Math 650, Winter 2001, Homework # 2
solve at least 7 out of 12 problems

1. Let $1 \le p \le 2$ and $1/p + 1/q = 1$. Show that if $(a_n)_{n\in\mathbb{Z}} \in l^p(\mathbb{Z})$ then there exists $f \in L^q(\mathbb{T})$ such that $a_n = \hat{f}(n)$. What is the estimate between $||f||_q$ and $||(a_n)||_p$. Hint: the Riesz-Thorin Theorem.

2. Show that $F(z) = \sum_{n=1}^{\infty} 2^{-n}[(z+n)^2 + n^{-1}]^{-1}$ is analytic on $\mathbb{R}$ and $F|_{\mathbb{R}} \in L^1 \cap L^\infty(\mathbb{R})$, but F is not holomorphic in any strip $\{z \in \mathbb{C} : |\Im z| < a\}$, $a > 0$.

3. Show that if $f \in L^1$, $g \in L^p$, $1 \le p \le 2$, then $h = f * g \in L^p$ and $\hat{h}(\xi) = \hat{f}(\xi)\hat{g}(\xi)$. Hint: Minkowski's inequality and $L^p \subset L^1 + L^2$, $1 \le p \le 2$.

4. Suppose $\phi \in L^1(\mathbb{R}^n)$ is such that $\int \phi(x)dx = 1$, $\phi(x) = O((1+|x|)^{-n-\delta})$, $\hat{\phi}(\xi) = O((1+|\xi|)^{-n-\delta})$ for some $\delta > 0$. Show that

$$\frac{1}{(2\pi)^n} \sum_{k\in\mathbb{Z}^n} \hat{\phi}(k/\lambda)\hat{f}(k)e^{i\langle x,k\rangle} \to f(x) \qquad \text{as } \lambda \to \infty,$$

where the convergence is in L^p, $1 \le p < \infty$ for any $f \in L^p(\mathbb{T}^n)$. If $f \in C(\mathbb{T}^n)$ then the convergence is uniform. Hint: the Poisson summation formula.

5. Suppose A is $n \times n$ non-singular real matrix and $b \in \mathbb{R}^n$. Show that for $f \in L^1(\mathbb{R}^n)$

$$f(\widehat{Ax - b})(\xi) = |\det A|^{-1}\hat{f}((A^{-1})^*\xi)e^{-i\langle\xi, A^{-1}b\rangle}.$$

6. Suppose μ, ν are finite Borel measure on $\mathbb{R}^n$ show that $\mu * \nu$ given by

$$\mu * \nu(A) = \int_{\mathbb{R}^n} \mu(A - x)d\nu(x), \qquad \text{for Borel } A \subset \mathbb{R}^n,$$

is also a finite Borel measure. Show that $\widehat{(\mu * \nu)}(\xi) = \hat{\mu}(\xi)\hat{\nu}(\xi)$ for all $\xi \in \mathbb{R}^n$.

7. Show that for any $p > 2$ there exists $f \in L^p$ such that $\hat{f}$ is **not** a regular distribution, i.e., it is not given as an integration against a locally integrable function. Hint: consider $f(x) = (1+i\delta)^{-1/2}e^{-x^2/(1+i\delta)}$ and use the Closed Graph Theorem.

8. Show that the Fourier transform does not map $L^1(\mathbb{R})$ **onto** $C_0(\mathbb{R})$.

9. Show that the Hermite functions $h_k(x) = e^{x^2/2}\frac{d^k}{dx^k}e^{-x^2}$, $k = 0, 1, \ldots$, are the eigenvectors of the Fourier transform. Hint: Using

$$h_k' - xh_k = h_{k+1} \qquad k = 0, 1, \ldots$$

show that $\hat{h}_k$ and $(-i)^k h_k$ obey the same recursion formula and $\hat{h}_0 = \sqrt{2\pi}h_0$.

10. Show that $\{h_k/||h_k||_2\}_{k=0,1,\ldots}$ forms an orthonormal basis of $L^2(\mathbb{R})$. Hint: consider a differential operator $f \mapsto f'' - x^2 f$ and use

$$h_k' + xh_k = -2kh_{k-1} \qquad k = 0, 1, \ldots$$

11. For $f \in \mathcal{S}(\mathbb{R})$ define define its Hilbert transform $Hf(x)$ by

$$Hf(x) = \frac{1}{\pi} \,\text{p. v.} \int_{-\infty}^{\infty} \frac{f(x-u)}{u} du = \frac{1}{\pi} \lim_{\epsilon \to 0^+} \int_{|u| \geq \epsilon} \frac{f(x-u)}{u} du.$$

Show that $\widehat{Hf}(\xi) = -i \frac{\xi}{|\xi|} \hat{f}(\xi)$. Moreover, show that H extends to a unitary operator on $L^2(\mathbb{R})$.

12. Show that the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ is separable. Hint: consider linear combinations of $e^{-|x-a|^2/q}$, $a \in \mathbb{Q}^n$, $0 < q \in \mathbb{Q}$, with rational coefficients.