J. reine angew. Math. **654** (2011), 219—244
DOI 10.1515/CRELLE.2011.035

Journal für die reine und
angewandte Mathematik
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Characterization of sequences of frame norms

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Abstract. We show that frames with frame bounds A and B are images of orthonormal bases under positive operators with spectrum contained in $\{0\} \cup [\sqrt{A}, \sqrt{B}]$. Then, we give an explicit characterization of the diagonals of such operators, which in turn gives a characterization of the sequences which are the norms of a frame. Our result extends the tight case result of Kadison [15], [16], which characterizes diagonals of orthogonal projections, to a non-tight case. We illustrate our main theorem by studying the set of possible lower bounds of positive operators with prescribed diagonal.

1. Introduction

Definition 1.1. A sequence $\{f_i\}_{i \in I}$ in a Hilbert space $\mathscr{H}$ is called a *frame* if there exist $0 < A \leqq B < \infty$ such that

$$A\|f\|^2 \leqq \sum |\langle f, f_i \rangle|^2 \leqq B\|f\|^2 \quad \text{for all } f \in \mathscr{H}. \tag{1.1}$$

The numbers A and B are called the *frame bounds*. The supremum over all A's and infimum over all B's which satisfy (1.1) are called the *optimal frame bounds*. If $A = B$, then $\{f_i\}$ is said to be a *tight frame*. In addition, if $A = B = 1$, then $\{f_i\}$ is called a *Parseval frame*.

The goal of this paper is to characterize all possible sequences of norms of a frame with prescribed optimal bounds A and B. This question can be reformulated to an equivalent problem asking for a characterization of diagonals of self-adjoint operators E with spectrum $A, B \in \sigma(E) \subseteq \{0\} \cup [A, B]$. This reformulation is due to Antezana, Massey, Ruiz, and Stojanoff [1] who established the relationship of the frame norm problem with the Schur–Horn theorem. Consequently, a characterization of norms of finite frames follows from the Schur–Horn theorem. The special tight case $A = B$ is a celebrated Pythagorean theorem of Kadison [15], [16], which gives a complete characterization of diagonals of projections.

The problem of characterizing norms of frames with prescribed frame operator attracted a significant number of researchers. Casazza and Leon [7], [8] gave explicit and

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The authors were partially supported by NSF grant DMS-0653881.

algorithmic construction of finite tight frames with prescribed norms. Moreover, Casazza, Fickus, Kovačević, Leon, and Tremain [9] characterized norms of finite tight frames in terms of their "fundamental frame inequality" using frame potential methods of Benedetto and Fickus [6]. An alternative approach using projection decomposition was undertaken by Kornelson and Larson [11], [19], which yields some necessary and some sufficient conditions for infinite dimensional Hilbert spaces. Antezana, Massey, Ruiz, and Stojanoff [1] established the connection of this problem with the infinite dimensional Schur–Horn problem and gave refined necessary conditions and sufficient conditions. Finally, Kadison [15], [16] gave the complete answer for Parseval frames, which easily extends to tight frames by scaling.

The equivalent problem of characterizing diagonals of self-adjoint operators remains open in infinite dimensions despite remarkable recent progress. The finite Schur–Horn theorem was extended to positive trace class operators by Gohberg and Markus [12] and by Arveson and Kadison [5], and to compact positive operators by Kaftal and Weiss [18]. Moreover, its extensions to II_1 factors [2], [3] and normal operators [4] were also studied. The infinite dimensional version of the Schur–Horn theorem due to Neumann [20], which is phrased in terms of ℓ^∞-completion of the convexity condition, is too crude for our purposes. For detailed survey of recent progress on infinite Schur–Horn majorization theorem we refer to the paper of Kaftal and Weiss [17].

Our main result can be thought as the infinite Schur–Horn theorem for a class of self-adjoint operators with prescribed lower and upper bounds and with the zero in the spectrum. Note that the assumption of $\{d_i\}$ being non-summable in Theorem 1.1 is not a true limitation. Indeed, the summable case requires more restrictive conditions reflected in Theorem 3.4.

Theorem 1.1. *Let $0 < A < B < \infty$ and $\{d_i\}_{i\in I}$ be a non-summable sequence in $[0, B]$. Define*

$$C = \sum_{d_i<A} d_i \quad \text{and} \quad D = \sum_{d_i\geqq A} (B - d_i). \tag{1.2}$$

Then, there is a positive operator E on a Hilbert space $\mathscr{H}$ with $\{A, B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A, B]$ and diagonal $\{d_i\}$ if and only if one of the following holds: (i) $C = \infty$, (ii) $D = \infty$, (iii) $C, D < \infty$ *and*

$$\exists n \in \mathbb{N} \cup \{0\} \quad nA \leqq C \leqq A + B(n - 1) + D. \tag{1.3}$$

As a corollary of Theorem 1.1 we obtain the characterization of sequences of frame norms.

Corollary 1.2. *Let $0 < A < B < \infty$ and $\{d_i\}$ be a non-summable sequence in $[0, B]$. There exists a frame $\{f_i\}$ for some Hilbert space with optimal frame bounds A and B and $d_i = \|f_i\|^2$ if and only if* (i), (ii), *or* (iii) *hold.*

We would like to emphasize that the non-tight case is not a mere generalization of the tight case $A = B$ established by Kadison [15], [16], see Theorem 4.2. Indeed, the non-tight case is qualitatively different from the tight case, since by setting $A = B$ in Theorem 1.1 we do not get the correct necessary and sufficient condition (4.2) previously discovered by

Kadison. Furthermore, the non-tight summable and non-summable cases require different characterization conditions. This is again unlike the tight case when the same condition (4.2) works in either case.

The proof of Theorem 1.1 breaks into 3 distinctive parts. The summable case does not require much new techniques since it reduces to the study of trace class operators. It can be deduced from the work of Arveson–Kadison [5] and Kaftal–Weiss [18]. However, the non-summable case is much more involved. The sufficiency part of Theorem 1.1 requires special techniques of "moving" diagonal entries to more favorable configurations, where it is possible to construct required operators. This is done in Section 4 by considering a variety of cases, some of which are tight in the sense that the required operator has a 3 point spectrum. It is worth adding that our construction is quite explicit and algorithmic leading always to diagonalizable operators. Finally, Section 5 contains the necessity proof of Theorem 1.1. This part is shown using arguments involving trace class operators and Kadison's Theorem 4.2.

Theorem 1.1 has an analogue for operators without the zero in the spectrum, see Theorem 6.3. This result is much easier to show and it leads to a characterization of norms of Riesz bases with prescribed bounds. Finally, in the last section we illustrate how our main theorem can be applied to determine the set $\mathscr{A}$ of possible lower bounds of positive operators with fixed diagonal $\{d_i\}$. While we show that it is always closed, $\mathscr{A}$ can take distinct configurations depending on the choice of a diagonal.

2. Reformulation of frames by positive operators

In this section we reformulate the problem of characterizing norms of frames to an equivalent problem of characterizing diagonals of positive operators with prescribed lower and upper bounds. We start with the following basic fact.

Proposition 2.1. *Let $\mathscr{H}$ be a Hilbert space with an orthonormal basis $\{e_i\}_{i\in I}$ and $0 < A \leqq B < \infty$. If E is a positive operator with $\sigma(E) \subseteqq \{0\} \cup [A, B]$, then $\{Ee_i\}$ is a frame for the Hilbert space $E(\mathscr{H})$ with frame bounds A^2 and B^2.*

Proof. Let $f \in E(\mathscr{H})$, then we have

$$\sum_{i\in I} |\langle f, Ee_i\rangle|^2 = \sum_{i\in I} |\langle Ef, e_i\rangle|^2 = \|Ef\|^2.$$

This clearly implies that B^2 is an upper frame bound. Since $f \in E(\mathscr{H})$ we have $\|Ef\| \geqq A\|f\|$, which shows that A^2 is a lower frame bound. □

Our goal is to establish the converse statement. That is, any frame in $\mathscr{H}$ is an image of an orthonormal basis of a larger Hilbert space $\mathscr{K} \supset \mathscr{H}$ under a positive operator. This generalizes the standard result saying that Parseval frames are images of orthonormal bases under orthogonal projections [13]. We will use the following standard terminology.

Definition 2.1. If $\{f_i\}_{i\in I}$ is a frame we call the operator $T : \mathscr{H} \to \ell^2(I)$, given by

$$Tf = \{\langle f, f_i\rangle\}_{i\in I} \tag{2.1}$$

the *analysis operator*. The adjoint $T^*: \ell^2(I) \to \mathscr{H}$ given by

$$T^*(\{a_i\}_{i\in I}) = \sum_{i\in I} a_i f_i \tag{2.2}$$

is called the *synthesis operator*. The operator $S = T^*T$ given by

$$Sf = \sum_{i\in I} \langle f, f_i \rangle f_i \tag{2.3}$$

is called the *frame operator*.

The following is a standard fact about frames [10]:

Proposition 2.2. *If $\{f_i\}_{i\in I}$ is a frame for $\mathscr{H}$ and S is the frame operator, then $\{S^{-1/2}f_i\}_{i\in I}$ is a Parseval frame for $\mathscr{H}$.*

The following result is an extension of the classical dilation theorem for Parseval frames due to Han and Larson [13], Proposition 1.1. Proposition 2.3 is essentially contained in the work of Antezana, Massey, Ruiz, and Stojanoff [1], Proposition 4.5. In particular, the authors of [1] established the relationship of our problem with the Schur–Horn theorem of majorization theory which we state in a convenient form in Theorem 2.4.

Proposition 2.3. *Let $\{f_i\}_{i\in I}$ be a frame for $\mathscr{H}$ with optimal frame bounds A^2 and B^2. Then, there exist an isometry $\Phi: \mathscr{H} \to \ell^2(I)$ and a positive operator $E: \ell^2(I) \to \Phi(\mathscr{H})$ such that $\{A, B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A, B]$ and $Ee_i = \Phi f_i$, where $\{e_i\}_{i\in I}$ is the coordinate basis of $\ell^2(I)$. If S is the frame operator of $\{f_i\}_{i\in I}$ and $\mathbf{0_e}$ is the zero operator on $\Phi(\mathscr{H})^\perp$, then E^2 is unitarily equivalent to $S \oplus \mathbf{0_e}$.*

Proof. Let S be the frame operator of $\{f_i\}$. By Proposition 2.2, $\{S^{-1/2}f_i\}$ is a Parseval frame. Set $p_i = S^{-1/2}f_i$, and let Φ be the analysis operator of $\{p_i\}$. Since $\{p_i\}$ is a Parseval frame, Φ is an isometry. Let P be the orthogonal projection onto $\Phi(\mathscr{H})$. As a consequence of the Han–Larson dilation theorem for Parseval frames [13], Proposition 1.1, we have $Pe_i = \Phi p_i$ for all $i \in I$. Hence, we also have $\Phi^* e_i = p_i$. Define the operator $E = \Phi S^{1/2} \Phi^*$. Clearly, E is a self-adjoint operator on $\ell^2(I)$. Observe that

$$Ee_i = \Phi S^{1/2}\Phi^* e_i = \Phi S^{1/2} p_i = \Phi S^{1/2} S^{-1/2} f_i = \Phi f_i.$$

Thus,

$$\|Ef\|^2 = \sum_{i\in I} |\langle Ef, e_i\rangle|^2 = \sum_{i\in I} |\langle f, Ee_i\rangle|^2 = \sum_{i\in I} |\langle f, \Phi f_i\rangle|^2.$$

Since Φ is unitary, $\{\Phi f_i\}_{i\in I}$ is a frame for $\Phi(\mathscr{H})$ with optimal frame bounds A^2 and B^2. The frame property now implies $A^2\|f\|^2 \leqq \|Ef\|^2 \leqq B^2\|f\|^2$, which in turn implies that $\{A, B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A, B]$.

Finally, define $U: \mathscr{H} \oplus \Phi(\mathscr{H})^\perp \to \ell^2(I)$ by

$$Uf = \begin{cases} \Phi f, & f \in \mathscr{H}, \\ f, & f \in \Phi(\mathscr{H})^\perp. \end{cases}$$

It is clear that U is unitary, since $\Phi : \mathcal{H} \to \Phi(\mathcal{H})$ is an isometric isomorphism. Note that $\Phi^*\Phi$ is the identity on $\mathcal{H}$, thus

$$E^2 = \Phi S^{1/2}\Phi^*\Phi S^{1/2}\Phi^* = \Phi S \Phi^*.$$

Finally, for $f \in \mathcal{H}$

$$E^2 Uf = E^2 \Phi f = \Phi S f = USf,$$

and for $f \in \Phi(\mathcal{H})^\perp$

$$E^2 Uf = E^2 f = \Phi S \Phi^* f = 0 = U\mathbf{0}_\mathbf{e} f.$$

This proves the last part of Proposition 2.3. □

One should remark that Han and Larson [13] gave a different extension of their frame dilation result than Proposition 2.3. [13], Proposition 1.6, says that any frame is an image of a Riesz basis under an orthogonal projection, and the frame and Riesz bounds are the same.

Theorem 2.4. *Suppose* $0 < A \leqq B < \infty$, $\mathcal{H}$ *is a Hilbert space, and* $\{e_i\}_{i\in I}$ *is the coordinate basis of* $\ell^2(I)$. *The following sets are equal*:

$$\mathcal{N} = \{\{\|f_i\|^2\}_{i\in I} \mid \{f_i\}_{i\in I} \text{ is a frame for } \mathcal{H} \text{ with optimal bounds } A \text{ and } B\},$$

$$\mathcal{D} = \{\{\langle Ee_i, e_i\rangle\}_{i\in I} \mid E \text{ is self-adjoint on } \ell^2(I) \text{ with rank} = \dim \mathcal{H}$$
$$\text{and } \{A,B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A,B]\}.$$

Proof. First we show $\mathcal{D} \subseteqq \mathcal{N}$. Let $\{d_i\}_{i\in I} \in \mathcal{D}$ be the diagonal of E. Since $E \geqq 0$, it has a positive square root $E^{1/2}$ with $\{\sqrt{A}, \sqrt{B}\} \subseteqq \sigma(E^{1/2}) \subseteqq \{0\} \cup [\sqrt{A}, \sqrt{B}]$. By Proposition 2.1 the sequence $\{E^{1/2}e_i\}_{i\in I}$ is a frame for the Hilbert space $E^{1/2}(\ell^2(I))$ with frame bounds A and B. Since $\{\sqrt{A}, \sqrt{B}\} \subseteqq \sigma(E^{1/2})$ it is clear that the bounds A and B are optimal. Since

$$\|E^{1/2}e_i\|^2 = \langle E^{1/2}e_i, E^{1/2}e_i\rangle = \langle Ee_i, e_i\rangle = d_i,$$

this shows that $\{d_i\} \in \mathcal{N}$.

Next, we will show that $\mathcal{N} \subseteqq \mathcal{D}$. Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with optimal frame bounds A and B. By Proposition 2.3 there is an isometry $\Phi : \mathcal{H} \to \ell^2(I)$ and a positive operator $E : \ell^2(I) \to \Phi(\mathcal{H})$ with $\{\sqrt{A}, \sqrt{B}\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [\sqrt{A}, \sqrt{B}]$ such that $Ee_i = \Phi f_i$. Since $\{A, B\} \subseteqq \sigma(E^2) \subseteqq \{0\} \cup [A, B]$, and

$$\langle E^2 e_i, e_i\rangle = \langle Ee_i, Ee_i\rangle = \|Ee_i\|^2 = \|\Phi f_i\|^2 = \|f_i\|^2,$$

this shows that $\{\|f_i\|^2\}_{i\in I} \in \mathcal{D}$. □

A similar result holds for Riesz bases.

Definition 2.2. A sequence $\{f_i\}_{i \in I}$ in a Hilbert space $\mathscr{H}$ is called a *Riesz basis* if it is complete and there exist $0 < A \leqq B < \infty$ such that

$$A \sum |a_i|^2 \leqq \left\| \sum a_i f_i \right\|^2 \leqq B \sum |a_i|^2 \tag{2.4}$$

for all finitely supported sequences $\{a_i\}_{i \in I}$. The numbers A and B are called the *Riesz bounds*. The supremum over all A's and infimum over all B's which satisfy (2.4) are called the *optimal Riesz bounds*.

Equivalently, a Riesz basis is a frame such that its synthesis operator T^*, and thus analysis operator T, is an isomorphism. Moreover, optimal Riesz and frame bounds are the same. Therefore, an analogue of Proposition 2.3 for Riesz bases involves operators E without zero in the spectrum. Consequently, we have the following analogue of Theorem 2.4:

Theorem 2.5. *Suppose* $0 < A \leqq B < \infty$, $\mathscr{H}$ *is a Hilbert space, and* $\{e_i\}_{i \in I}$ *is the coordinate basis of* $\ell^2(I)$. *The following sets are equal*:

$$\mathscr{N} = \{\{\|f_i\|^2\}_{i \in I} \mid \{f_i\}_{i \in I} \text{ is a Riesz basis for } \mathscr{H} \text{ with optimal bounds } A \text{ and } B\},$$

$$\mathscr{D} = \{\{\langle Ee_i, e_i \rangle\}_{i \in I} \mid E \text{ is self-adjoint on } \ell^2(I) \text{ and } \{A, B\} \subseteqq \sigma(E) \subseteqq [A, B]\}.$$

3. The summable case

The goal of this section is to establish the summable case of our main Theorem 1.1. This special case can be deduced from a finite rank version of the Schur–Horn theorem.

Theorem 3.1 (Schur–Horn theorem). *Let* $\{\lambda_i\}_{i=1}^N$ *and* $\{d_i\}_{i=1}^N$ *be real sequences with non-increasing order. If*

$$\begin{aligned} &\sum_{i=1}^n d_i \leqq \sum_{i=1}^n \lambda_i \quad \forall n = 1, \dots, N, \\ &\sum_{i=1}^N \lambda_i = \sum_{i=1}^N d_i, \end{aligned} \tag{3.1}$$

then there is a self-adjoint operator $E : \mathbb{R}^N \to \mathbb{R}^N$ *with eigenvalues* $\{\lambda_i\}$ *and diagonal* $\{d_i\}$.

Conversely, if $E : \mathbb{C}^N \to \mathbb{C}^N$ *is a self-adjoint operator with eigenvalues* $\{\lambda_i\}$ *and diagonal* $\{d_i\}$, *then* (3.1) *holds.*

The analogue of the Schur–Horn theorem for trace class operators was proved by Arveson and Kadison in [5]. It was further generalized to compact operators by Kaftal and Weiss in [18]. The following is a special case of Arveson–Kadison theorem [5], Theorem 4.1, for finite rank operators. Theorem 3.2 can also be deduced from the Kaftal and Weiss infinite dimensional extension of the Schur–Horn theorem [18], Theorem 6.1, which also considers the real case.

Theorem 3.2 (finite rank Horn's theorem). *Let $\{\lambda_i\}_{i=1}^N$ and $\{d_i\}_{i=1}^\infty$ be positive non-increasing sequences. If*

$$\text{(3.2)} \qquad \begin{aligned} \sum_{i=1}^n d_i &\leqq \sum_{i=1}^n \lambda_i \quad \forall n \leqq N, \\ \sum_{i=1}^\infty d_i &= \sum_{i=1}^N \lambda_i, \end{aligned}$$

then there is a positive rank N operator E on a real Hilbert space $\mathscr{H}$ with eigenvalues $\{\lambda_i\}_{i=1}^N$ and diagonal $\{d_i\}_{i=1}^\infty$.

The necessity of majorization condition (3.2) is a classical result of Schur [21].

Theorem 3.3 (Schur). *If $E : \mathscr{H} \to \mathscr{H}$ is a positive compact operator with eigenvalues (with multiplicity) $\{\lambda_i\}_{i=1}^\infty$ in non-increasing order, then for any orthonormal basis $\{e_i\}_{i=1}^\infty$ of $\mathscr{H}$ we have*

$$\text{(3.3)} \qquad \sum_{i=1}^n \langle Ee_i, e_i\rangle \leqq \sum_{i=1}^n \lambda_i \quad \forall n \in \mathbb{N}.$$

Using Theorems 3.2 and 3.3 we can prove the summable variant of Theorem 1.1.

Theorem 3.4. *Suppose $0 < A \leqq B < \infty$ and $M \in \mathbb{N} \cup \{\infty\}$. Let $\{d_i\}_{i=1}^M$ be a summable sequence in $[0, B]$. There is a positive, rank $N+1$ operator E on a Hilbert space $\mathscr{H}$ with diagonal $\{d_i\}$ and $\{A, B\} \subseteq \sigma(E) \subseteq \{0\} \cup [A, B]$ if and only if*

$$\text{(3.4)} \qquad \sum_{i=1}^M d_i \in [AN + B, A + BN],$$

$$\text{(3.5)} \qquad \sum_{d_i < A} d_i \geqq A(N - m_0 + 1), \quad m_0 = |\{i : d_i \geqq A\}|.$$

Proof. Assume an operator E is as in Theorem 3.4. Because each of the $N+1$ non-zero eigenvalues of E is at most B, and A is an eigenvalue, we have $\sum d_i = \operatorname{tr}(E) \leqq A + BN$. Similarly, since each of the $N+1$ non-zero eigenvalues of E is at least A, and B is an eigenvalue, we have $\sum d_i = \operatorname{tr}(E) \geqq AN + B$. After rearranging $\{d_i\}$ in non-increasing order Theorem 3.3 yields

$$\sum_{d_i < A} d_i = \sum_{i=m_0+1}^\infty d_i \geqq \sum_{i=m_0+1}^\infty \lambda_i = \sum_{i=m_0+1}^{N+1} \lambda_i \geqq A(N - m_0 + 1),$$

where $\{\lambda_i\}$ are eigenvalues of E in non-increasing order (with multiplicity). This shows that (3.4) and (3.5) are necessary.

Conversely, assume we have a sequence $\{d_i\}_{i=1}^M$ which satisfies (3.4) and (3.5). If $\sum d_i < A + BN$ then there exist unique $n_0 \in \{1, 2, \ldots, N\}$ and $x \in [A, B)$ such that

$$\text{(3.6)} \qquad \sum_{i=1}^M d_i = A(N - n_0) + x + Bn_0.$$

We set

$$\lambda_i = \begin{cases} B, & i \in \{1, \dots, n_0\}, \\ x, & i = n_0 + 1, \\ A, & i \in \{n_0 + 2, \dots, N + 1\}. \end{cases}$$

If $\sum d_i = A + BN$, simply let $n_0 = N - 1$ and $x = B$. By Theorem 3.2, we only need to check that the majorization property (3.2) holds for $\{d_i\}$ and $\{\lambda_i\}$.

Combining (3.5) and (3.6), we have

$$\sum_{i=1}^{m_0} d_i \leqq Bn_0 + x + A(m_0 - n_0 - 1). \tag{3.7}$$

For $m \leqq m_0$, we have

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{m_0} d_i - \sum_{i=m+1}^{m_0} d_i \leqq \sum_{i=1}^{m_0} d_i + A(m - m_0).$$

For $m_0 < m \leqq N + 1$, we have

$$\sum_{i=1}^{m} d_i = \sum_{i=1}^{m_0} d_i + \sum_{i=m_0+1}^{m} d_i \leqq \sum_{i=1}^{m_0} d_i + A(m - m_0).$$

In either case, combining these with (3.7), yields

$$\sum_{i=1}^{m} d_i \leqq Bn_0 + x + A(m - n_0 - 1) \leqq \sum_{i=1}^{m} \lambda_i \quad \text{for } n_0 + 1 \leqq m \leqq N + 1.$$

Finally, for $m > N + 1$ and $m < n_0 + 1$ the majorization property is trivial. $\square$

As a corollary of Theorems 2.4 and 3.4 we have

Corollary 3.5. *Suppose $0 < A \leqq B < \infty$ and $M \in \mathbb{N} \cup \{\infty\}$. Let $\{d_i\}_{i=1}^{M}$ be a summable sequence in $[0, B]$. There exists a frame $\{f_i\}$ for an $(N + 1)$-dimensional space with optimal frame bounds A and B and $d_i = \|f_i\|^2$ if and only if* (3.4) *and* (3.5) *hold.*

In the non-summable case the condition (3.4) makes no sense. However, we can give an alternate set of conditions which will generalize.

Theorem 3.6. *Suppose $0 < A \leqq B < \infty$ and $M \in \mathbb{N} \cup \{\infty\}$. Let $\{d_i\}_{i=1}^{M}$ be a summable sequence in $[0, B]$. Define the numbers*

$$C = \sum_{d_i < A} d_i \quad \text{and} \quad D = \sum_{d_i \geqq A} (B - d_i).$$

There is a positive, rank $N+1$ *operator* E *on a Hilbert space* $\mathscr{H}$ *with diagonal* $\{d_i\}$ *and* $\{A,B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A,B]$ *if and only if*

$$C \in [A(N-m_0+1), A+B(N-m_0)+D], \quad m_0 = |\{i : d_i \geqq A\}|, \tag{3.8}$$

$$\sum_{i=1}^{M} d_i \geqq AN + B. \tag{3.9}$$

Proof. Assuming (3.4) and (3.5) we have

$$C - D = \sum_{d_i<A} d_i - \sum_{d_i \geqq A} (B-d_i) = \sum_{i=1}^{M} d_i - m_0 B \leqq A + BN - m_0 B$$

which shows $C \leqq A + B(N-m_0) + D$, and the other parts of (3.8) and (3.9) are obvious. Similarly, assuming (3.8) and (3.9) we see

$$\sum_{i=1}^{M} d_i = C - D + m_0 B \leqq A + B(N-m_0) + D - D + m_0 B = A + BN,$$

and the other parts of (3.4) and (3.5) are obvious. □

Note that if $\{d_i\}$ is not summable, then (3.9) is trivially satisfied. Thus it is a reasonable and correct guess that a variant of (3.8) is the necessary and sufficient condition.

4. The non-summable case of Carpenter's theorem

The goal of this section is to prove the sufficiency part of our main theorem. In the terminology of Kadison [15], [16], this is a non-tight version of Carpenter's theorem.

Theorem 4.1. *Suppose* $0 < A < B < \infty$. *Let* $\{d_i\}_{i \in I}$ *be a non-summable sequence in* $[0,B]$ *and*

$$C = \sum_{d_i<A} d_i, \quad D = \sum_{d_i \geqq A} (B-d_i).$$

If

$$C \in \bigcup_{n=0}^{\infty} [An, A+B(n-1)+D] \cup \{\infty\}, \tag{4.1}$$

then there is a positive diagonalizable operator E *on a Hilbert space* $\mathscr{H}$ *with* $\{A,B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A,B]$ *and diagonal* $\{d_i\}_{i \in I}$.

Remark 4.1. In Theorem 4.1, the index set I may or may not be countable and $\mathscr{H}$ may or may not be separable. The case of $\mathscr{H}$ being non-separable can be reduced to the separable case. We will use the convention that a "sequence" $\{d_i\}_{i \in I}$ can have an indexing set of any cardinality. Note that, if $D = \infty$, then the first interval in the union is $[0, \infty]$

so (4.1) is always satisfied. Similarly, if $C = \infty$, then (4.1) is always satisfied. Moreover, if $A - B + D < 0$, then we interpret the interval $[0, A - B + D]$ to be $\emptyset$, which means that if $D < B - A$, then $C = 0$ does not satisfy (4.1). Finally, note that the set in (4.1) reduces to a finite union of intervals since it always contains an infinite interval $[(n+1)A, \infty]$, where $n = \lceil B/(B - A) \rceil$.

In the tight case $A = B$, the condition (4.1) is necessary but not sufficient. The correct condition was discovered by Kadison [15], [16]. We state it in a form convenient for our purposes, since it plays a prominent role in our arguments. The necessity part of Theorem 4.2 is referred in [15], [16] as the Pythagorean theorem, whereas the sufficiency part is the Carpenter theorem.

Theorem 4.2 (Kadison). *Let $\{d_i\}_{i \in I}$ be a sequence in $[0, B]$. For $\alpha \in (0, B)$ define*

$$a = \sum_{d_i < \alpha} d_i, \quad b = \sum_{d_i \geqq \alpha} (B - d_i).$$

Then, there is an orthogonal projection P such that BP has a diagonal $\{d_i\}_{i \in I}$ if and only if

$$a - b \in B\mathbb{Z} \cup \{\pm\infty\} \tag{4.2}$$

with the convention that $\infty - \infty = 0$.

Remark 4.2. Note that the condition (4.2) is independent of the choice of $\alpha \in (0, B)$. That is, if (4.2) holds for some α, then it must hold for all $\alpha \in (0, B)$. To see that conditions (4.1) and (4.2) are different in the tight case, consider the sequence $\{d_i\}$ which contains the terms $\{n^{-2}\}_{n=2}^{\infty}$ and $\{1 - 2^{-n}\}_{n=1}^{\infty}$. For $B = 1$ and $\alpha = 1/2$ we have $a = \dfrac{\pi^2 - 6}{6}$ and $b = 1$, thus $b - a \notin \mathbb{Z}$. By Theorem 4.2 there is no projection with diagonal $\{d_i\}$, although (4.1) is satisfied since $C = \infty$.

Secondly, note that the indexing set I is not assumed to be countable. In [15], [16] the possibility that I is an uncountable set is addressed in all but the most difficult case where $\{d_i\}$ and $\{B - d_i\}$ are non-summable [16], Theorem 15. However, the case where I is uncountable is a simple extension of the countable case, as we will now explain.

Proof of the reduction of Theorem 4.2 *to the countable case.* By normalizing, we may assume $B = 1$. First, we consider a projection P with diagonal $\{d_i\}_{i \in I}$ with respect to some orthonormal basis $\{e_i\}$. If a or b is infinite then there is nothing to show, so we may assume $a, b < \infty$. Set $J = \{i \in I \mid d_i = 0\} \cup \{i \in I \mid d_i = 1\}$, and let P' be the operator P acting on $\overline{\text{span}}\{e_i\}_{i \in I \setminus J}$. Since e_i is an eigenvector for each $i \in J$, P' is a projection with diagonal $\{d_i\}_{i \in I \setminus J}$. The assumption that $a, b < \infty$ implies $I \setminus J$ is at most countable. Thus, the countable case of Theorem 4.2 applied to the operator P' yields $a - b \in \mathbb{Z}$. This shows that (4.2) is necessary.

To show that (4.2) is sufficient, we claim that it is enough to assume that all of the d_i's are in $(0, 1)$. If we can find a projection P with only these d_i's, then we take I to be the identity and $\mathbf{0}$ the zero operator on Hilbert spaces with dimensions chosen so that $P \oplus I \oplus \mathbf{0}$ has diagonal $\{d_i\}$. Since a and b do not change when we restrict to $(0, 1)$, we may assume that $\{d_i\}_{i\in I}$ has uncountably many terms and is contained in $(0,1)$. There is some $n\in\mathbb{N}$ such that $J=\{i\in I \mid 1/n<d_i<1-1/n\}$ has the same cardinality as I. Thus, we can partition I into a collection of countable infinite sets $\{I_k\}_{k\in K}$ such that $I_k\cap J$ is infinite for each $k\in K$. Each sequence $\{d_i\}_{i\in I_k}$ contains infinitely many terms bounded away from 0 and 1, thus (4.2) holds with a or b infinite. Again, by the countable case of Theorem 4.2 for each $k\in K$ there is a projection P_k with diagonal $\{d_i\}_{i\in I_k}$. Thus, $\bigoplus_{k\in K} P_k$ is a projection with diagonal $\{d_i\}_{i\in I}$. □

The following elementary "moving toward 0-1" lemma plays a key role in the proof of Theorem 4.1.

Lemma 4.3. *Let $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^M$ be sequences in $[0,B]$ with $\max\{a_i\}\leqq\min\{b_i\}$. Let $\eta_0\geqq 0$ and*

$$\eta_0\leqq\min\left\{\sum_{i=1}^N a_i,\sum_{i=1}^M(B-b_i)\right\}.$$

Then, there exist sequences $\{\tilde{a}_i\}_{i=1}^N$ and $\{\tilde{b}_i\}_{i=1}^M$ in $[0,B]$ satisfying

$$\tilde{a}_i\leqq a_i,\quad i=1,\dots,N,\qquad\text{and}\qquad b_i\leqq\tilde{b}_i,\quad i=1,\dots,M, \tag{4.3}$$

$$\eta_0+\sum_{i=1}^N\tilde{a}_i=\sum_{i=1}^N a_i\quad\text{and}\quad\eta_0+\sum_{i=1}^M(B-\tilde{b}_i)=\sum_{i=1}^M(B-b_i). \tag{4.4}$$

Proof. By scaling the sequences, we can reduce Lemma 4.3 to the case $B=1$. Set

$$\{a_i^{(0)}\}_{i=1}^N=\{a_i\}_{i=1}^N\quad\text{and}\quad\{b_i^{(0)}\}_{i=1}^M=\{b_i\}_{i=1}^M.$$

Define a series of new sequences by applying the following algorithm:

Step i. If $\eta_{i-1}=0$ then we are done. Otherwise set

$$a_{n_i}^{(i-1)}=\max\{a_n^{(i-1)}\}\quad\text{and}\quad b_{m_i}^{(i-1)}=\min\{b_m^{(i-1)}\}.$$

Then define

$$\delta_i=\min\{a_{n_i}^{(i-1)},1-b_{m_i}^{(i-1)},\eta_{i-1}\}.$$

Now define the sequences $\{a_n^{(i)}\}$ and $\{b_m^{(i)}\}$ by

$$a_n^{(i)}=\begin{cases}a_{n_i}^{(i-1)}-\delta_i, & n=n_i,\\ a_n^{(i-1)} & \text{otherwise,}\end{cases}\qquad b_m^{(i)}=\begin{cases}b_{m_i}^{(i-1)}+\delta_i, & m=m_i,\\ b_m^{(i-1)} & \text{otherwise.}\end{cases}$$

Define

$$\eta_i=\eta_{i-1}-\delta_i$$

and proceed to step $i+1$.

We claim that the above algorithm will stop after $K \leqq N + M - 1$ steps. Notice that if $\delta_i = \eta_{i-1}$, then $\eta_i = 0$ and the algorithm stops. So, assume $\delta_i = a_{n_i}$ or $1 - b_{m_i}$ for all i. If $\delta_i = a_{n_i}$, then the sequence $\{a_n^{(i)}\}$ will have one more zero than $\{a_n^{(i-1)}\}$. If $\delta_i = 1 - b_{m_i}$, then the sequence $\{b_m^{(i)}\}$ will have one more 1 than $\{b_m^{(i-1)}\}$. If $\{a_n^{(i)}\}$ is a sequence of zeros then the algorithm must have stopped, since $\eta_i \leqq \sum_{n=1}^{N} a_n^{(i)}$. Similarly, if $\{b_m^{(i)}\}$ is a sequence of ones, then the algorithm must have stopped, since $\eta_i \leqq \sum_{m=1}^{M} (1 - b_m^{(i)})$. Thus, the algorithm can continue for at most $N + M - 1$ steps. Finally, set $\tilde{a}_i = a_i^{(K)}$ and $\tilde{b}_j = b_j^{(K)}$ for all i and j. □

The operational version of the "moving toward 0-1" lemma takes the following form:

Lemma 4.4. *Let $\{a_i\}$, $\{b_i\}$, $\{\tilde{a}_i\}$, and $\{\tilde{b}_i\}$ be sequences in $[0, B]$ as in Lemma 4.3. If there is a self-adjoint operator $\tilde{E}$ with diagonal*

$$\{\tilde{a}_1, \ldots, \tilde{a}_N, \tilde{b}_1, \ldots, \tilde{b}_M, c_1, c_2, \ldots\},$$

then there exists an operator E on $\mathscr{H}$ unitarily equivalent to $\tilde{E}$ with diagonal

$$\{a_1, \ldots, a_N, b_1, \ldots, b_M, c_1, c_2, \ldots\}.$$

Here, $\{c_i\}$ is either a finite or an infinite bounded sequence of real numbers.

Proof. Let $\{e_i\}$ be the orthonormal basis, with respect to which $\tilde{E}$ has diagonal

$$\{\tilde{b}_1, \ldots, \tilde{b}_M, \tilde{a}_1, \ldots, \tilde{a}_N, c_1, c_2, \ldots\}.$$

We may assume $\{\tilde{b}_1, \ldots, \tilde{b}_M, \tilde{a}_1, \ldots, \tilde{a}_N\}$ is written in non-increasing order. Let P be the orthogonal projection onto the finite dimensional Hilbert space $\mathscr{H}_0 = \operatorname{span}\{e_i\}_{i=1}^{N+M}$, and let $\tilde{E}_0 : \mathscr{H}_0 \to \mathscr{H}_0$ be the operator $P\tilde{E}$ restricted to $\mathscr{H}_0$. In other words, $\tilde{E}_0$ is the $(N + M) \times (N + M)$ corner of $\tilde{E}$ with diagonal $\{\tilde{b}_1, \ldots, \tilde{b}_M, \tilde{a}_1, \ldots, \tilde{a}_N\}$.

Let $\{\lambda_i\}_{i=1}^{N+M}$ be the eigenvalues of $\tilde{E}_0$, written in non-increasing order. By Theorem 3.1 we have the majorization property (3.1) majorization for the diagonal of $\tilde{E}_0$ and $\{\lambda_i\}$. Using (4.3) and (4.4) yields

$$\sum_{i=1}^{k} b_i \leqq \sum_{i=1}^{k} \tilde{b}_i \quad \text{for } k = 1, \ldots, M,$$

$$\sum_{i=1}^{M} b_i + \sum_{i=1}^{k} a_i = \sum_{i=1}^{M} \tilde{b}_i - \eta_0 + \sum_{i=1}^{k} a_i \leqq \sum_{i=1}^{M} \tilde{b}_i + \sum_{i=1}^{k} \tilde{a}_i \quad \text{for } k = 1, \ldots, N.$$

This shows that the majorization property also holds for $\{b_1, \ldots, b_M, a_1, \ldots, a_N\}$ and $\{\lambda_i\}$. By Theorem 3.1 there is an operator $E_0 : \mathscr{H}_0 \to \mathscr{H}_0$ with diagonal $\{b_1, \ldots, b_M, a_1, \ldots, a_N\}$ and eigenvalues $\{\lambda_i\}$, and thus there is a unitary $U_0 : \mathscr{H}_0 \to \mathscr{H}_0$ such that $E_0 = U_0^* \tilde{E}_0 U_0$.

Define the unitary $U = U_0 \oplus I$, where I is the identity operator on $\overline{\operatorname{span}}\{e_i\}_{i>N+M}$. Hence, the operator $E = U^* \tilde{E} U$ has diagonal

$$\{a_1, \ldots, a_N, b_1, \ldots, b_M, c_1, c_2, \ldots\}. \quad \square$$

We are ready to give the proof of Theorem 4.1 which breaks into several cases.

Proof of Theorem 4.1. Throughout this proof let $\{a_i\}$ and $\{b_i\}$ be the subsequences of d_i's in $[0, A)$ and $[A, B]$, respectively.

Case 1. Assume $C = \infty$.

Partition $\{a_i\}$ into a countable number of sequences $\{a_i^{(k)}\}$ for each $k \in \mathbb{N}$, each with infinite sum. For each $k \in \mathbb{N}$ we apply Theorem 4.2 on $\left[0, A + \frac{B-A}{k}\right]$ with $\alpha = A$. Since

$$\sum_{a_i^{(k)} < \alpha} a_i^{(k)} = \infty,$$

there is a projection $P_k \neq I$ on a Hilbert space $\mathscr{H}_k$ such that the diagonal of $\left(A + \frac{B-A}{k}\right) P_k$ is $\{a_i^{(k)}\}$. Let S be the diagonal operator with the diagonal $\{b_i\}$ on a Hilbert space $\mathscr{H}_0$. Then, the operator

$$E = S \oplus \bigoplus_{k=1}^{\infty} \left(A + \frac{B-A}{k}\right) P_k$$

on the Hilbert space $\mathscr{H} = \bigoplus_{k=0}^{\infty} \mathscr{H}_k$ has diagonal $\{d_i\}$. By construction $\sigma(E)$ is the closure of $\{0\} \cup \left\{A + \frac{B-A}{k} : k \in \mathbb{N}\right\} \cup \{b_i\}$, thus $\{A, B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A, B]$.

Case 2. Assume $D = \infty$.

First, suppose that A is not an accumulation point of $\{b_i\}$. Partition $\{b_i\}$ into two sequences $\{b_i^{(1)}\}$ and $\{b_i^{(2)}\}$ such that

$$\sum_{i=1}^{\infty} (B - b_i^{(k)}) = \infty \quad \text{for } i = 1, 2. \tag{4.5}$$

Let $\{c_i\}$ be the sequence consisting of $\{a_i\}$ and $\{b_i^{(1)}\}$. By Theorem 4.2 on $[0, B]$ with $\alpha = A$ and

$$\sum_{c_i \geqq \alpha} (B - c_i) = \infty,$$

there is a projection P_1 on a Hilbert space $\mathscr{H}_1$ such that BP_1 has diagonal $\{c_i\}$.

Define

$$k_i = \frac{b_i^{(2)} - A}{B - A}.$$

The sequence $\{k_i\}$ is in $[0,1]$ and 0 is not an accumulation point. Thus, there exists $\alpha \in (0,1)$ such that

$$\sum_{k_i \geqq \alpha} (1 - k_i) = \infty.$$

By Theorem 4.2 there is a projection P_2 on a Hilbert space $\mathscr{H}_2$ with diagonal $\{k_i\}$. The operator $S = (B - A)P_2 + AI$ is diagonalizable with eigenvalues A and B, and diagonal $\{b_i^{(2)}\}$. Thus, the operator $E = BP_1 \oplus S$ on $\mathscr{H}_1 \oplus \mathscr{H}_2$ has diagonal $\{d_i\}$ and $\sigma(E) = \{0, A, B\}$.

Finally, suppose that A is an accumulation point of $\{b_i\}$. Partition $\{b_i\}$ into two infinite sequences $\{b_i^{(1)}\}$ and $\{b_i^{(2)}\}$ each with infimum A. Then, (4.5) holds. Let P_1 be a projection on $\mathscr{H}_1$ as before. Let S be the diagonal operator on a Hilbert space $\mathscr{H}_2$ with $\{b_i^{(2)}\}$ on the diagonal. The operator $E = BP_1 \oplus S$ has diagonal $\{d_i\}$. Clearly, $\sigma(S) \subseteqq [A, B]$, and since $\inf\{b_i^{(2)}\} = A$ we have $A \in \sigma(S)$. We also have $\{0, B\} = \sigma(BP_1)$, thus $\{A, B\} \subseteqq \sigma(E) = \sigma(BP_1) \cup \sigma(S) \subseteqq \{0\} \cup [A, B]$, as desired.

Case 3. Assume $C, D < \infty$ and $C \in [An, A + B(n-1) + D]$ for some $n \in \mathbb{N}$.

We claim that it is enough to prove Case 3 when $\{d_i\}$ is countable. The fact that $C, D < \infty$ implies that the sequence $\{d_i\}$ contains at most countably many terms in $(0, B)$. Assume that there exists an operator E with the desired spectrum and diagonal consisting of only the terms of $\{d_i\}$ in $(0, B)$. Let I be the identity operator on a Hilbert space of dimension $|\{i : d_i = B\}|$, and let $\mathbf{0}$ be the zero operator on a Hilbert space of dimension $|\{i : d_i = 0\}|$. The operator $E \oplus BI \oplus \mathbf{0}$ has the same spectrum as E and diagonal $\{d_i\}$. However, it may happen that the sequence of terms contained in $(0, B)$ is summable. This would imply that $\{d_i\}$ must contain infinitely many terms equal to B (since $\{d_i\}$ is assumed to be non-summable). In this case we consider the sequence of terms in $(0, B)$ together with a countable infinite sequence of B's. If we can find an operator E with this diagonal sequence and the desired spectrum, then $E \oplus BI \oplus \mathbf{0}$ is again the desired operator. This proves our claim.

Let $n \in \mathbb{N}$ be the largest such that $C \in [An, A + B(n-1) + D]$. Since $\{d_i\}$ is not summable, $\{b_i\}$ is an infinite sequence. First, assume $C = An$. By Theorem 4.2 on $[0, A]$ there is a projection P on a Hilbert space $\mathscr{H}_1$ such that AP has diagonal $\{a_i\}$. Let $\mathscr{H}_2$ be an infinite dimensional Hilbert space, and S be a diagonal operator with $\{b_i\}$ on the diagonal. Since $\{b_i\}$ is an infinite sequence in $[A, B]$ and $D < \infty$ we clearly have $B \in \sigma(S)$ and thus $E = AP \oplus S$ is the desired operator.

Next, assume $C \in \big(An, A + B(n-1)\big]$ and set $C = An + x$. Since $\sup\{b_i\} = B$, there is some $i_0 \in \mathbb{N}$ such that $b_{i_0} + x \geqq B$. Define the sequence $\{\tilde{a}_i\}$ to be the sequence consisting of $\{a_i\}$ and b_{i_0}. This sequence is summable and

$$\sum \tilde{a}_i = C + b_{i_0} = An + x + b_{i_0} \geqq An + B,$$

$$\sum \tilde{a}_i = C + b_{i_0} \leqq A + B(n-1) + b_{i_0} \leqq A + Bn.$$

Since

$$\sum_{\tilde{a}_i < A} \tilde{a}_i = C \geqq nA,$$

and there is exactly one term in $\{\tilde{a}_i\}$ which is $\geqq A$, the sequence meets the conditions of Theorem 3.4. Thus there is an operator S_1 with A and B as eigenvalues,

$$\sigma(S_1) \subseteqq \{0\} \cup [A, B]$$

and diagonal $\{\tilde{a}_i\}$. Define $\{\tilde{b}_i\}$ to be the sequence $\{b_i\}_{i \neq i_0}$. Let S_2 be the diagonal operator with $\{\tilde{b}_i\}$ on the diagonal. The operator $E = S_1 \oplus S_2$ is the desired operator.

Next, assume $C \in \big(A + B(n-1), A + B(n-1) + D\big)$ and set $C = A + B(n-1) + x$. Since $x < D$ and $x < C$, there are $N, M \in \mathbb{N}$ such that

$$\sum_{i=1}^{N} a_i \geqq x \quad \text{and} \quad \sum_{i=1}^{M} (B - b_i) \geqq x.$$

Apply Lemma 4.3 to the sequences $\{a_i\}_{i=1}^N$ and $\{b_i\}_{i=1}^M$ with $\eta_0 = x$ to get new sequences $\{\tilde{a}_i\}_{i=1}^N$ and $\{\tilde{b}_i\}_{i=1}^M$ satisfying (4.3) and (4.4). Let $\{\tilde{b}_i\}_{i=1}^\infty$ be the sequence consisting of $\{\tilde{b}_i\}_{i=1}^N$ and $\{b_i\}_{i=N+1}^\infty$ and similarly define $\{\tilde{a}_i\}$. We purposely omit indexing for $\{\tilde{a}_i\}$ since the original sequence $\{a_i\}$ might be either finite or infinite. Set

$$\tilde{C} = \sum \tilde{a}_i \quad \text{and} \quad \tilde{D} = \sum (B - \tilde{b}_i). \tag{4.6}$$

We have $\tilde{C} = A + B(n-1)$ and we can apply the previous case to get an operator $\tilde{E}$ with $\{A, B\} \subseteqq \sigma(\tilde{E}) \subseteqq \{0\} \cup [A, B]$ with diagonal consisting of $\{\tilde{a}_i\}$ and $\{\tilde{b}_i\}$. Then, Lemma 4.4 yields an operator E with the same spectrum as $\tilde{E}$ and diagonal $\{a_i\} \cup \{b_i\}$.

Finally, assume $C = A + B(n-1) + D$. First, we look at the case where $C = A$. This implies $n = 1$ and $D = 0$. Thus, $\{b_i\}$ is an infinite sequence of B's. By Theorem 4.2 there is a projection P, such that AP has diagonal $\{a_i\}$. Let F be the diagonal operator with $\{b_i\}$ on the diagonal, then $AP \oplus F$ has the desired spectrum and diagonal. Now, we may assume $C > A$.

Arrange the sequence $\{a_i\}$ in non-increasing order and define

$$M_0 = \max\left\{m : \sum_{i=1}^{m} a_i \leqq A\right\} \quad \text{and} \quad x = A - \left(\sum_{i=1}^{M_0} a_i\right).$$

Observe that $M_0 \geqq 1$ and there is $N \geqq M_0 + 1$ such that

$$\sum_{i=M_0+1}^{N} a_i \geqq x.$$

It is also clear that

$$\sum_{i=1}^{M_0} (A - a_i) \geqq x.$$

Apply Lemma 4.3 to the sequences $\{a_i\}_{i=M_0+1}^{N}$ and $\{a_i\}_{i=1}^{M_0}$ on the interval $[0, A]$ with $\eta_0 = x$ to get new sequences $\{\tilde{a}_i\}_{i=M_0+1}^{N}$ and $\{\tilde{a}_i\}_{i=1}^{M_0}$ satisfying (4.3) and (4.4). Let $\{\tilde{a}_i\}$ be the sequence consisting of $\{\tilde{a}_i\}_{i=1}^{N}$ and $\{a_i\}_{i\geqq N+1}$. By (4.4) observe that

$$\sum_{i=1}^{M_0} \tilde{a}_i = \sum_{i=1}^{M_0} a_i + x = A, \quad \text{and} \quad \sum_{i\geqq M_0+1} \tilde{a}_i = \sum_{i\geqq M_0+1} a_i - x = C - A.$$

Thus, by Theorem 4.2 we can construct a rank one projection P such that the operator AP has diagonal $\{\tilde{a}_i\}_{i=1}^{M_0}$. Define

$$a = \sum_{i\geqq M_0+1} \tilde{a}_i \quad \text{and} \quad b = \sum_{i=1}^{\infty}(B - b_i),$$

and note that $a - b = C - A - D = (n-1)B$. Thus, by Theorem 4.2 there is a projection Q such that BQ has diagonal consisting of $\{\tilde{a}_i\}_{i\geqq M_0+1}$ and $\{b_i\}$. Now, $\tilde{E} = AP \oplus BQ$ has diagonal consisting of $\{\tilde{a}_i\}$ and $\{b_i\}$ and $\sigma(\tilde{E}) = \{0, A, B\}$. Then, Lemma 4.4 yields an operator E with the same spectrum as $\tilde{E}$ and diagonal $\{a_i\} \cup \{b_i\}$.

Case 4. Assume $C \in [0, A - B + D]$.

Using the same argument as in Case 3, it suffices to consider only countable sequences $\{d_i\}$. Note that it is implicitly assumed that $D \geqq B - A > 0$. First, assume $D = B - A$, this implies $C = 0$ and all a_i's are 0. Since $\sum(B - b_i) = B - A$, by Theorem 4.2, there exists a projection P such that $(B - A)P$ has diagonal $\{B - b_i\}$. Thus, $E = BI - (B - A)P$ has the desired spectrum and diagonal $\{b_i\}$. For the rest of Case 4 we may assume $D > B - A$.

Now, assume $C = 0$. Let $b_1 = \min\{b_i\}$ and $\eta_0 = b_1 - A$. We have

$$\sum_{i=2}^{\infty}(B - b_i) = D - (B - b_1) > B - A - B + b_1 = \eta_0.$$

So there is some N such that

$$\sum_{i=2}^{N}(B - b_i) \geqq \eta_0.$$

Apply Lemma 4.3 to $\{b_1\}$ and $\{b_i\}_{i=2}^{N}$ on the interval $[0, B]$ to obtain new sequences $\{\tilde{b}_1 = A\}$ and $\{\tilde{b}_i\}_{i=2}^{N}$. Let $\{\tilde{b}_i\}_{i=1}^{\infty}$ be the sequence consisting of $\{\tilde{b}_i\}_{i=1}^{N}$ and $\{b_i\}_{i=N+1}^{\infty}$. Let $\tilde{E}$ be the operator with $\{\tilde{b}_i\}$ on the diagonal (recall all a_i's are 0). Clearly, $\{A, B\} \subseteqq \sigma(\tilde{E}) \subseteqq [A, B]$. Using Lemma 4.4 there exists an operator E with the desired diagonal and spectrum.

Finally, we assume $C > 0$. Again, let $b_1 = \min\{b_i\}$. Fix any $0 < \varepsilon < \min(A, C)$. Since

$$\varepsilon + B - A < C + B - A \leqq D = \sum_{i=2}^{\infty}(B - b_i) + (B - b_1),$$

by subtracting $(B - b_1)$ from both sides we have

$$\varepsilon + b_1 - A < \sum_{i=2}^{\infty}(B - b_i).$$

Thus, there exists $M \geqq 2$ such that

$$\sum_{i=2}^{M}(B - b_i) > \varepsilon + b_1 - A.$$

Apply Lemma 4.3 to the sequences $\{b_1\}$ and $\{b_i\}_{i=2}^{M}$ on the interval $[0, B]$, with $\eta_0 = \varepsilon + b_1 - A$, to obtain sequences $\{\tilde{b}_1\}$ and $\{\tilde{b}_i\}_{i=2}^{M}$. By Lemma 4.3 we have

$$\tilde{b}_1 = b_1 - (\varepsilon + b_1 - A) = A - \varepsilon,$$

and

$$\sum_{i=2}^{M}(B - \tilde{b}_i) = \sum_{i=2}^{M}(B - b_i) - (\varepsilon + b_1 - A).$$

Let $\{\tilde{d}_i\}$ be the sequence consisting of $\{a_i\}$, $\{\tilde{b}_i\}_{i=1}^{M}$, and $\{b_i\}_{i=M+1}^{\infty}$. Set

$$\tilde{C} = \sum_{\tilde{d}_i < A} \tilde{d}_i = C + A - \varepsilon,$$

and

$$\begin{aligned}\tilde{D} = \sum_{\tilde{d}_i \geqq A}(B - \tilde{d}_i) &= \sum_{i=2}^{M}(B - \tilde{b}_i) + \sum_{i=M+1}^{\infty}(B - b_i)\\ &= D - (B - b_1) - (\varepsilon + b_1 - A) = D - B + A - \varepsilon.\end{aligned}$$

Observe that $\tilde{C} > A$. We also have

$$\tilde{C} = A + \varepsilon = \tilde{D} - D + B + C \leqq \tilde{D} - D + B + A - B + D = \tilde{D} + A,$$

so that $\tilde{C} \in [A, A + \tilde{D}]$. By the argument in Case 3, there is an operator $\tilde{E}$ with diagonal $\{\tilde{d}_i\}$ and the desired spectrum. By Lemma 4.4 there is an operator E unitarily equivalent to E with diagonal $\{d_i\}$. This completes the proof of Theorem 4.1. □

5. The non-summable case of the Pythagorean theorem

The goal of this section is to prove the necessity part of our main theorem. The summable case was already shown in Section 3. The non-summable case requires special arguments involving trace-class operators and Kadison's Theorem 4.2. In the terminology of Kadison [15], [16], this is a non-tight version of the Pythagorean theorem.

Theorem 5.1. *Suppose* $0 < A < B < \infty$. *Let* E *be a positive operator with* $\{A, B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A, B]$. *Let* $\{e_i\}_{i \in I}$ *be an orthonormal basis for* $\mathscr{H}$ *and* $d_i = \langle Ee_i, e_i \rangle$. *If*

$$C = \sum_{d_i < A} d_i < \infty \quad \text{and} \quad D = \sum_{d_i \geqq A} (B - d_i) < \infty, \tag{5.1}$$

then

$$C \in \bigcup_{n=0}^{\infty} [nA, A + B(n-1) + D].$$

Furthermore, $K := B(I - P) - E$ *is a positive trace class operator on* $\mathscr{H}$, *where* P *is the orthogonal projection onto* $\ker(E) \subseteqq \ker(K)$.

Observe that Theorem 5.1 does not require the assumption that $\{d_i\}$ is non-summable. However, if $\{d_i\}$ is summable, Theorem 5.1 gives only necessary, but not sufficient, condition, see Theorem 3.6.

Proof. We claim that it is sufficient to consider the case where $\{d_i\}$ is at most countable. The condition (5.1) implies that the sequence $\{d_i\}$ contains at most countably many terms in $(0, B)$. Thus, we only need consider sequences $\{d_i\}$ which contain an uncountable number of 0's or B's. Let $\{e_i\}_{i \in I}$ be the orthonormal basis with respect to which E has diagonal $\{d_i\}_{i \in I}$. Let $J = \{i : d_i = 0\} \cup \{i : d_i = B\}$. Since E is a positive operator with $\|E\| = B$, for each $i \in J$, e_i is an eigenvector of E. Let E' be E acting on $\overline{\operatorname{span}}\{e_i\}_{i \in I \setminus J}$. Note that E acting on $\overline{\operatorname{span}}\{e_i\}_{i \in J}$ is B times some projection Q. Thus, we have the orthogonal decomposition $E = E' \oplus BQ$. The operator E' has countable (possibly finite) diagonal consisting of the terms of $\{d_i\}$ contained in $(0, B)$. Thus, E' has the same values of C and D as E. If the conclusions of the theorem hold for E', then by $E = E' \oplus BQ$, they also hold for E.

By the above, we can take an indexing set to be $I = \mathbb{Z} \setminus \{0\}$. For convenience, we reorder the basis so that $d_i \in [A, B]$ for $i > 0$ and $d_i \in [0, A)$ for $i < 0$. The case when there are only finitely many $d_i \in [A, B]$, or $d_i \in [0, A)$, does not cause any extra difficulties, and it is left to the reader.

Let $k_i = \langle Ke_i, e_i \rangle$ and $n_i = \langle Pe_i, e_i \rangle$ be the diagonal entries of K and P, respectively. Observe that K is a positive operator and thus

$$k_i = B(1 - n_i) - d_i \geqq 0 \quad \text{for all } i \in \mathbb{Z} \setminus \{0\}. \tag{5.2}$$

Since $Bn_i \leqq B - d_i$, we have

$$\sum_{i=1}^{\infty} Bn_i \leqq \sum_{i=1}^{\infty} (B - d_i) \leqq D < \infty.$$

Hence,

$$\sum_{i=1}^{\infty} k_i = \sum_{i=1}^{\infty} (B - d_i) - B \sum_{i=1}^{\infty} n_i \leqq D < \infty. \tag{5.3}$$

Since $\sigma(E) \subseteqq \{0\} \cup [A, B]$, we have $A(I - P) \leqq E$. Thus, $B - Bn_i \leqq (B/A)d_i$, which immediately shows

$$\sum_{i=1}^{\infty}(B - Bn_{-i}) \leqq \frac{B}{A}\sum_{i=1}^{\infty} d_{-i} = \frac{BC}{A} < \infty.$$

Using (5.2),

$$\sum_{i=1}^{\infty} k_{-i} = \sum_{i=1}^{\infty}\big(B(1 - n_{-i}) - d_{-i}\big) \leqq \frac{BC}{A} - C < \infty. \tag{5.4}$$

Since K is a positive operator, (5.3) and (5.4) show that K is a trace class. Observe that the diagonal entries of P satisfy

$$a = \sum_{i=1}^{\infty} n_i < \infty \quad \text{and} \quad b = \sum_{i=1}^{\infty}(1 - n_{-i}) < \infty.$$

Despite the fact that the above splitting of $\{n_i\}$ may not be the same as in Theorem 4.2, it differs only by a finite number of terms from the standard splitting such that $n_i < \alpha$ for $i < 0$ and $n_i \geqq \alpha$ for $i > 0$, where $0 < \alpha < 1$. And this change does not affect the property of $a - b$ being an integer. Thus, by Theorem 4.2 applied to the projection P we have $n_0 := b - a \in \mathbb{Z}$. Using (5.3), and (5.4) again we have

$$\operatorname{tr}(K) = \sum_{i \in \mathbb{Z}\setminus\{0\}} k_i = D - C + Bn_0 \leqq D + \frac{BC}{A} - C. \tag{5.5}$$

This immediately yields the lower bound for C:

$$An_0 \leqq C. \tag{5.6}$$

Since $A \in \sigma(E)$ we know that $B - A$ is an eigenvalue of K and thus $(B - A) \leqq \operatorname{tr}(K)$. Again using (5.5) we see that

$$B - A \leqq D - C + Bn_0.$$

This yields the upper bound

$$C \leqq A + B(n_0 - 1) + D. \tag{5.7}$$

If $n_0 \geqq 0$ then (5.6) and (5.7) show that $C \in [n_0 A, A + B(n_0 - 1) + D]$ as desired. If $n_0 \leqq -1$ then $B(n_0 - 1) \leqq -B$ and thus (5.7) and the fact that $C \geqq 0$ shows $C \in [0, A - B + D]$ as desired. This completes the proof of Theorem 5.1. □

As a corollary of Theorems 2.4, 4.1, and 5.1 we obtain

Corollary 5.2. *Let $0 < A < B < \infty$ and $\{d_i\}_{i \in I}$ be a non-summable sequence in $[0, B]$. The following are equivalent*:

(i) $\{d_i\}_{i \in I}$ *satisfies* (4.1).

(ii) *There is a positive operator E on a Hilbert space* $\ell^2(I)$ *with*

$$\{A, B\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A, B]$$

and diagonal $\{d_i\}_{i \in I}$.

(iii) *There exists a frame* $\{f_i\}_{i \in I}$ *for some infinite dimensional Hilbert space* $\mathscr{H}$ *with optimal frame bounds A and B and* $d_i = \|f_i\|^2$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) follows directly from Theorems 4.1 and 5.1. Assume (ii). By Theorem 2.4, there exists a frame $\{f_i\}_{i \in I}$ with optimal frame bounds A and B and $d_i = \|f_i\|^2$. This frame lives on a Hilbert space $\mathscr{H}$ with $\dim \mathscr{H}$ equal to the rank of E. Since E is positive with infinite trace, $\mathscr{H}$ is infinite dimensional, which shows (iii). The implication (iii) $\Rightarrow$ (ii) similarly follows from Theorem 2.4. □

6. Without zero in the spectrum

The goal of this section is to establish an analogue of Theorem 1.1 for positive operators without zero in the spectrum. This result turns out to be less involved than our main theorem. As a consequence, we obtain a characterization of norms of Riesz bases with optimal bounds A and B. In the finite case, we obtain this result immediately from the Schur–Horn theorem.

Theorem 6.1. *Let* $0 < A \leqq B < \infty$. *Let* $\{d_i\}_{i=1}^{N+1}$ *be a sequence in* $[A, B]$. *There is a positive operator* $E : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ *with* $\{A, B\} \subseteqq \sigma(E) \subseteqq [A, B]$ *with diagonal* $\{d_i\}$ *if and only if*

$$\sum_{i=1}^{N+1} d_i \in [AN + B, A + BN]. \tag{6.1}$$

Without zero in the spectrum the diagonal must be in $[A, B]$, and thus there is no summable infinite dimensional case. We can reformulate the condition (6.1) to something that generalizes to the infinite dimensional case.

Corollary 6.2. *Let* $0 < A \leqq B < \infty$. *Let* $\{d_i\}_{i=1}^{N+1}$ *be a sequence in* $[A, B]$. *Define the numbers*

$$C = \sum_{i=1}^{N+1} (d_i - A), \quad D = \sum_{i=1}^{N+1} (B - d_i). \tag{6.2}$$

There is a positive operator $E : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}$ *with* $\{A, B\} \subseteqq \sigma(E) \subseteqq [A, B]$ *with diagonal* $\{d_i\}$ *if and only if*

$$C, D \geqq B - A. \tag{6.3}$$

Proof. The condition (6.1) implies

$$C = \sum_{i=1}^{N+1} d_i - (N+1)A \geqq AN + B - NA - A = B - A,$$

$$D = (N+1)B - \sum_{i=1}^{N+1} d_i \geqq NB + B - A - NB = B - A.$$

Conversely, it is also clear that these inequalities imply (6.1). □

We can now state the infinite dimensional case.

Theorem 6.3. *Let $0 < A \leqq B < \infty$. Let $\{d_i\}_{i \in I}$ be a sequence in $[A, B]$. Define*

$$C = \sum_{i \in I} (d_i - A), \quad D = \sum_{i \in I} (B - d_i). \tag{6.4}$$

There is a positive operator E with $\{A, B\} \subseteqq \sigma(E) \subseteqq [A, B]$ with diagonal $\{d_i\}$ if and only if

$$C, D \geqq B - A. \tag{6.5}$$

Proof. We can assume that I is countable, since the non-separable case follows from simple modifications as in the proof of Theorem 4.1. Suppose that E is a positive operator as in Theorem 6.3. First, we assume $D < \infty$. The operator $BI - E$ is a positive trace class with trace D. This implies that $D = \sum (B - \lambda)$, where the sum runs over all eigenvalues λ of E, repeated according to multiplicity. We also see that each $\lambda \in \sigma(E) \backslash \{B\}$ is an eigenvalue of E. Thus, A is an eigenvalue of E and $D \geqq B - A$. Next, we assume $C < \infty$. The operator $E - AI$ is a trace class with trace C. Since B is in the spectrum of E, it is an eigenvalue of E, and thus $C \geqq B - A$. Finally, if $C = D = \infty$, then (6.5) trivially holds.

Conversely, suppose that $\{d_i\}$ is a sequence in $[A, B]$ satisfying (6.5). If we assume $C, D > B - A$, then we can find some $N \in \mathbb{N}$ such that both

$$\sum_{i=1}^{N+1} (B - d_i) \geqq B - A \quad \text{and} \quad \sum_{i=1}^{N+1} (d_i - A) \geqq B - A.$$

By Corollary 6.2, there is an operator E_1 on an $N+1$-dimensional Hilbert space $\mathscr{H}_{N+1}$ such that $\{A, B\} \subseteqq \sigma(E_1) \subseteqq [A, B]$ and diagonal $\{d_i\}_{i=1}^{N+1}$. Let E_2 be the diagonal operator on the infinite dimensional Hilbert space $\mathscr{H}_\infty$ with $\{d_i\}_{i=N+2}^{\infty}$ on the diagonal. Now, $E = E_1 \oplus E_2$ on $\mathscr{H}_{N+1} \oplus \mathscr{H}_\infty$ is the desired operator. Next, we assume $D = B - A$. By Theorem 4.2 there is a rank 1 operator K with eigenvalue $B - A$ and diagonal $\{B - d_i\}_{i=1}^{\infty}$. Then, $E = BI - K$ is the desired operator. Finally, assume $C = B - A$. By Theorem 4.2 there is a rank 1 operator K with eigenvalue $B - A$ and diagonal $\{d_i - A\}_{i=1}^{\infty}$. Then, $E = K + AI$ is the desired operator. □

As a corollary of Theorem 2.5 we have

Corollary 6.4. *Let $0 < A \leqq B < \infty$ and $\{d_i\}$ be a sequence in $[A, B]$. There exists a Riesz basis $\{f_i\}$ with optimal bounds A and B and $d_i = \|f_i\|^2$ if and only if* (6.5) *holds.*

7. Examples

The goal of this section is to illustrate our main theorem. We start with the definition of the set of possible lower bounds of positive operators with a fixed diagonal.

Definition 7.1. Let $\{d_i\}_{i\in\mathbb{N}}$ be a given non-summable sequence in $[0,1]$. Define

$$\mathscr{A} = \{A \in (0,1] : \exists E \text{ positive with diagonal } \{d_i\}_{i\in\mathbb{N}} \text{ and } A \in \sigma(E) \subseteqq \{0\} \cup [A,1]\}.$$

Without loss of generality we can assume that $\sup d_i = 1$. Indeed, if $\sup d_i < 1$, then by Theorem 1.1 there exists a positive operator E with diagonal $\{d_i\}$ and $\{A,1\} \subseteqq \sigma(E) \subseteqq \{0\} \cup [A,1]$ for any $0 < A \leqq 1$. This fact can also be deduced from a result of Kornelson and Larson [19], Theorem 6. Thus, we have always $\mathscr{A} = (0,1]$ and this case is not interesting.

Example 1. Take any $0 < \beta < 1$ and define $d_i = 1 - \beta^i$ for $i \in \mathbb{N}$. First, we determine the set $\mathscr{A}$ near 0. We claim that

$$(7.1) \qquad \begin{aligned} (0, 1-\beta] &\subseteqq \mathscr{A} \quad \text{for } 1/2 \leqq \beta < 1, \\ \mathscr{A} \cap (0, 1-\beta] &= [(1-2\beta)/(1-\beta), 1-\beta] \quad \text{for } 0 < \beta < 1/2. \end{aligned}$$

Indeed, if $A \in (0, 1-\beta]$, we have $C = 0$ and $D = \sum_{i=1}^{\infty} \beta^i = \beta/(1-\beta)$. The condition (4.1) holds if and only if $A - 1 + D \geqq 0$ and thus $A \geqq (1-2\beta)/(1-\beta)$. This shows the first claim. Next, we claim

$$(7.2) \qquad \exists \delta = \delta(\beta) > 0 \quad (1-\delta, 1) \cap \mathscr{A} = \emptyset.$$

Moreover, $1 \in \mathscr{A}$ if and only if β is of the form $\beta = N/(N+1)$ for some $N \in \mathbb{N}$ by a simple application of Theorem 4.2.

Indeed, assume that $A \in (1-\beta^i, 1-\beta^{i+1}]$ for some $i \in \mathbb{N}$. Then,

$$C = i + \frac{\beta^{i+1} - \beta}{1-\beta}, \quad D = \frac{\beta^{i+1}}{1-\beta}.$$

Suppose that $C \in [nA, A + B(n-1) + D]$ for some $n \in \mathbb{N}$. Then,

$$(1-\beta^i)n \leqq An \leqq C \leqq A + B(n-1) + D \leqq n + \frac{\beta^{i+2}}{1-\beta}.$$

The upper bound on C yields $i \leqq n + \frac{\beta}{1-\beta} - \beta^{i+1}$ and thus $i \leqq n + \left\lfloor \frac{\beta}{1-\beta} \right\rfloor$. On the other hand, the lower bound $(1-\beta^i)\left(i - \left\lfloor \frac{\beta}{1-\beta} \right\rfloor\right) \leqq C$ yields

$$(7.3) \qquad i \geqq \left\{ \frac{\beta}{1-\beta} \right\} (\beta^{-i} - 1),$$

where $\{\cdot\}$ is the fractional part. Obviously, (7.3) must fail for sufficiently large i provided that $\beta \neq N/(N+1)$ for some $N \in \mathbb{N}$. In the special case of $\beta = N/(N+1)$, the upper bound on C actually yields $i \leqq n + \frac{\beta}{1-\beta} - 1$. A similar argument as before shows that the lower bound for C must fail for sufficiently large i (depending on β). Therefore, in either case we have (7.2).

Finally, we claim that

$$\mathscr{A} = [(1-2\beta)/(1-\beta), 1-\beta] \quad \text{for } 0 < \beta < 1/2. \tag{7.4}$$

By (7.1), it suffices to consider $A > 1-\beta$. Since (7.3) fails for $0 < \beta < 1/2$ and $i \geqq 2$, we have that $(1-\beta^2, 1) \cap \mathscr{A} = \emptyset$. Moreover, $1 \notin \mathscr{A}$ by Theorem 4.2. Finally, if $A \in (1-\beta, 1-\beta^2]$, then $C = 1-\beta$, $D = \frac{\beta^2}{1-\beta}$. It is easy to see that $A - 1 + D < C < A$. Thus, $\mathscr{A} \cap (1-\beta, 1-\beta^2] = \emptyset$, which shows (7.4).

Example 2. Let $\beta \approx 0.57$ be the real root of $\beta^3 - (1-\beta)^2 = 0$, and take $d_i = 1 - \beta^i$ for $i \in \mathbb{N}$. We will show that

$$\mathscr{A} = (0, 1-\beta] \cup \left[1-\beta^2, \frac{1}{3}(2+2\beta-\beta^2)\right]. \tag{7.5}$$

By previous consideration we have $(0, 1-\beta] \subseteqq \mathscr{A}$. Moreover, a simple numerical calculation shows that the inequality (7.3) fails for $i \geqq 5$. Thus, $(1-\beta^5, 1] \cap \mathscr{A} = \emptyset$.

Assume that $A \in (1-\beta, 1-\beta^2)$. We have $C = 1-\beta$ and $D = \frac{\beta^2}{1-\beta}$. Note that $C < A$, but

$$A - 1 + D < \frac{\beta^2}{1-\beta} - \beta^2 = \frac{\beta^3}{1-\beta} = 1 - \beta = C$$

and thus $\mathscr{A} \cap (1-\beta, 1-\beta^2) = \emptyset$. But, if $A = 1-\beta^2$ then we have $A - 1 + D = C$, so that $1 - \beta^2 \in \mathscr{A}$.

Next, assume that $A \in (1-\beta^2, 1-\beta^3]$. We have

$$C = 2 - \beta - \beta^2 \quad \text{and} \quad D = \frac{\beta^3}{1-\beta} = 1 - \beta.$$

Since $\beta < 3/5$ we see that $2\beta < 2 - \beta$ and

$$A \leqq 1 - \beta^3 = 2\beta - \beta^2 < 2 - \beta - \beta^2 = C.$$

Now,

$$A + D \geqq 1 - \beta^2 + 1 - \beta = C$$

so that $C \in [A, A+D]$ and $(1-\beta^2, 1-\beta^3] \subseteqq \mathscr{A}$. A similar calculation shows that $(1-\beta^3, 1-\beta^4] \subseteqq \mathscr{A}$.

Now, assume that $A \in \left(1-\beta^4, \frac{1}{3}(2+2\beta-\beta^2)\right]$, we have $C = 2+2\beta-\beta^2$, so that $3A \leqq C$. We have $D = 2\beta - 1$, and using the fact that $\beta > 1/2$ we easily see that

$$A + 2 + D \geqq 1 - \beta^4 + 2 + 2\beta - 1 = 1 + 3\beta + \beta^2 \geqq 2 + 2\beta - \beta^2 = C.$$

Thus $C \in [3A, A + 2B + D]$ and $\left(1-\beta^4, \frac{1}{3}(2+2\beta-\beta^2)\right] \subseteqq \mathscr{A}$. Finally, assume

$$A \in \left(\frac{1}{3}(2+2\beta-\beta^2), 1-\beta^5\right].$$

Again, we have $C = 2+2\beta-\beta^2$, so that $3A > C$. Using the numerical estimates $\beta \in \left(\frac{1}{2}, \frac{3}{5}\right)$ we easily obtain $2A \leqq C$. However,

$$A + 1 + D \leqq 1 - \beta^5 + 1 + 2\beta - 1 = 2 - \beta + 2\beta^2 < 2 + 2\beta - \beta^2 = C$$

which shows that $C \in (A+1+D, 3A)$ and thus $\left(\frac{1}{3}(2+2\beta-\beta^2), 1-\beta^5\right] \cap \mathscr{A} = \emptyset$. This shows (7.5).

In general, determining the set $\mathscr{A}$ for sequences satisfying (5.1) is not an easy task since it boils down to checking condition (4.1) for all possible values of $0 < A < 1$. This often leads to computing countably many infinite series (1.2) and verifying whether (4.1) holds or not. In the above examples involving geometric series this task actually reduces to checking a finite number of conditions using properties (7.1) and (7.2). Nevertheless, we have the following general fact about $\mathscr{A}$:

Theorem 7.1. *Let* $\{d_i\}_{i\in\mathbb{N}} \subseteqq [0,1]$ *with* $\sup d_i = 1$. *The set* $\mathscr{A} \cup \{0,1\}$ *is closed.*

Proof. For any $A \in (0,1]$ define the numbers

$$C(A) = \sum_{d_i < A} d_i, \quad D(A) = \sum_{d_i \geqq A} (1 - d_i).$$

By Theorem 1.1, $A \notin \mathscr{A}$ if and only if $C(A), D(A) < \infty$ and

$$\exists n \in \mathbb{N} \quad A + n - 2 + D(A) < C(A) < An. \tag{7.6}$$

Let $A_0 \in (0,1) \backslash \mathscr{A}$. First, assume $A_0 \neq d_i$ for all $i \in \mathbb{N}$. This implies there is some $\varepsilon > 0$ such that for all $A \in (A_0 - \varepsilon, A_0 + \varepsilon)$ we have $C(A) = C(A_0)$ and $D(A) = D(A_0)$. By continuity, there exists $\delta > 0$ such that (7.6) holds for $|A - A_0| < \delta$. Thus, $(A_0 - \delta, A_0 + \delta) \cap \mathscr{A} = \emptyset$.

Now, assume $A_0 = d_i$ for some $i \in \mathbb{N}$, and let $k \in \mathbb{N}$ be the number of terms in the sequence $\{d_i\}$ equal to A_0. There is some $\varepsilon > 0$ such that $(A_0 - \varepsilon, A_0 + \varepsilon)$ contains no

$d_i \neq A_0$. Note that for $A \in (A - \varepsilon, A_0]$ we have $C(A) = C(A_0)$ and $D(A) = D(A_0)$. The same argument as above shows that there is some $\delta > 0$ such that $(A_0 - \delta, A_0] \cap \mathscr{A} = \emptyset$. Finally, for each $A \in (A_0, A_0 + \varepsilon)$ we have

$$C(A) = C(A_0) + kA_0 \quad \text{and} \quad D(A) = D(A_0) - k + kA_0,$$

and (7.6) is equivalent to

$$\exists n \in \mathbb{N} \quad A + n - k - 2 + D(A_0) < C(A_0) < A(n-k) + (A - A_0)k.$$

Since (7.6) holds for $A = A_0$ with $n = n_0$, the above holds with $n = n_0 + k$ and $A \in (A_0, A_0 + \delta)$ for some $\delta > 0$. This shows that $(A_0, A_0 + \delta) \cap \mathscr{A} = \emptyset$. □

We end this section by comparing our results with the characterization of the closure of the collection of diagonals of a self-adjoint operator due to A. Neumann [20]. While Neumann's results also apply to non-diagonalizable operators, see [20], Section 4, they take the simplest form for diagonalizable operators.

Suppose that E is a diagonalizable operator with the eigenvalue list $\Lambda(E) = \{\lambda_i\}_{i \in \mathbb{N}}$. Let $\mathfrak{S} = \mathfrak{S}(\mathbb{N})$ be the group of bijections on $\mathbb{N}$. Let $\mathscr{D}(E)$ be the set of all possible diagonals of E. Then, Neumann's result asserts that

$$\overline{\mathscr{D}(E)}^{\infty} = \overline{\operatorname{conv} \mathfrak{S}.\Lambda(E)}^{\infty}, \tag{7.7}$$

with the closure taken in ℓ^∞-norm. For example, take an operator E_0 with 3 point spectrum $\{0, A, B\}$, such that the eigenvalue A has finite multiplicity, and 0 and B have infinite multiplicities. A simple calculation shows that the sequence $d = (0, B, 0, B, \ldots)$ belongs to the closure of the convex hull in (7.7). However, d can not be a diagonal of E_0 in light of Theorem 1.1. In fact, any operator E with $\sigma(E) \subset [0, B]$ and diagonal d is actually a diagonal operator. Thus, $A \notin \sigma(E) = \{0, B\}$.

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Eingegangen 22. April 2009, in revidierter Fassung 29. Januar 2010