Math 650, Winter 2001, Homework # 1
solve at least 7 out of 12 problems

1. (Vitali) Show that an orthonormal system $\{e_i\}_{i\in I}$ in $L^2[a,b]$ is complete $\iff$

$$\sum_{i\in I}\left|\int_a^x e_i(t)dt\right|^2 = x-a \qquad \text{for a.e. } x\in[a,b].$$

2. (Dalzell) Show that an orthonormal system $\{e_i\}_{i\in I}$ in $L^2[a,b]$ is complete $\iff$

$$\sum_{i\in I}\int_a^b\left|\int_a^x e_i(t)dt\right|^2 = \frac{(b-a)^2}{2}.$$

3. Show that a frame that ceases to be a frame when any of its elements is removed is a Riesz basis.

4. Prove that for any $(b_n)_{n\in\mathbb{Z}}$ such that $b_n\to 0$ as $|n|\to\infty$ there exists $(a_n)_{n\in\mathbb{Z}}$ such that $a_n\to 0$ as $|n|\to\infty$, $a_n=a_{-n}\ge 0$, $a_{n+1}+a_{n-1}-2a_n\ge 0$ for $n>0$, and $b_n/a_n\to 0$ as $n\to\infty$.

5. Show that $f(t)=\sum_{n=2}^{\infty}\frac{\sin(nt)}{\log(n)}$ converges for all $t\in\mathbb{R}$. Hint: Abel summation method.

6. Show that f defined in 5. does not belong to $L^1(\mathbb{T})$. (This problem is harder than it seems; it turns out that $f(t)\sim(t\ln(1/|t|))^{-1}$ as $t\to 0$).

7. Suppose that $f\in L^1(\mathbb{T})$ and $\hat{f}(n)=O(|n|^{-k})$ as $|n|\to\infty$. Show that f is m-differentiable with $f^{(m)}\in L^2(\mathbb{T})$ provided $k-m>1/2$.

8. Show that the Lebesgue constants satisfy $L_n=4/\pi^2\ln n+O(1)$ as $n\to\infty$. Hint:

$$2\pi L_n = 2\int_0^{\pi}\left|\frac{\sin(n+1/2)t}{\sin(t/2)}\right|dt = 4\sum_{j=1}^{n-1}\int_{\frac{j\pi}{n+1/2}}^{\frac{(j+1)\pi}{n+1/2)}}\frac{|\sin(n+1/2)t|}{t}dt+O(1).$$

9. Suppose that $(a_n)_{n\in\mathbb{Z}}$ satisfies $a_n\to 0$ as $|n|\to\infty$, $a_n=a_{-n}\ge 0$, $a_{n+1}+a_{n-1}-2a_n\ge 0$ for $n>0$. Prove that the partial sums $S_N(f)$ of the Fourier series $f(t)=\sum_{n\in\mathbb{Z}}a_ne^{int}$ are bounded in $L^1(\mathbb{T})$ if and only if $a_n\ln n=O(1)$ and the series converges in $L^1(\mathbb{T})$ if and only if $\lim_{n\to\infty}a_n\ln n=0$.

10. Suppose $f\in L^1(\mathbb{T})$ is Lipschitz at $t=t_0$. Show that the partial Fourier sums $S_n(f,t_0)$ converge to $f(t_0)$ as $n\to\infty$. Hint: if $t_0=0$ then consider $g(t)=(f(2t)-f(0))/\sin t$.

11. Show that each of the systems $\{\sqrt{2/\pi}\cos(n+1/2)x\}_{n=0,1,\dots}$ and $\{\sqrt{2/\pi}\sin(n+1/2)x\}_{n=0,1,\dots}$ is an orthonormal basis for $L^2(0,\pi)$.

12. Let $\{\lambda_n\}$ be lacunary and let $f\sim\sum a_n\cos(\lambda_n t)$ satisfies a Lip_α condition with $0<\alpha<1$ at $t=t_0$. Show that $a_n=O(\lambda_n^{-\alpha})$ as $n\to\infty$. Deduce that $f\in\mathrm{Lip}_\alpha(\mathbb{T})$.

If you get stuck in any of the problems 4–12 you can always look up Zygmund's book "Trigonometric series".