Math 457/557, Discrete Dynamical Systems
Prof. Bownik, Practice Midterm Exam

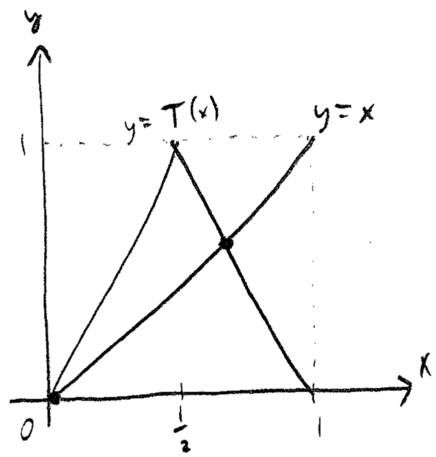
Your name: SOLUTIONS 5/6/2009

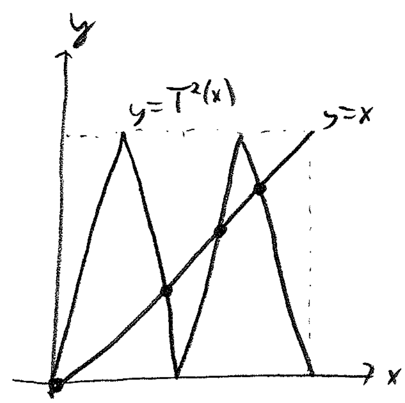
Instructions: Show all your work. Partial credit is available for many problems, but it can only be given if you try to explain your reasoning. There is no need to simplify your answers unless you are told so. Good luck!

1. (20 pts) Define the tent map T on $[0,1]$ by

$$T(x) = \begin{cases} 2x & 0 \le x < 1/2, \\ 2 - 2x & 1/2 \le x \le 1. \end{cases}$$

Sketch the graphs of T and T^2. Find all fixed points of T and T^2.





$T(x) = x$

$2x = x \Rightarrow \boxed{x = 0}$

$2 - 2x = x \Rightarrow \boxed{x = \frac{2}{3}}$

$$T^2(x) = \begin{cases} 4x & 0 \le x < \frac{1}{4} \\ 2 - 4x & \frac{1}{4} \le x < \frac{1}{2} \\ -2 + 4x & \frac{1}{2} \le x < \frac{3}{4} \\ 4 - 4x & \frac{3}{4} \le x \le 1 \end{cases}$$

$4x = x \Rightarrow x = 0$

$2 - 4x = x \Rightarrow x = \frac{2}{5}$

$4 - 4x = x \Rightarrow x = \frac{4}{5}$

$-2 + 4x = x \Rightarrow x = \frac{2}{3}$

2. (20 pts.) Find all fixed points of tent map T and classify them as attracting, repelling, or neutral.

$x = 0$

$x = \frac{2}{3}$ See   Problem 1

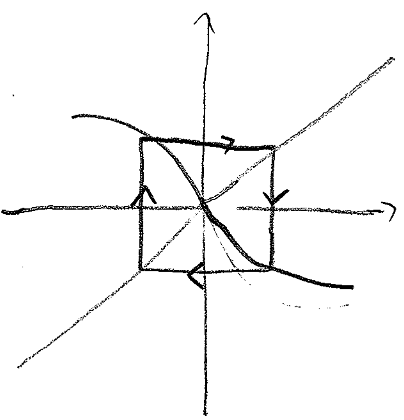
$T'(0) = 2$

$T'(\frac{2}{3}) = -2$ $\Rightarrow$ $x = 0$, $x = \frac{2}{3}$ repelling fixed points

3. (20 pts.) The function

$$F_\lambda(x) = \lambda \sin(x)$$

undergoes a bifurcation of fixed points at $\lambda = -1$. Use algebraic or graphical methods to identify type of this bifurcation. Sketch 3 phase portraits for typical parameter values below, at, and above the bifurcation value.

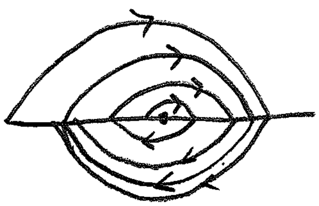


$\lambda < -1$

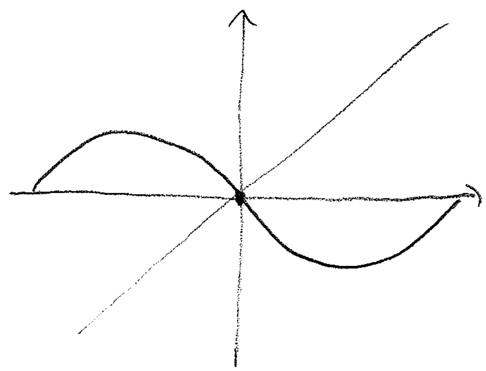
x = 0 repelling fixed point

& attracting

2 cycle



x = 0 repelling

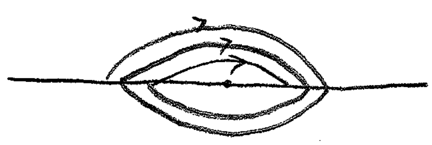


$\lambda = -1$

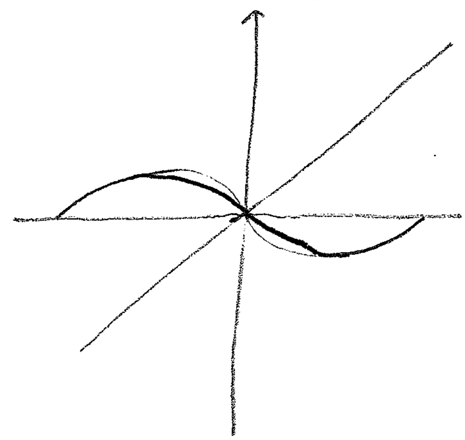
x = 0 neutral

fixed point

since $F_\lambda'(0) = -1$



x = 0 weakly attracting



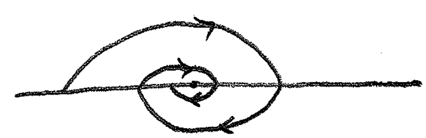
$\lambda > -1$

x = 0 attracting

fixed point

$|F_\lambda'(0)| < 1$

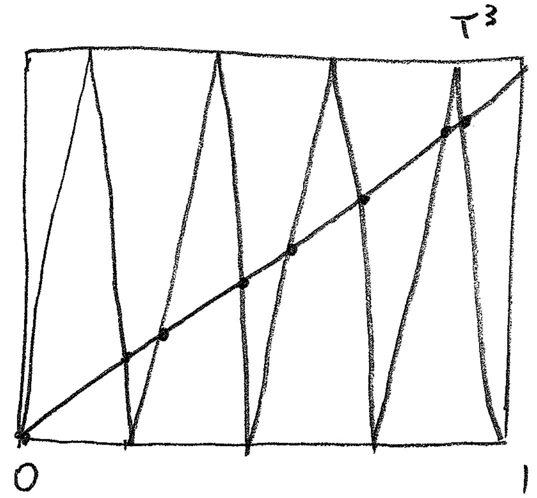
no 2-cycles



x = 0 attracting

4. (20 pts) Prove that the tent map T has at least 2^n periodic points of (not necessarily prime) period n in the interval $[0,1]$.

The Tent map $T: [0,1] \to [0,1]$ covers each point in $[0,1)$ exactly twice. Hence, graph of T^3 is



has 8 fixed points

In general graph of T^n looks like sawtooth with 2^{n-1} teeth. The diagonal intersects each tooth at 2 points $\Rightarrow$ there are $2 \cdot 2^{n-1} = 2^n$ fixed points

5. (20 pts) Find an orbit of T of prime period 2 and classify this orbit as attracting, repelling, or neutral.

Problem 1 $\Rightarrow$ $\frac{2}{5}, \frac{4}{5}, \frac{2}{5}, \frac{4}{5}, \ldots$ is orbit with period $= 2$.

Clearly $(T^2)'(\frac{2}{5}) = -4$

$\Rightarrow$ this orbit is repelling.

6. (Bonus 10 points) Find all initial points x_0 such such that their orbits under T are eventually fixed.

Answer: $\{ x = \frac{k}{2^j} : 0 \leq k \leq 2^j, j \in \mathbb{N} \}$

DYADIC POINTS in $[0,1]$