**Math 650, Winter 2003, Homework #2**
**Problems on Fourier transforms and distributions**

Directions: solve at least 8 out of 16 problems.

1. Let $1 \le p \le 2$ and $1/p + 1/q = 1$. Show that if $(a_k)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$ then there exists $f \in L^q(\mathbb{T})$ such that $a_k = \hat{f}(k)$. What is the estimate between $||f||_q$ and $||(a_k)||_p$. Hint: apply Riesz-Thorin Theorem.

2. Show that $F(z) = \sum_{n=1}^{\infty} 2^{-n}[(z+n)^2 + n^{-1}]^{-1}$ is analytic on $\mathbb{R}$ and $F|_{\mathbb{R}} \in L^1 \cap L^\infty(\mathbb{R})$, but $F$ is not holomorphic in a strip $\{z \in \mathbb{C} : |\text{Im } z| < a\}$ for any $a > 0$.

3. Show that if $f \in L^1$, $g \in L^p$, $1 \le p \le 2$, then $h = f * g \in L^p$ and $\hat{h}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. Hint: Minkowski's inequality and $L^p \subset L^1 + L^2$, $1 \le p \le 2$.

4. Suppose $\phi \in L^1(\mathbb{R}^n)$ is such that $\int \phi(x)dx = 1$, $|\phi(x)| \le C(1+|x|)^{-n-\delta}$, $|\hat{\phi}(\xi)| \le C(1+|\xi|)^{-n-\delta}$ for some $\delta > 0$. Show that for any $f \in L^p(\mathbb{T}^n)$, $\mathbb{T}^n = (-\pi, \pi]^n$, $1 \le p < \infty$,

$$\frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n} \hat{\phi}(k/\lambda)\hat{f}(k)e^{i\langle x,k\rangle} \to f(x) \qquad \text{as } \lambda \to \infty,$$

where the convergence is in $L^p(\mathbb{T}^n)$. Hint: the Poisson summation formula.

5. Under the assumptions of Problem 4, show that if $f : \mathbb{R}^n \to \mathbb{C}$ is continuous and $2\pi\mathbb{Z}^n$-periodic, then the convergence in Problem 4 is uniform.

6. Suppose $A$ is $n \times n$ non-singular real matrix and $b \in \mathbb{R}^n$. Show that for $f \in L^1(\mathbb{R}^n)$

$$\widehat{f(Ax-b)}(\xi) = |\det A|^{-1}\hat{f}((A^{-1})^*\xi)e^{-i\langle\xi, A^{-1}b\rangle}.$$

7. Suppose $\mu, \nu$ are finite Borel measure on $\mathbb{R}^n$ show that $\mu * \nu$ given by

$$\mu * \nu(A) = \int_{\mathbb{R}^n} \mu(A - x)d\nu(x), \qquad \text{for Borel } A \subset \mathbb{R}^n,$$

is also a finite Borel measure. Show also that

$$\widehat{(\mu * \nu)}(\xi) = \hat{\mu}(\xi)\hat{\nu}(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n.$$

8. Show that for any $p > 2$ there exists $f \in L^p$ such that $\hat{f}$ is **not** a regular tempered distribution, i.e., it is not given as an integration against a locally integrable function. Hint: consider $f(x) = (1 + i\delta)^{-1/2}e^{-x^2/(1+i\delta)}$ as $\delta \to \infty$ and use the Closed Graph Theorem.

9. Show that the Fourier transform does not map $L^1(\mathbb{R})$ **onto**

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : f(x) \to 0 \text{ as } |x| \to \infty\}.$$

10. Show that the Hermite functions $h_k(x) = e^{x^2/2} \frac{d^k}{dx^k} e^{-x^2}$, $k = 0, 1, \ldots$, are the eigenvectors of the Fourier transform. Hint: show that $\hat{h}_0 = \sqrt{2\pi} h_0$, $\{h_k\}$ satisfy

$$h_k' - x h_k = h_{k+1} \qquad k = 0, 1, \ldots,$$

and $\{(-i)^k h_k\}$ obey the same recursion formula.

11. Show that $\{h_k / \|h_k\|_2\}_{k=0,1,\ldots}$ forms an orthonormal basis of $L^2(\mathbb{R})$. Hint: consider the differential operator $f \mapsto f'' - x^2 f$ and show that

$$h_k' + x h_k = -2k h_{k-1} \qquad k = 0, 1, \ldots$$

12. For $f \in \mathcal{S}(\mathbb{R})$ define define its Hilbert transform $Hf(x)$ by

$$Hf(x) = \frac{1}{\pi} \text{p. v.} \int_{-\infty}^{\infty} \frac{f(x-u)}{u} du = \frac{1}{\pi} \lim_{\epsilon \to 0+} \int_{|u| \geq \epsilon} \frac{f(x-u)}{u} du.$$

Show that $\widehat{Hf}(\xi) = -i \frac{\xi}{|\xi|} \hat{f}(\xi)$.

13. Use Problem 12 to show that $H$ extends to a unitary operator on $L^2(\mathbb{R})$.

14. Show that

$$\mathcal{D}(\mathbb{R}^n) = \{\varphi \in C^\infty(\mathbb{R}^n) : \operatorname{supp} \varphi \text{ is compact}\}$$

is a dense subspace of $\mathcal{S}(\mathbb{R}^n)$.

15. Show that the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ as a metrizable vector space is separable. Hint: show that linear combinations of $e^{-|x-a|^2/q}$, $a \in \mathbb{Q}^n$, $0 < q \in \mathbb{Q}$, with rational coefficients are dense.

16. Suppose that $\{f_i\}_{i \in \mathbb{N}}$ is a sequence of distributions in $\mathcal{S}'(\mathbb{R}^n)$ and $d \geq 0$ is an integer. Assume that for every multi-index $\gamma$ with $|\gamma| = d + 1$ the sequence of partial derivatives $\{\partial^\gamma f_i\}$ converges in $\mathcal{S}'$ as $i \to \infty$. Show that there exists a sequence of polynomials $\{P_i\}_{i \in \mathbb{N}}$ with $\deg P_i \leq d$ such that $\{f_i + P_i\}$ converges to some distribution $f \in \mathcal{S}'$ as $i \to \infty$.

## References


H. Dym, H. McKean, "Fourier series and integrals".

W. Rudin, "Functional analysis".

E. Stein, G. Weiss, "Introduction to Fourier Analysis on Euclidean Spaces".