimes n}).$$

We have $v_1 = v$. On the diagonal of $v^{\otimes 2}$ there are $2 \cdot 2 + 1 \cdot 1 = 5$ entries equal to 1 and $2 \cdot 1 + 1 \cdot 2 = 4$ entries equal to -1. So it is unitarily equivalent to v_2. Inductively, $v^{\otimes n}$ is unitarily equivalent to v_n. So the action γ has the tracial Rokhlin property.

The tracial Rokhlin property is common

$$\gamma = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \right)$$

has the tracial Rokhlin property.

In fact, it turns out to be hard to write down a product type action of $\mathbb{Z}_2$ using conjugation by matrices of the form

$$\mathrm{diag}\big(1, 1, \ldots, 1, -1, -1, \ldots, -1\big)$$

which doesn't have the tracial Rokhlin property. Either eventually they all have to be ± 1 (in which case the action is inner), or the matrix sizes must go to infinity. (Example: Next slide.)

In fact, any product type action of $\mathbb{Z}_2$ is conjugate to one of this form.

Some other actions with the tracial Rokhlin property

- The actions on irrational rotation algebras coming from finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ have the tracial Rokhlin property.
- The action of $\mathbb{Z}_n$ on an irrational rotation algebra generated by $u \mapsto e^{2\pi i/n} u$ and $v \mapsto v$ has the tracial Rokhlin property.
- The tensor flip on any UHF algebra has the tracial Rokhlin property.
- An action of a finite group on a unital Kirchberg algebra has the tracial Rokhlin property if and only if it is pointwise outer (essentially due to Nakamura).

Except for the last, none of these actions has the Rokhlin property.

Some of the proofs are complicated.

It is true (and easy) that the tracial Rokhlin property implies pointwise outerness in complete generality, but the converse is false. Counterexample:

$$\bigotimes_{n=1}^{\infty} \mathrm{Ad}(\mathrm{diag}(-1, 1, 1, \ldots, 1)) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_{2^n+1}.$$

Crossed products by actions with the tracial Rokhlin property

The tracial Rokhlin property is good for understanding the structure of crossed products.

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

This is important because tracial rank zero is a hypothesis in a major classification theorem (due to Lin).

There are examples (such as the one of Elliott mentioned above) which show that this theorem fails if one weakens the condition on the action to pointwise outerness.

Crossed products by actions with the tracial Rokhlin property (continued)

Theorem

Let A be a simple separable unital C*-algebra with tracial rank zero. Let G be a finite group, and let $\alpha\colon G \to \mathrm{Aut}(A)$ have the tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has tracial rank zero.

The idea of the proof is essentially the same as for crossed products of AF algebras by actions with the Rokhlin property. The definitions of both tracial rank zero and the tracial Rokhlin property allow a "small" (in trace) error projection. One must show that the sum of two "small" error projections is again "small".

There is one additional difficulty. The hypotheses give an error which is "small" relative to A. One must prove that it is also "small" relative to $C^*(G, A, \alpha)$. This uses a theorem of Jeong and Osaka. We give the ideas of a proof.

"Small" in $C^*(G, A, \alpha)$ vs. "small" in A

The following result implies that one can make the error projection "small" relative to $C^*(G, A, \alpha)$ by requiring that it be "small" relative to A.

Theorem

Let A be an infinite dimensional simple separable unital C*-algebra which has Property (SP) (every nonzero hereditary subalgebra contains a nonzero projection), and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group which has the tracial Rokhlin property. Then for every nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$, there is a nonzero projection $p \in D$ which is Murray-von Neumann equivalent to a projection $q \in A$.

So, to ensure the error projection $1 - e$ is Murray-von Neumann equivalent to a projection in D, it is enough to require that $1 - e \precsim q$ in A.

Much weaker conditions suffice: provided one uses $C^*_{\mathrm{r}}(G, A, \alpha)$, one can allow any pointwise outer action of a discrete group (Jeong-Osaka).

Tracial rank zero is known to imply Property (SP).

Kishimoto's condition

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A.

We use what we call Kishimoto's condition (from his paper on simplicity of reduced crossed products). Here is the version for finite groups:

Definition

Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on a C*-algebra A. We say that α satisfies *Kishimoto's condition* if for every $x \in A_+$ with $\|x\| = 1$, for every finite set $S \subset A$, and for every $\varepsilon > 0$, there is $a \in A_+$ with $\|a\| = 1$ such that:

1. $\|axa\| > 1 - \varepsilon$.
2. $\|ab\alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$ and $b \in S$.

For general discrete groups, one uses finite subsets of $G \setminus \{1\}$. Kishimoto shows that it holds for pointwise outer actions on simple C*-algebras (in fact, under weaker assumptions).

Getting Kishimoto's condition

For $F \subset A$ finite and $\delta > 0$, the tracial Rokhlin property gives mutually orthogonal projections e_g such that (omitting a condition we won't need):

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\|e_g b - b e_g\| < \delta$ for all $g \in G$ and all $b \in F$.
3. With $e = \sum_{g \in G} e_g$, we have $\|exe\| > 1 - \varepsilon$.

Given $x \in A_+$ with $\|x\| = 1$, and a finite set $S \subset A$, we want:

1. $\|axa\| > 1 - \varepsilon$.
2. $\|ab\alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$ and $b \in S$.

Apply the tracial Rokhlin property with $F = S \cup \{x\}$. Then $exe \approx \sum_{h \in G} e_h x e_h$, so, provided δ is small enough, there is some $h \in G$ such that $\|e_h x e_h\| > 1 - \varepsilon$. Now $g \neq 1$ implies

$$e_h b \alpha_g(e_h) \approx b e_h \alpha_g(e_h) = b e_h e_{gh} = 0.$$

Using Kishimoto's condition

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A. We assume α satisfies Kishimoto's condition.

Step 1: Choose a nonzero positive element $c = \sum_{g \in G} c_g u_g \in D$. We claim $c_1 \geq 0$ and we can arrange to have $\|c_1\| = 1$.

To see this, write $c = yy^*$ with $y = \sum_{g \in G} y_g u_g$. Multiply out, getting $c_1 = \sum_{g \in G} y_g u_g u_g^* y_g^* = \sum_{g \in G} y_g y_g^* \geq 0$. If this is zero, then $y_g = 0$ for all g, so $y = 0$, so $c = 0$.

Now multiply c by a suitable scalar.

Step 2: Apply Kishimoto's condition, with suitable $\varepsilon > 0$, with $x = c_1$, and using the finite set of coefficients c_g for $g \in G$, getting $a \in A_+$ with $\|a\| = 1$ such that:

1. $\|a c_1 a\| > 1 - \varepsilon$.
2. $\|a c_g \alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$.

Using Kishimoto's condition (continued)

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A.

We have $c = \sum_{g \in G} c_g u_g \in D_+$ and $a \in A_+$ with $\|c_1\| = \|a\| = 1$ and

1. $\|a c_1 a\| > 1 - \varepsilon$.
2. $\|a c_g \alpha_g(a)\| < \varepsilon$ for all $g \in G \setminus \{1\}$.

Step 3: Using $u_g a u_g^* = \alpha_g(a)$ at the second step and (2) at the third step:

$$aca = \sum_{g \in G} a c_g u_g a = \sum_{g \in G} a c_g \alpha_g(a) u_g \approx a c_1 a u_1 = a c_1 a.$$

Step 4: Choose $f \colon [0,1] \to [0,1]$ such that $f = 0$ on $[0, 1 - 2\varepsilon]$ and $f(1 - \varepsilon) = 1$. Then $f(a c_1 a) \neq 0$ by (1). By Property (SP), there is a nonzero projection p in the hereditary subalgebra of A generated by $f(a c_1 a)$. One can show that $p a c_1 a p \approx p$.

Using Kishimoto's condition (continued)

We want to show that a nonzero hereditary subalgebra $D \subset C^*(G, A, \alpha)$ contains a nonzero projection equivalent to a projection in A.

We have $c \in D_+$ and $a \in A_+$, and a nonzero projection $p \in A$, all satisfying

$$aca \approx a c_1 a \quad \text{and} \quad p a c_1 a p \approx p.$$

Step 5: So

$$pacap \approx p.$$

Define

$$s_0 = c^{1/2} a p.$$

Then $s_0^* s_0 \approx p$ and is in $pC^*(G, A, \alpha)p$. So we can form $s = s_0 (s_0^* s_0)^{-1/2}$ (functional calculus in $pC^*(G, A, \alpha)p$), getting

$$s^* s = p \quad \text{and} \quad s s^* = c^{1/2} a p \big((s_0^* s_0)^{-1/2} \big)^2 p a c^{1/2} \in \overline{c C^*(G, A, \alpha) c} \subset D.$$

Thus ss^* is a projection in D equivalent to the nonzero projection $p \in A$. This is what we want.

An application: Crossed products of rotation algebras by finite groups

Theorem (Joint with Echterhoff, Lück, and Walters; known previously for the order 2 case)

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Let $\alpha \colon G \to \mathrm{Aut}(A_\theta)$ be the action on A_θ of one of the finite subgroups of $\mathrm{SL}_2(\mathbb{Z})$ (of order 2, 3, 4, or 6). Then $C^*(G, A_\theta, \alpha)$ is an AF algebra.

This solved a problem open for some years. The result is initially unexpected, since A_θ itself is not AF. It was suggested by K-theory computations done for rational θ.

The proof uses the tracial Rokhlin property for the action to show that $C^*(G, A_\theta, \alpha)$ has tracial rank zero. One then applies classification theory (specifically, Lin's classification theorem), but one must compute the K-theory of $C^*(G, A_\theta, \alpha)$ and show that it satisfies the Universal Coefficient Theorem. This is done using known cases of the Baum-Connes conjecture.

An application: Higher dimensional noncommutative tori

Theorem

Every simple higher dimensional noncommutative torus is an AT algebra.

A higher dimensional noncommutative torus is a version of the rotation algebra using more unitaries as generators. An AT algebra is a direct limit of finite direct sums of C*-algebras of the form $C(S^1, M_n)$.

Elliott and Evans proved that A_θ is an AT algebra for θ irrational. A general simple higher dimensional noncommutative torus can be obtained from some A_θ by taking repeated crossed products by $\mathbb{Z}$. If all the intermediate crossed products are simple, the theorem follows from a result of Kishimoto. Using classification theory and the tracial Rokhlin property for actions generalizing the one on A_θ generated by

$$u \mapsto e^{2\pi i/n} u \quad \text{and} \quad v \mapsto v,$$

one can reduce the general case to the case proved by Kishimoto.

Directions for further work

- Infinite discrete groups. This is a vast area, with almost nothing known beyond $\mathbb{Z}^d$, and not that much beyond $\mathbb{Z}$.
- Actions of finite dimensional quantum groups (recently started by Kodaka, Osaka, Teruya).
- What if there are few or no projections? Some things are known, but there is lots to do.
- The nonsimple case: Almost nothing is known, not even the right version of Kishimoto's theorem on crossed products by pointwise outer actions.

For many open problems for actions of finite groups, see the survey article:

N. C. Phillips, *Freeness of actions of finite groups on C*-algebras*, pages 217–257 in: *Operator structures and dynamical systems*, M. de Jeu, S. Silvestrov, C. Skau, and J. Tomiyama (eds.), Contemporary Mathematics vol. 503, Amer. Math. Soc., Providence RI, 2009.

Two problems on the nonsimple case

Problem

Find a suitable definition of the tracial Rokhlin property for actions on nonsimple unital C*-algebras, and possibly a different definition of tracial rank zero, so that crossed products of C*-algebras with tracial rank zero by tracially Rokhlin actions again have tracial rank zero.

There is some small unpublished progress here.

Problem

Is there a suitable version of pointwise outerness of an action $\alpha \colon G \to \mathrm{Aut}(A)$ (with G finite) which guarantees that every ideal in $C^*(G, A, \alpha)$ has the form $C^*(G, J, \alpha)$ for some G-invariant ideal $J \subset A$?

Appendix: More on directions for further work

I will be very sketchy here, but I want to demonstrate that there is still much to be done even just for actions of finite groups.

There is far more to be done for other group actions. The finite group case is just the most elementary case of a huge research area, but, to keep the length down, we discuss only actions of finite groups. There are also generalizations in a different direction, to actions of finite dimensional quantum groups (Kodaka, Osaka, Teruya); again, I will say nothing more.

What if there are few or no projections? There are several competing suggested conditions generalizing the tracial Rokhlin property, and several theorems, but the situation is far from clear. There is also a kind of higher dimensional Rokhlin property, but there is no known analog of the ordinary Rokhlin property for actions on C*-algebras with no nontrivial projections.

What should be done in the nonunital case? There is at least one suggested answer, but as far as I know almost no theorems.

Appendix: More on directions for further work (continued)

The result on crossed products of AF algebras by Rokhlin actions did not need simplicity. There is a definition of tracial rank zero for unital nonsimple algebras, but some things go wrong: it no longer implies either real rank zero or stable rank one. (There are theorems relating tracial rank zero to tracial quasidiagonality of extensions, which suggest that, nevertheless, this definition may be the "right" one.)

Problem

Find a suitable definition of the tracial Rokhlin property for actions on nonsimple unital C*-algebras, and possibly a different definition of tracial rank zero, so that crossed products of C*-algebras with tracial rank zero by tracially Rokhlin actions again have tracial rank zero.

There is some small unpublished progress here.

Appendix: More on directions for further work (continued)

What about simplicity of crossed products? When the algebra is simple, pointwise outerness is enough, but is not necessary. It is necessary that there be no nontrivial invariant ideals, so the right question is really as follows:

Problem

Is there a suitable version of pointwise outerness of an action $\alpha\colon G \to \mathrm{Aut}(A)$ (with G finite) which guarantees that every ideal in $C^*(G, A, \alpha)$ has the form $C^*(G, J, \alpha)$ for some G-invariant ideal $J \subset A$?

There is a necessary and sufficient condition in terms of the noncommutative Connes spectrum, but this is hard to compute.

One needs at least that for every subgroup $H \subset G$ and every H-invariant subquotient J/I of A, the induced action of H on J/I is pointwise outer. (There is a finite dimensional counterexample if one doesn't consider subgroups.)

Appendix: More on directions for further work (continued)

A problem on crossed products of rotation algebras by finite groups:

$C^*(G, A_\theta, \alpha)$ is AF for finite subgroups $G \subset \mathrm{SL}_2(\mathbb{Z})$. For $G = \mathbb{Z}_2$, much more direct methods are known. For the other cases, our proof, using several different pieces of heavy machinery (the Elliott classification program and the Baum-Connes conjecture), is the only one known.

Problem

Prove that $C^*(G, A_\theta, \alpha)$ for $G \subset \mathrm{SL}_2(\mathbb{Z})$ of order 3, 4, and 6, by explictly writing down a direct system of finite dimensional C*-algebras and proving directly that its direct limit is $C^*(G, A_\theta, \alpha)$.

Appendix: More on directions for further work (continued)

Finally, here is an old problem on stable rank:

Problem

Let A be a simple C*-algebra with stable rank one, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group on A. Does it follow that $C^*(G, A, \alpha)$ has stable rank one?

Without simplicity, there is a counterexample (Blackadar).