Math 650, Winter 2003, Homework # 3
solve at least 5 out of 12 problems

1. Let $h(x)$ be the Haar wavelet, $h(x) = \mathbf{1}_{(0,1/2)} - \mathbf{1}_{(1/2,1)}$. Let X be the closed linear span of $\{h_{j,k}(x) = 2^{j/2}h(2^j x - k)\}_{j,k\in\mathbb{Z}}$ in $L^1(\mathbb{R})$. Show that $\mathbf{1}_{[0,1]} \notin X$ and $h(x+k+1/2) \notin X$ for all $k \in \mathbb{Z}$.

2. Suppose that f is a Hölder continuous function with exponent $0 < \alpha \le 1$, i.e., there exists C such that for all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \le C|x-y|^\alpha.$$

Let $h(x)$ be the Haar wavelet. Show that for all $j, k \in \mathbb{Z}$

$$|\langle f, h_{j,k}\rangle| \le C2^{-j(\alpha+1/2)}.$$

3. Suppose φ is an orthonormal scaling function associated with an MRA $(V_j)_{j\in\mathbb{Z}}$ and φ is an even function. Show that there is an even orthonormal wavelet associated with this MRA. Show also that there is an odd orthonormal wavelet associated with this MRA.

4. Define

$$V_0 = \{f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R}) : \forall k \in \mathbb{Z}\ f'(k) = 0 \text{ and } f|_{[k,k+1]} \text{ is a polynomial with } \deg \le 3\}.$$

Show that there is an MRA generated by this V_0.

5. Given $n \ge 1$, let

$$\mathcal{S}^n(\mathbb{Z}) = \{f \in L^2(\mathbb{R}) \cap C^{n-1}(\mathbb{R}) : f|_{[k,k+1]} \text{ is a polynomial with } \deg \le n\}.$$

Show that there is no compactly supported $g \in \mathcal{S}^n(\mathbb{Z})$ such that $\{g(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal sequence.

6. Let $\phi \in \mathcal{S}^n(\mathbb{Z})$ be such that $||\phi||_2 = 1$, $\phi|_{(1,\infty)} = 0$ and $\phi \perp f$ for all $f \in \mathcal{S}^n(\mathbb{Z})$ with $f|_{(0,\infty)} = 0$. Show that such ϕ exists and is unique up to a unimodular constant. Show that $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $\mathcal{S}^n(\mathbb{Z})$ and $\phi(x)$ has exponential decay.

7. Let $\phi(x)$ be a compactly supported continuous function on $\mathbb{R}$ with $\hat{\phi}(0) \neq 0$. Show that the following are equivalent:

(i) for every polynomial $p(x)$ of degree $\le m$ there exists a sequence $(a_k)_{k\in\mathbb{Z}}$ such that $p(x) = \sum_{k\in\mathbb{Z}} a_k\phi(x-k)$ for all $x \in \mathbb{R}$,

(ii) $\frac{d^j}{d\xi^j}\hat{\phi}(k) = 0$ for all $j = 0, \dots, m$ and all $k \in \mathbb{Z}\setminus\{0\}$.

8. Let $\psi \in L^2(\mathbb{R})$ be an MSF wavelet, i.e., $|\hat{\psi}(\xi)| = \mathbf{1}_K(\xi)$ for some measurable $K \subset \mathbb{R}$. Show that ψ is associated with an MRA if and only if $\{k + \bigcup_{j\le -1} 2^j K\}_{k\in\mathbb{Z}}$ partitions $\mathbb{R}$ modulo null sets.

9. Let $\phi \in L^2(\mathbb{R})$. Show that the system $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is a frame with constants A and B (for its closed span) if and only if

$$A\mathbf{1}_K(\xi) \le \sum_{k\in\mathbb{Z}} |\hat{\phi}(\xi+2\pi k)|^2 \le B\mathbf{1}_K(\xi) \qquad \text{for a.e. } \xi \in [-\pi,\pi],$$

where $K \subset [-\pi,\pi]$ is some measurable set of non-zero measure.

10. Suppose m is 2π-periodic C^1 function on $\mathbb{R}$ such that $m(0)=1$ and

$$|m(\xi)|^2 + |m(\xi+\pi)|^2 = 1 \qquad \text{for all } \xi \in \mathbb{R}.$$

Let $\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m(\xi/2^j)$. Show that $\psi \in L^2(\mathbb{R})$ given by

$$\hat{\psi}(\xi) = e^{i\xi/2}\overline{m(\xi/2+\pi)}\hat{\varphi}(\xi/2)$$

is a tight frame wavelet with constant 1.

11. Under assumptions of Problem 10, we say that m satisfies Cohen's condition if there exists a compact set $K \subset \mathbb{R}$ containing a neighborhood of 0 such that $\{K+2\pi k\}_{k\in\mathbb{Z}}$ partitions $\mathbb{R}$ (modulo null sets) and

$$\inf_{j\ge 1}\inf_{\xi\in K} |m(2^{-j}\xi)| > 0.$$

Show that the Cohen condition implies that ψ as above is an orthonormal wavelet.

12. Show the converse to Problem 11 under assumptions of Problem 10. That is, if ψ is an orthonormal wavelet then m necessarily satisfies Cohen's condition.

A collection of easily stated open problems in wavelet theory

1. Suppose ψ is an orthonormal wavelet such that ψ belongs to the Schwartz class. Is $\hat{\psi}(\xi)$ necessarily compactly supported ?

2. Does there exist a Riesz wavelet ψ for

$$H^2(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \hat{f}(\xi) = 0 \quad \text{for } \xi \le 0\}$$

such that ψ belongs to the Schwartz class ?

3. Is it true that for any orthonormal wavelet $\psi \in L^2(\mathbb{R})$, there exists an MSF wavelet $\tilde{\psi}$ such that such that $\operatorname{supp}\hat{\tilde{\psi}} \subset \operatorname{supp}\hat{\psi}$?

4. Is the collection of all orthonormal wavelets in $L^2(\mathbb{R})$ connected in the topology of $L^2(\mathbb{R})$?

5. Is the collection of all Riesz wavelets dense in $L^2(\mathbb{R})$?

6. For what values of $\pi/4 < b \le \pi/3$, is ψ_b a frame wavelet, where $\hat{\psi}_b = \mathbf{1}_{(-2\pi,-b)\cup(b,2\pi)}$?

References

[HW] E. Hernández, G. Weiss, "A first course on wavelets".
[W] P. Wojtaszczyk, "A mathematical introduction to wavelets".