**Math 650, Winter 2003, Homework #1**
**Problems on Fourier series and frames**

Directions: solve at least 10 out of 17 problems.

**Problem 1.** (Vitali) Show that an orthonormal system $\{e_i\}_{i\in I}$ in $L^2[a,b]$ is complete $\iff$

$$\sum_{i\in I}\left|\int_a^x e_i(t)dt\right|^2 = x-a \qquad \text{for a.e. } x\in[a,b].$$

**Problem 2.** (Dalzell) Show that an orthonormal system $\{e_i\}_{i\in I}$ in $L^2[a,b]$ is complete $\iff$

$$\sum_{i\in I}\int_a^b\left|\int_a^x e_i(t)dt\right|^2 = \frac{(b-a)^2}{2}.$$

**Problem 3.** Show that a frame that ceases to be a frame when any of its elements is removed is a Riesz basis.

**Problem 4.** Let $\{e_k\}_{k\in\mathbb{Z}}$ be the standard orthonormal basis of $\ell^2(\mathbb{Z})$. Determine the values of $c\in\mathbb{C}$ for which $\{f_k\}_{k\in\mathbb{Z}}$, $f_k = e_k + ce_{k+1}$, is a frame for $\ell^2(\mathbb{Z})$.

**Problem 5.** Prove that for any $(b_n)_{n\in\mathbb{Z}}$ such that $b_n\to 0$ as $|n|\to\infty$ there exists $(a_n)_{n\in\mathbb{Z}}$ such that $a_n\to 0$ as $|n|\to\infty$, $a_n = a_{-n}\geq 0$, $a_{n+1}+a_{n-1}-2a_n\geq 0$ for $n>0$, and $b_n/a_n\to 0$ as $n\to\infty$.

**Problem 6.** Show that $f(t)=\sum_{n=2}^\infty \frac{\sin(nt)}{\log(n)}$ converges for all $t\in\mathbb{T}$. Hint: use Abel summation method.

**Problem 7.** Give a rigorous proof that $f(t)=\sum_{n=2}^\infty \frac{\sin(nt)}{\log(n)}$ defined in Problem 5 does not belong to $L^1(\mathbb{T})$.
Remark: This problem is more subtle than it may seem. From the lecture, we know that $\sum_{n=2}^\infty \frac{\sin(nt)}{\log(n)}$ is not a Fourier series of a function in $L^1(\mathbb{T})$. But, does this imply that $f(t)\notin L^1(\mathbb{T})$? In fact, it turns out that $f(t)\sim (t\ln(1/|t|))^{-1}$ as $t\to 0$.

**Problem 8.** Suppose that $f\in L^1(\mathbb{T})$ and $\hat{f}(n)=O(|n|^{-k})$ as $|n|\to\infty$. Show that $f$ is $m$-differentiable with $f^{(m)}\in L^2(\mathbb{T})$ provided $k-m>1/2$.

**Problem 9.** Show that the Lebesgue constants, given by $L_n=\frac{1}{2\pi}\int_0^{2\pi}|D_n(t)|dt$, where $D_n(t)$ is the $n$th Dirichlet kernel, satisfy $L_n = 4/\pi^2\ln n + O(1)$ as $n\to\infty$. Hint:

$$2\pi L_n = 2\int_0^\pi\left|\frac{\sin(n+1/2)t}{\sin(t/2)}\right|dt = 4\sum_{j=1}^{n-1}\int_{\frac{j\pi}{n+1/2}}^{\frac{(j+1)\pi}{n+1/2}}\frac{|\sin(n+1/2)t|}{t}dt + O(1).$$

**Problem 10.** Suppose that $(a_n)_{n\in\mathbb{Z}}$ satisfies $a_n \to 0$ as $|n| \to \infty$, $a_n = a_{-n} \geq 0$, $a_{n+1} + a_{n-1} - 2a_n \geq 0$ for $n > 0$. Prove that the partial sums $S_n(f)$ of the Fourier series of $f(t) \sim \sum_{n\in\mathbb{Z}} a_n e^{int}$ are bounded in $L^1(\mathbb{T})$ if and only if $a_n \ln n = O(1)$ as $n \to \infty$.

**Problem 11.** Suppose that $(a_n)_{n\in\mathbb{Z}}$ is the same as in Problem 9. Prove that the partial sums $S_n(f)$ of the Fourier series of $f(t) \sim \sum_{n\in\mathbb{Z}} a_n e^{int}$ converge in $L^1(\mathbb{T})$ if and only if $\lim_{n\to\infty} a_n \ln n = 0$.

**Problem 12.** Suppose $f \in L^1(\mathbb{T})$ is Lipschitz at $t_0 \in \mathbb{T}$, i.e., $\exists C > 0$ s.t. $|f(t) - f(t_0)| \leq C|t - t_0|$ for all $t \in \mathbb{T}$. Show that the partial Fourier sums $S_n(f, t_0) \to f(t_0)$ as $n \to \infty$. Hint: for $t_0 = 0$, consider $g(t) = (f(2t) - f(0))/\sin t$.

**Problem 13.** Suppose $f \in L^1(\mathbb{T})$ satisfies Dini condition at $t_0 \in \mathbb{T}$, i.e.,

$$\int_{-1}^{1} \left| \frac{f(t + t_0) - f(t_0)}{t} \right| dt < \infty.$$

Show that the partial Fourier sums $S_n(f, t_0) \to f(t_0)$ as $n \to \infty$.

**Problem 14.** Show that for any countable sequence $\{t_i\}_{i=1}^{\infty} \subset \mathbb{T}$, there exists a continuous function $f(t)$ whose Fourier series diverges at each $t_i$.

**Problem 15.** Show that each of the systems $\{1/\sqrt{\pi}\} \cup \{\sqrt{2/\pi}\cos(nt)\}_{n=1}^{\infty}$ and $\{\sqrt{2/\pi}\sin(nt)\}_{n=1}^{\infty}$ is an orthonormal basis for $L^2(0, \pi)$.

**Problem 16.** Show that each of the systems $\{\sqrt{2/\pi}\cos((n + 1/2)t)\}_{n=0}^{\infty}$ and $\{\sqrt{2/\pi}\sin((n + 1/2)t)\}_{n=0}^{\infty}$ is an orthonormal basis for $L^2(0, \pi)$.

**Problem 17.** Suppose the sequence $\{\lambda_n\}_{n=1}^{\infty}$ is lacunary and $f(t) \sim \sum a_n \cos(\lambda_n t)$ satisfies $\mathrm{Lip}_\alpha$ condition with $0 < \alpha < 1$ at $t_0 \in \mathbb{T}$, i.e., $\exists C > 0$ s.t. $|f(t) - f(t_0)| \leq C|t - t_0|^\alpha$ for all $t \in \mathbb{T}$. Show that $a_n = O(\lambda_n^{-\alpha})$ as $n \to \infty$. Next, deduce that $f$ satisfies $\mathrm{Lip}_\alpha$ for every $t_0 \in \mathbb{T}$.

## References

Y. Katznelson "An introduction to harmonic analysis"
R. Young "An introduction to nonharmonic Fourier series"
A. Zygmund "Trigonometric series"