Lecture 2: Crossed Products of AF Algebras by Actions with the Rokhlin Property

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- Lecture 1 (7 Dec. 2011): Crossed Products of C*-Algebras by Finite Groups.
- Lecture 2 (8 Dec. 2011): Crossed Products of AF Algebras by Actions with the Rokhlin Property.
- Lecture 3 (9 Dec. 2011): Crossed Products of Tracially AF Algebras by Actions with the Tracial Rokhlin Property.

Motivation for the Rokhlin property

Let X be the Cantor set, let G be a finite group, and let G act freely on X. (Freeness of the action means that every $g \in G \setminus \{1\}$ acts on X with no fixed points.)

Fix $x_0 \in X$. Then the points $g x_0$, for $g \in G$, are all distinct, so by continuity and total disconnectedness of the space, there is a compact open set $K \subset X$ such that $x_0 \in K$ and the sets gK, for $g \in G$, are all disjoint.

By repeating this process, one can find a compact open set $L \subset X$ such that the sets $L_g = gL$, for $g \in G$, are all disjoint, and such that their union is X.

The Rokhlin property

Definition

Let A be a unital C*-algebra, and let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say that α has the *Rokhlin property* if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\| \alpha_g (e_h) - e_{gh} \| < \varepsilon$ for all $g, h \in G$.
2. $\| e_g a - a e_g \| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

For C*-algebras, this goes back to about 1980, and is adapted from earlier work on von Neumann algebras (Ph.D. thesis of Vaughan Jones). The Rokhlin property for actions of $\mathbb{Z}$ goes back further.

The original use of the Rokhlin property was for understanding the structure of group actions. Application to the structure of crossed products is much more recent.

The Rokhlin property (continued)

The conditions in the definition:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

The projections e_g are the analogs of the characteristic functions of the compact open sets gL from the Cantor set example.

Condition (1) is an approximate version of $gL_h = L_{gh}$. (Recall that $L_g = gL$.)

Condition (3) is the requirement that X be the disjoint union of the sets L_g.

Condition (2) is vacuous for a commutative C*-algebra. In the noncommutative case, without it the inner action of $\mathbb{Z}_2$ on M_2 generated by $\mathrm{Ad}\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ would have the Rokhlin property. We don't want this, essentially because M_2 is simple but the crossed product isn't.

An example

Are there any actions with the Rokhlin property?
From Lecture 1, recall the product type action of $\mathbb{Z}_2$ generated by

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \quad \text{on} \quad A = \bigotimes_{n=1}^{\infty} M_2.$$

We show that this action has the Rokhlin property.

In fact, we will use an action conjugate to this one: we will use $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in place of $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Conjugacy (next slide) is isomorphism of dynamical systems.

Reasons for using $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ will appear in Lecture 3.

Conjugacy

Definition

Let G be a group, let A and B be C*-algebras, and let $\alpha\colon G \to \mathrm{Aut}(A)$ and $\beta\colon G \to \mathrm{Aut}(B)$ be actions of G on A and B. We say that α and β are *conjugate* if there exists an isomorphism $\varphi\colon A \to B$ such that $\varphi \circ \alpha_g \circ \varphi^{-1} = \beta_g$ for all $g \in G$.

Recall the condition for equivariance: $\varphi \circ \alpha_g = \beta_g \circ \varphi$. Thus, the condition on φ is that it be equivariant and bijective, so that it is an isomorphism of dynamical systems.

As a side note: We saw in Lecture 1 that an equivariant homomorphism $\varphi\colon A \to B$ induces a homomorphism

$$\overline{\varphi}\colon C^*(G, A, \alpha) \to C^*(G, B, \beta),$$

just by applying φ to the coefficients. When φ is an isomorphism, then so is $\overline{\varphi}$. So conjugate actions give isomorphic crossed products.

An example (continued)

We had

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v) \quad \text{with} \quad v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

One can check that there is a unitary $c \in M_2$ such that

$$cvc^* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Call the right hand side w. Then $\mathrm{Ad}(c) \circ \mathrm{Ad}(v) \circ \mathrm{Ad}(c)^{-1} = \mathrm{Ad}(w)$, that is, $\mathrm{Ad}(v)$ is conjugate to $\mathrm{Ad}(w)$.

Taking infinite tensor products, one also finds that $\varphi = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(c)$ defines a conjugacy from

$$\alpha = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v) \quad \text{to} \quad \beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w).$$

So we show that the action generated by β has the Rokhlin property.

An example (continued)

We had

$$w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Define projections $p_0, p_1 \in M_2$ by

$$p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad p_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$w p_0 w^* = p_1, \quad w p_1 w^* = p_0, \quad \text{and} \quad p_0 + p_1 = 1.$$

The action is generated by $\beta = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(w)$ on $A = \bigotimes_{n=1}^{\infty} M_2$. Also, $w p_0 w^* = p_1$, $w p_1 w^* = p_0$, and $p_0 + p_1 = 1$.

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\|\beta_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Since the union of the subalgebras $(M_2)^{\otimes n} = A_n$ is dense, we can assume $F \subset A_n$ for some n.

For $g = 0, 1 \in \mathbb{Z}_2$, take

$$e_g = 1_{A_n} \otimes p_g \in A_n \otimes M_2 = A_{n+1} \subset A.$$

Clearly $e_0 + e_1 = 1$. Check that $\beta(e_0) = e_1$ and $\beta(e_1) = e_0$, and that e_0 and e_1 actually commute with everything in F. (Proofs: See the next slide.) This proves the Rokhlin property.

An example (continued)

The projections e_0 and e_1 actually commute with everything in F, essentially because the nontrival parts are in different tensor factors.

Explicitly: Everything is in $A_{n+1} = M_{2^{n+1}}$, which we identify with $M_{2^n} \otimes M_2$. In this tensor factorization, elements of F have the form

$$a \otimes 1,$$

and

$$e_g = 1 \otimes p_g.$$

Clearly these commute.

For $\beta(e_0) = e_1$: we have $\beta|_{A_{n+1}} = \mathrm{Ad}\big(w^{\otimes n} \otimes w\big)$, so

$$\beta(e_0) = \big(w^{\otimes n} \otimes w\big)(1 \otimes p_0)\big(w^{\otimes n} \otimes w\big)^* = 1 \otimes w p_0 w^* = 1 \otimes p_1 = e_1.$$

The proof that $\beta(e_1) = e_0$ is the same.

Some other actions with the Rokhlin property

Let G be a finite group, and set $n = \mathrm{card}(G)$. Let $g \mapsto v_g$ be the left regular representation of G on $l^2(G)$, identify $L(l^2(G))$ with M_n, and let $A = \bigotimes_{n=1}^{\infty} M_n$ be the n^{∞} UHF algebra. Then the action

$$g \mapsto \alpha_g = \bigotimes_{n=1}^{\infty} \mathrm{Ad}(v_g)$$

of G on A has the Rokhlin property.

The example we just did is the case $G = \mathbb{Z}_2$, and the proof in the general case is the same.

Yet more actions with the Rokhlin property

Let G be a finite group, and set $n = \mathrm{card}(G)$.

Let $\mathcal{O}_n$ be the Cuntz algebra, but call its generators s_g for $g \in G$. The relations are thus

$$s_g^* s_g = 1$$

for all $g \in G$, and

$$\sum_{g \in G} s_g s_g^* = 1.$$

There is an action $\gamma \colon G \to \mathrm{Aut}(\mathcal{O}_n)$ such that

$$\gamma_g(s_h) = s_{gh}$$

for $g, h \in G$. This action turns out to have the Rokhlin property (Izumi).

The action of $\mathbb{Z}_2$ on $\mathcal{O}_2$ generated by $s_1 \mapsto s_1$ and $s_2 \mapsto -s_2$ is conjugate to the one gotten using $G = \mathbb{Z}_2$ above, so also has the Rokhlin property.

Exactly permuting the projections

Recall the conditions in the definition of the Rokhlin property. $F \subset A$ is finite, $\varepsilon > 0$, and we want projections e_g such that:

1. $\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

Theorem (2011)

Let $\alpha \colon G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Then α has the Rokhlin property if and only if for every finite set $F \subset A$ and every $\varepsilon > 0$, there are mutually orthogonal projections $e_g \in A$ for $g \in G$ such that:

1. $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$.
2. $\|e_g a - a e_g\| < \varepsilon$ for all $g \in G$ and all $a \in F$.
3. $\sum_{g \in G} e_g = 1$.

The difference is in (1).

Exactly permuting the projections (continued)

In the definition of the Rokhlin property, one can replace "$\|\alpha_g(e_h) - e_{gh}\| < \varepsilon$ for all $g, h \in G$" with "$\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$".

The proof uses methods unrelated to those here. This result simplifies some proofs by replacing some approximate equalities by equalities, so we will assume it, but it makes no real difference.

(This simplification has not been made in the crossed product notes—proving the theorem is more complicated than doing without it.)

Crossed products by actions with the Rokhlin property

A structure theorem for products by actions with the Rokhlin property:

Theorem

Let A be a unital AF algebra. Let G be a finite group, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. Then $C^*(G, A, \alpha)$ is AF.

Crossed products by actions of finite groups with the Rokhlin property preserve many other structural properties of C*-algebras. (See below.)

The basic idea: Let $e_g \in A$, for $g \in G$, be Rokhlin projections, with $\alpha_g(e_h) = e_{gh}$ for all $g, h \in G$. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g. Then $v_{g,h} = e_g u_{gh^{-1}}$ defines a system of matrix units in $C^*(G, A, \alpha)$. Using the homomorphism $M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ given by $v_{g,h} \otimes d \mapsto v_{g,1} d v_{1,h}$, one can approximate $C^*(G, A, \alpha)$ by matrix algebras over corners of A.

A future modification of the argument

We want to approximate elements of $C^*(G, A, \alpha)$ using unital homomorphisms from $M_n \otimes e_1 A e_1$ to $C^*(G, A, \alpha)$.

In Lecture 3, we are going to need the same argument again, but under slightly weaker conditions. We will still assume that the projections e_g are orthogonal, are exactly permuted by the group action, and can be chosen to approximately commute with a given finite subset of A. However, the sum $e = \sum_{g \in G} e_g$ will no longer necessarily be equal to 1.

We can nevertheless carry out the same argument; we get unital homomorphisms from $M_n \otimes e_1 A e_1$ to $e C^*(G, A, \alpha) e$, and we just get the weaker conclusion that we can approximate a finite set in $e C^*(G, A, \alpha) e$, rather than one in $C^*(G, A, \alpha)$, by a matrix algebra over a corner of A.

Crossed products by actions with the Rokhlin property (continued)

Recall the conclusion of the theorem: $C^*(G, A, \alpha)$ is AF.

To prove the theorem, we prove that for every finite set $S \subset C^*(G, A, \alpha)$ and every $\varepsilon > 0$, there is an AF subalgebra $D \subset C^*(G, A, \alpha)$ such that every element of S is within ε of an element of D. Let $u_g \in C^*(G, A, \alpha)$ be the canonical unitary implementing the automorphism α_g. Thus, a general element has the form $\sum_{g \in G} c_g u_g$, with $c_g \in A$ for $g \in G$. It suffices to consider a finite set of the form $S = F \cup \{u_g : g \in G\}$, where F is a finite subset of A. So let $F \subset A$ be a finite subset and let $\varepsilon > 0$.

Set

$$n = \mathrm{card}(G) \quad \text{and} \quad \delta = \frac{\varepsilon}{n(n-1)}.$$

Crossed products by actions with the Rokhlin property (continued)

We had: $S = F \cup \{u_g : g \in G\}$, with F a finite subset of A.

Apply the Rokhlin property to α with F as given and with δ in place of ε, obtaining projections $e_g \in A$ for $g \in G$ such that $\alpha_g(e_h) = e_{gh}$ for $g, h \in G$, $\|e_g a - a e_g\| < \delta$ for $g \in G$ and $a \in F$, and $\sum_{g \in G} e_g = 1$.

Define $v_{g,h} = e_g u_{gh^{-1}}$ for $g, h \in G$. In particular, $v_{g,g} = e_g$, so the $v_{g,g}$ are orthogonal projections which add up to 1.

We claim that the $v_{g,h}$ form a system of $n \times n$ matrix units in $C^*(G, A, \alpha)$. Recall for comparison: when proving that $C^*(G, C(G)) \cong M_n$, we used the matrix units $v_{g,h} = \chi_{\{g\}} u_{gh^{-1}}$. The computation here is exactly the same as there, so we don't repeat it.

Crossed products by actions with the Rokhlin property (continued)

We had: $(v_{g,h})_{g,h \in G}$ is an $n \times n$ system of matrix units in $C^*(G, A, \alpha)$.

Let $(w_{g,h})_{g,h \in G}$ be a system of matrix units for M_n. There is a unital homomorphism $\varphi_0 \colon M_n \to C^*(G, A, \alpha)$ such that $\varphi_0(w_{g,h}) = v_{g,h}$ for all $g, h \in G$. In particular, $\varphi_0(w_{g,g}) = e_g$ for all $g \in G$.

Now define a unital homomorphism $\varphi \colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ by $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$ for $g, h \in G$ and $d \in e_1 A e_1$.

Corners of AF algebras are AF, and φ is injective, so $D = \varphi(M_n \otimes e_1 A e_1)$ is an AF subalgebra of $C^*(G, A, \alpha)$. We complete the proof by showing that every element of S is within ε of an element of D. Recall that $S = F \cup \{u_g : g \in G\}$, and F is a finite subset of A.

Crossed products by actions with the Rokhlin property (continued)

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$ by elements of $D = \varphi(M_n \otimes e_1 A e_1)$.

We first consider u_g with $g \in G$. In fact, for u_g no approximation is necessary. Recall that $v_{g,h} = e_g u_{gh^{-1}}$. We have

$$\varphi\left(\sum_{h \in G} w_{h,g^{-1}h}\right) = \varphi_0\left(\sum_{h \in G} w_{h,g^{-1}h}\right) = \sum_{h \in G} v_{h,g^{-1}h} = \sum_{h \in G} e_h u_g = u_g.$$

Crossed products by actions with the Rokhlin property (continued)

We have to approximate elements of $S = F \cup \{u_g : g \in G\}$, with $F \subset A$ finite, by elements of $D = \varphi(M_n \otimes e_1 A e_1)$. Recall that $\varphi\colon M_n \otimes e_1 A e_1 \to C^*(G, A, \alpha)$ is defined by $\varphi(w_{g,h} \otimes d) = v_{g,1} d v_{1,h}$ for $g, h \in G$ and $d \in e_1 A e_1$. We already took care of u_g.

Let $a \in F$. The obvious first step in approximating a is to use

$$\sum_{g \in G} e_g a e_g.$$

In fact, one needs to (implicitly) use this approximation in the form

$$\sum_{g \in G} \alpha_g\big(e_1 \alpha_g^{-1}(a) e_1\big).$$

This happens because the definition of φ sends $w_{g,h} \otimes d$, for $d \in e_1 A e_1$, to an element obtained by using the action of the group elements g and h.

Set

$$b = \sum_{g \in G} w_{g,g} \otimes e_1 \alpha_g^{-1}(a) e_1 \in M_n \otimes e_1 A e_1.$$

For $g \neq h$, we have

$$\|e_g a e_h\| \leq \|e_g a - a e_g\| + \|a e_g e_h\| = \|e_g a - a e_g\| < \delta.$$

We then get

$$\left\| a - \sum\nolimits_{g \in G} e_g a e_g \right\| \leq \sum\nolimits_{g \neq h} \|e_g a e_h\| < n(n-1)\delta.$$

We use this, and the relations

$$v_{g,1} e_1 = e_g u_g e_1 = e_g u_g \quad \text{and} \quad e_1 \alpha_g^{-1}(a) e_1 = \alpha_g^{-1}(e_g a e_g),$$

to get

$$\begin{aligned} \|a - \varphi(b)\| &= \left\| a - \sum\nolimits_{g \in G} v_{g,1} e_1 \alpha_g^{-1}(a) e_1 e_{1,g} \right\| \\ &= \left\| a - \sum\nolimits_{g \in G} e_g u_g \alpha_g^{-1}(a) u_g^* e_g \right\| = \left\| a - \sum\nolimits_{g \in G} e_g a e_g \right\| \\ &< n(n-1)\delta = \varepsilon. \end{aligned}$$

This completes the proof of the theorem.

Other structural consequences of the Rokhlin property

Crossed products by actions of finite groups with the Rokhlin property preserve various other classes of C*-algebras. In many cases, the proofs are similar to what we did for AF algebras. Some examples of such classes:

1. Simple unital C*-algebras.
2. Various classes of unital but not necessarily simple countable direct limit C*-algebras using semiprojective building blocks. (With Osaka.)
3. Simple unital AH algebras with slow dimension growth and real rank zero. (With Osaka.)
4. D-absorbing separable unital C*-algebras for a strongly self-absorbing C*-algebra D. (Hirshberg-Winter.)
5. Separable nuclear unital C*-algebras whose quotients all satisfy the Universal Coefficient Theorem. (With Osaka.)
6. Separable unital approximately divisible C*-algebras. (Hirshberg-Winter.)
7. Unital C*-algebras with the ideal property and unital C*-algebras with the projection property. (With Pasnicu.)

Freeness and the Rokhlin property

A free action of a finite group on the Cantor set has the Rokhlin property.

Free actions on connected spaces don't, since there are no nontrivial projections.

We consider mostly simple C*-algebras with many projections. Recall that irrational rotation algebras, UHF algebras, and Cuntz algebras all have real rank zero.

(If there are not enough projections, finite group actions are much less well understood, although there has been some recent progress. In the nonsimple case, rather little is known.)

Tracial states and the Rokhlin property

Suppose A has a unique tracial state. (This is true for both UHF algebras and irrational rotation algebras.) Let G be finite, and let $\alpha \colon G \to \mathrm{Aut}(A)$ have the Rokhlin property. In the definition take $\varepsilon = 1$ and $F = \varnothing$. We get projections e_g such that, in particular:

- $\alpha_g(e_1) = e_g$ for all $g \in G$.
- $\sum_{g \in G} e_g = 1$.

Since τ is unique, we have $\tau \circ \alpha_g = \tau$ for all $g \in G$. So $\tau(e_g) = \tau(e_1)$. It follows that

$$\tau(e_1) = \frac{1}{\mathrm{card}(G)}.$$

The Rokhlin property and A_θ

Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and recall that A_θ is generated by unitaries u and v satisfying $vu = e^{2\pi i \theta} uv$. Further recall the action $\alpha \colon \mathbb{Z}_n \to \mathrm{Aut}(A_\theta)$ generated by

$$u \mapsto e^{2\pi i/n} u \quad \text{and} \quad v \mapsto v.$$

This is a noncommutative version of the free action of $\mathbb{Z}_n$ on $S^1 \times S^1$ given by rotation by $e^{2\pi i/n}$ in the first coordinate. Moreover, A_θ has many projections, like $C(X)$ when X is the Cantor set. So one would hope that α has the Rokhlin property.

The Rokhlin property and A_θ (continued)

In fact, *no* action of *any* nontrivial finite group on A_θ has the Rokhlin property! The reason is that there is no projection $e \in A_\theta$ with $\tau(e) = \frac{1}{n}$, for any $n \geq 2$. (Recall that the tracial state τ defines an isomorphism $\tau_* \colon K_0(A_\theta) \to \mathbb{Z} + \theta\mathbb{Z}$.)

For similar reasons, no action of $\mathbb{Z}_2$ on $D = \bigotimes_{n=1}^{\infty} M_3$ has the Rokhlin property. (The tracial state τ defines an isomorphism $\tau_* \colon K_0(D) \to \mathbb{Z}[\frac{1}{3}]$.)

There are more subtle obstructions to the Rokhlin property. For example, $M_2 \otimes \bigotimes_{n=1}^{\infty} M_3$ does have projections with trace $\frac{1}{2}$, but there are still no actions of $\mathbb{Z}_2$ with the Rokhlin property. (One at least needs a copy of M_{2^n}) for every n.)

More examples with no Rokhlin actions

Consider the Cuntz algebras $\mathcal{O}_n$, for $n \in \{2, 3, \ldots, \infty\}$. The group $K_0(\mathcal{O}_n)$ is generated by $[1]$. Therefore every automorphism of $\mathcal{O}_n$ is the identity on $K_0(\mathcal{O}_n)$.

It follows that if $\mathrm{card}(G) = n$ and $\alpha \colon G \to \mathrm{Aut}(\mathcal{O}_n)$ has the Rokhlin property, then the class of a Rokhlin projection e_1 satisfies $n[e_1] = [1]$.

This can never happen in $\mathcal{O}_\infty$, so there is no action of any nontrivial finite group on $\mathcal{O}_\infty$ with the Rokhlin property.

For $\mathrm{card}(G) = n$, a Rokhlin action of G on $\mathcal{O}_n$ was described earlier (without proof). However, for example, there is no Rokhlin action of $\mathbb{Z}_2$ on $\mathcal{O}_3$, because $[1] \in K_0(\mathcal{O}_3)$ generates $K_0(\mathcal{O}_3) \cong \mathbb{Z}_2$, and so can't be written in the form $2[e]$.

There are stronger cohomological obstructions to the Rokhlin property (Izumi). In fact, the Rokhlin property is very rare.

In the next lecture, we will discuss the tracial Rokhlin property. There are many actions which have this property.