Math 352, Winter 2014, Practice Midterm Exam
Numerical Analysis II, Prof. Bownik

Your name: SOLUTIONS 2/14/2014

Instructions: Show all your work. Partial credit is available for many problems, but it can only be given if you try to explain your reasoning. There is no need to simplify your answers unless you are told so. Each problem is worth 20 points for the total of 100 points. Good luck!

1. Solve the following system using Gaussian elimination with scaled partial pivoting

$$\begin{pmatrix} 2 & 2 & 1 \\ 0 & 7 & 2 \\ 4 & 11 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}.$$

Show intermediate steps including scale and index vectors.

scale $s = [2 \ 7 \ 11]$

initial $l = [1 \ 2 \ 3]$

ratios $= \left[\frac{2}{2}, \frac{0}{7}, \frac{4}{11}\right]$ → row I - PIVOT

$$\xrightarrow{\text{III} - 2\text{I}} \left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 0 & 7 & 2 & -3 \\ 0 & 7 & 3 & -1 \end{array}\right]$$

ratio $= \left\{\frac{7}{7}, \frac{7}{11}\right\}$ → row II - PIVOT

$l = [1 \ 2 \ 3]$

$$\xrightarrow{\text{III} - \text{II}} \left[\begin{array}{ccc|c} 2 & 2 & 1 & 2 \\ 0 & 7 & 2 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$l = [1 \ 2 \ 3]$

back substitution $x_3 = 2$

$7x_2 + 2x_3 = -3 \quad 7x_2 = -7 \Rightarrow x_2 = -1$

$2x_1 + 2x_2 + x_3 = 2 \quad 2x_1 = 2 \Rightarrow x_1 = 1$

Answer $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

2. Find the matrix factorization $A = LU$, where L is unit lower triangular and U is upper triangular, for

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 7 & 2 \\ 4 & 11 & 5 \end{pmatrix}.$$

$$\xrightarrow{\text{III} - 2\text{I}} \begin{pmatrix} 2 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & 7 & 3 \end{pmatrix} \xrightarrow{\text{III} - \text{II}} \underbrace{\boxed{\begin{pmatrix} 2 & 2 & 1 \\ 0 & 7 & 2 \\ 0 & 0 & 1 \end{pmatrix}}}_{U}$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}} \qquad L = M_1^{-1} M_2^{-1}$$

3. (a) Write down the formula for Jacobi iterative method in the matrix form for the linear system $Ax = b$, where

$$A = \begin{pmatrix} 7 & 1 & -2 \\ 1 & 8 & 0 \\ 4 & -3 & -6 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$$

$$Q = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -6 \end{bmatrix} \qquad Q^{-1} = \begin{bmatrix} \frac{1}{7} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix}$$

$$Q^{-1}A = \begin{bmatrix} \frac{7}{7} & \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{8} & \frac{8}{8} & \frac{0}{8} \\ -\frac{4}{6} & +\frac{3}{6} & \frac{6}{6} \end{bmatrix}$$

So $I - Q^{-1}A = \begin{bmatrix} 0 & -\frac{1}{7} & \frac{2}{7} \\ -\frac{1}{8} & 0 & 0 \\ \frac{2}{3} & -\frac{1}{2} & 0 \end{bmatrix}$ $\qquad Q^{-1}b$

$$\boxed{\vec{x}^{(k+1)} = \underbrace{\begin{bmatrix} 0 & -\frac{1}{7} & \frac{2}{7} \\ -\frac{1}{8} & 0 & 0 \\ \frac{2}{3} & -\frac{1}{2} & 0 \end{bmatrix}}_{(I - Q^{-1}A)} \vec{x}_k + \begin{bmatrix} \frac{3}{7} \\ -\frac{5}{8} \\ -\frac{4}{6} \end{bmatrix}}$$

(b) (Multiple choice) What hypotheses guarantee that the Jacobi method works?
(i) A is diagonally dominant,
(ii) $I - A$ is diagonally dominant,
(iii) $I - Q^{-1}A$ is nonsingular,
(iv) $I - Q^{-1}A$ has spectral radius less than 1,
(v) none of these.

4. Find the quadratic spline S interpolating the data:

	t_0	t_1	t_2
x	-1	0	1
y	0	-2	5
	y_0	y_1	y_2

Assume that $z_0 = S'(-1) = 0$.

Recall:

$$\begin{cases} z_{i+1} = -z_i + 2\dfrac{y_{i+1}-y_i}{t_{i+1}-t_i} \\ Q_i(x) = \dfrac{z_{i+1}-z_i}{2(t_{i+1}-t_i)}(x-t_i)^2 + z_i(x-t_i) + y_i \end{cases}$$

$$z_1 = 0 + 2\,\frac{-2-0}{1} = -4$$

$$z_2 = 4 + 2\,\frac{5+2}{1} = 18$$

$$S(x) = \begin{cases} Q_0(x) = \dfrac{-4}{2}(x+1)^2 + 0 + 0 = -2(x+1)^2 & -1 \leq x < 0 \\ Q_1(x) = \dfrac{22}{2}x^2 + -4x - 2 = 11x^2 - 4x - 2 & 0 \leq x < 1 \end{cases}$$

5. Determine the coefficients such that the function

$$S(x) = \begin{cases} x^2 + x^3 & 0 \le x \le 1, \\ a + bx + cx^2 + dx^3 & 1 \le x \le 2. \end{cases} \quad \begin{matrix} S_0(x) \\ S_1(x) \end{matrix}$$

is a cubic spline which satisfies $S'''(2) = 12$.

$S_1'(x) = b + 2cx + 3dx^2$

$S_1''(x) = 2c + 6dx$

$S_1'''(x) = 6d$

$$S'''(2) = 6d = 12$$

$$\Rightarrow \boxed{d = 2}$$

$$\left\{\begin{array}{lll} S_0(1) = S_1(1) & \Rightarrow & 2 = a + b + c + d \\ S_0'(1) = S_1'(1) & \Rightarrow & 5 = b + 2c + 3d \\ S_0''(1) = S_1''(1) & \Rightarrow & 8 = 2c + 6d \end{array}\right.$$

$S_0'(x) = 2x + 3x^2$

$S_0''(x) = 2 + 6x$

$8 = 2c + 12 \Rightarrow 2c = -4 \quad \boxed{c = -2}$

$5 = b + 2\cdot(-2) + 3\cdot 2 \quad b = 5 - 2 = 3 \quad \boxed{b = 3}$

$2 = a + 3 + (-2) + 2 \quad \boxed{a = -1}$

6. BONUS

$S_1(2) = S_2(2) \Rightarrow b + c + d = e \quad b + 3 = e$

$S_1'(2) = S_2'(2) \Rightarrow 2b + c = -2e \quad 2b + 2 = -2e$

$4b + 6 + 2 = 0$

$\boxed{b = -2}$

$\boxed{e = 1}$

6. (Bonus 10 pts) Find a non-zero quadratic spline S with knots at $0, 1, 2, 3$ such that $S(0) = S'(0) = S(3) = S'(3) = 0$.

$$S(x) = \begin{cases} a x^2 & 0 \le x < 1 \\ b(x-1)^2 + c(x-1) + d & 1 \le x < 2 \\ e(x-3)^2 & 2 \le x < 3 \end{cases}$$

Take $\boxed{a = 1}$

$S_0(1) = S_1(1) \Rightarrow \boxed{d = 1}$

$S_0'(1) = S_1'(1) \Rightarrow \boxed{2 = c}$

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Instructions: 20% of your grade will come from a programming or research project due on the last day of classes. You can work individually or in teams of two. If working with another student, you should turn in a single project for both of you. You should decide on a topic within a week and then discuss with me either in person or by e-mail. I recommend that you turn in a rough draft before the last week of classes in order to receive a feedback.

Computer programming: Topics can be written in any language, such as Java, C, C++, or Mathematica. You must submit your source code, which should be thoroughly commented, and also a text file with a description of the program.

Research project: Projects should be 4–5 typed pages and should include a discussion of the theory, examples of computations, and a complete bibliography. You might consider writing your research project as a Mathematica notebook, since it has built-in formula formatting and you can use it for your calculations.

Topics: You can choose any of "Student Research Projects" listed in the textbook, an assortment of computer problems from the textbook, or another project of your own design in consultation with me. Most importantly, choose a topic that you find interesting. For example, a topic that ties Numerical Analysis to your other studies or interests. Here are a few suggested projects.

(1) **Gaussian elimination** Implement on a computer Gaussian elimination with scaled partial pivoting and investigate its numerical stability on some examples such as Hilbert matrix. Alternatively, write a research projection on algorithms for solving dense linear systems such as Gauss-Huard algorithm, see problem 7.2C:24.
(2) **Eigenvalues and eigenvectors.** Investigate some methods for computing eigenvalues and eigenvectors, see §8.3.
(3) **Singular Value Decomposition.** Investigate singular value decomposition, see §8.3.
(4) **History of splines.** Investigate history of splines, see problem 9.1:28.
(5) **Splines.** Implement an algorithm for interpolation using B splines, or for Bézier curves, see §9.3.
(6) **Ordinary Differential Equations.** Implement an adaptive Runge-Kutta algorithm described in §10.3.
(7) **Random number generators.** Investigate algorithms for generating random numbers, see problem 13.1C:22.
(8) **Minimization of functions.** Investigate algorithms for minimizing functions of several variables, see §16.2.
(9) ℓ^1 **minimization.** Investigate methods for solving inconsistent linear systems through ℓ^1 minimization, see §17.3.