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The Closure of the Set of Tight Frame Wavelets

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Abstract We study properties of the closure of the set of tight frame wavelets. We give a necessary condition and a sufficient condition for a function to be in this closure. In particular, we show that the collection of tight frame wavelets is not dense in $L^2(\mathbb{R}^n)$, which answers a question posed by D. Han and D. Larson (Preprint, 2008).

Keywords Tight frame wavelet · Framelet · Expansive dilation

Mathematics Subject Classification (2000) Primary: 42C40

1 Introduction

One of the fundamental areas of the theory of wavelets is the investigation of properties of the collection of all wavelets as a subset of $L^2(\mathbb{R})$. The most prominent problem in this area asks whether the collection of all orthonormal wavelets (as a subset of the unit sphere) is connected in $L^2(\mathbb{R})$ norm, see [7, 15, 16, 19, 20]. A related problem of connectivity of Parseval wavelets also attracted lots of attention, see [9–11, 17, 18].

While these two problems remain open so far, there have been recent progress on the study of the collection $\mathcal{W}$ of all frame wavelets (which are not necessarily tight). The author [4] proved that the set $\mathcal{W}$ is path connected in the usual $L^2(\mathbb{R})$ norm and also in "multiplicative" $L^2(\mathbb{R}, d\xi/|\xi|)$ norm. Recall that the space $L^2(\mathbb{R}, d\xi/|\xi|)$ consists of all functions f such that

$$\|f\|_{L^2(\mathbb{R},d\xi/|\xi|)} = \left(\int_{\mathbb{R}} |\hat{f}(\xi)|^2 \frac{d\xi}{|\xi|}\right)^{1/2} < \infty.$$

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Moreover, the set of frame wavelets $\mathcal{W}$ was shown to be dense as a subset $L^2(\mathbb{R})$ and simultaneously as a subset of $L^2(\mathbb{R}, d\xi/|\xi|)$. A similar density result was independently obtained by Cabrelli and Molter [5]. Finally, Han and Larson [12] recently showed that any $f \in L^2(\mathbb{R})$ can be approximated in $L^2(\mathbb{R})$-norm by a sequence $\{\psi_k\}_{k\in\mathbb{N}} \subset \mathcal{W}$ of asymptotically tight frame wavelets. Namely, if $0 < c_k \le d_k < \infty$ denote the lower and the upper frame bounds of ψ_k, then $d_k/c_k \to 1$ as $k \to \infty$. This naturally leads to the question of density of tight frame wavelets in $L^2(\mathbb{R})$.

In this paper we show that the set of tight frame wavelets is not dense in $L^2(\mathbb{R})$. In fact, we show that a necessary condition for $f \in L^2(\mathbb{R})$ to be in the closure of this set is one of familiar equations characterizing Parseval wavelets, i.e.,

$$\sum_{j=0}^{\infty} \hat{f}(2^j\xi)\overline{\hat{f}(2^j(\xi+q))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R} \text{ and } q \in 2\mathbb{Z}+1.$$

While we do not know whether this condition is also sufficient, we give a partial answer to this problem. We prove that any $f \in L^2(\mathbb{R})$ with sufficiently small support in the Fourier domain, i.e.,

$$|W \cap (k+W)| = 0 \quad \text{for all } k \in \mathbb{Z}\setminus\{0\}, \text{ where } W = \operatorname{supp} \hat{f},$$

belongs to the closure of the set of tight frame wavelets. Finally, we show that in the setting of real dilation factors a such that $a^j \notin \mathbb{Q}$ for all $j \in \mathbb{N}$, the above condition actually characterizes the closure of the set of tight framelets associated with a.

2 Results

Despite that our results are motivated by the classical case of dyadic wavelets in $\mathbb{R}$, we will consider a more general setting of real expansive dilations in higher dimension. An $n \times n$ real matrix A is said to be *expansive* if all eigenvalues λ of A satisfy $|\lambda| > 1$. A frame wavelet (aka framelet) is a function $\psi \in L^2(\mathbb{R}^n)$, such that the system

$$\psi_{j,k}(x) = |\det A|^{j/2}\psi(A^j x - k), \quad j \in \mathbb{Z},\ k \in \mathbb{Z}^n,$$

forms a frame for $L^2(\mathbb{R}^n)$. Hence, we require the existence of constants $0 < c_0 \le c_1 < \infty$ such that

$$c_0\|f\|^2 \le \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^n} |\langle f, \psi_{j,k}\rangle|^2 \le c_1\|f\|^2 \quad \text{for all } f \in L^2(\mathbb{R}^n). \tag{2.1}$$

We say that ψ is a tight frame wavelet if $c_0 = c_1$ or a Parseval wavelet if $c_0 = c_1 = 1$.

Let $\mathcal{W} = \mathcal{W}_A$ be the set of all framelets

$$\mathcal{W}_A = \{\psi \in L^2(\mathbb{R}^n) : \psi \text{ is a tight framelet}\}.$$

Every $\psi \in \mathcal{W}_A$ with a frame constant $c > 0$ is characterized by the two fundamental equations:

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(B^j\xi)|^2 = c \quad \text{for a.e. } \xi \in \mathbb{R}^n, \tag{2.2}$$

$$\sum_{j\in\mathbb{Z}:\ \alpha\in B^j\mathbb{Z}^n} \hat{\psi}(B^{-j}\xi)\overline{\hat{\psi}(B^{-j}(\xi+\alpha))} = 0 \quad \text{for a.e. } \xi \in \mathbb{R}^n \text{ and all } \alpha \in \mathbb{Z}^n\setminus\{0\}, \tag{2.3}$$

where $B = A^T$. The proof of this fact in various degrees of generality can be found in several works such as [8, 14] for dyadic dilations, [1] for integer dilations, and [6, 13] for real dilations.

Our first result gives a necessary condition for a function $f \in L^2(\mathbb{R}^n)$ to be in the closure of the set of tight framelets $\overline{\mathcal{W}_A}$. It is a consequence of the following useful lemma which can be traced back to the work of Garrigós and Speegle [9].

Lemma 2.1 *Let $\alpha \in \mathbb{R}^n \setminus \{0\}$. For any $f \in L^2(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, define*

$$T_\alpha(f)(\xi) = \sum_{j\in\mathbb{Z}:\ \alpha \in B^j\mathbb{Z}^n} \hat{f}(B^{-j}\xi)\overline{\hat{f}(B^{-j}(\xi+\alpha))}. \tag{2.4}$$

Then, $T_\alpha : L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is a well-defined continuous (non-linear) operator.

Proof For any $f \in L^2(\mathbb{R}^n)$ and $\alpha \in \mathbb{R}^n$, and $J \in \mathbb{N}$,

$$\int_{\mathbb{R}^n} \sum_{j\le J} |\hat{f}(B^{-j}(\xi+\alpha))|^2 d\xi = \int_{\mathbb{R}^n} \sum_{j\le J} |\det A|^j |\hat{f}(\xi)|^2 d\xi = \frac{|\det A|^{J+1}}{|\det A|-1}\|f\|^2 < \infty. \tag{2.5}$$

By $2|zw| \le |z|^2 + |w|^2$ for $z, w \in \mathbb{C}$, we have for a.e. $\xi \in \mathbb{R}^n$,

$$\begin{aligned} |T_\alpha(f)(\xi)| &\le \sum_{j\in\mathbb{Z}:\ \alpha \in B^j\mathbb{Z}^n} |f(B^{-j}\xi)\overline{\hat{f}(B^{-j}(\xi+\alpha))}| \\ &\le \frac{1}{2}\sum_{j\in\mathbb{Z}:\ \alpha \in B^j\mathbb{Z}^n} |\hat{f}(B^{-j}\xi)|^2 + \frac{1}{2}\sum_{j\in\mathbb{Z}:\ \alpha \in B^j\mathbb{Z}^n} |\hat{f}(B^{-j}(\xi+\alpha))|^2. \end{aligned} \tag{2.6}$$

Since the dilation B is expansive, there exists $J \in \mathbb{N}$ (depending only on the choice of $\alpha \ne 0$) such that $j \in \mathbb{Z}$ and $\alpha \in B^j\mathbb{Z}^n$ implies that $j \le J$. Therefore, the sums in (2.6) can admit only $j \in \mathbb{Z}$ such that $j \le J$. Integrating (2.6) over $\mathbb{R}^n$ and using (2.5) we have

$$\|T_\alpha(f)\|_{L^1(\mathbb{R}^n)} \le \frac{|\det A|^{J+1}}{|\det A|-1}\|f\|^2.$$

Thus, the series defining $T_\alpha(f)(\xi)$ converges absolutely for a.e. ξ, and the operator $T_\alpha : L^2(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is bounded. □

Theorem 2.1 *Assume that A is an expansive real dilation. Then, the closure of $\mathcal{W}_A$ in the $L^2(\mathbb{R}^n)$-norm satisfies*

$$\overline{\mathcal{W}_A} \subset \mathcal{Z}_A := \{f \in L^2(\mathbb{R}^n) : T_\alpha(f) = 0 \text{ for all } \alpha \in \mathbb{Z}^n \setminus \{0\}\}.$$

Proof The inclusion $\subset$ follows immediately by Lemma 2.1. Indeed, for any $\psi \in \mathcal{W}_A$ we have $T_\alpha(\psi) = 0$ for all $\alpha \in \mathbb{Z}^n \setminus \{0\}$ due to (2.3). Thus, the same must hold for any function in the closure due to the boundedness of T_α's. □

As an immediate corollary of Theorem 2.1 we can answer a question posed by Han and Larson [12].

Corollary 2.1 *The collection $\mathcal{W}_A$ is not dense in $L^2(\mathbb{R}^n)$.*

Proof Take any function $f \in L^2(\mathbb{R}^n)$ such that $\hat{f}(\xi) \geq 0$ for a.e. ξ, and

$$|W \cap (\alpha + W)| > 0 \quad \text{for some } \alpha \in \mathbb{Z}^n \setminus \{0\}, \text{ where } W = \operatorname{supp} \hat{f}.$$

Then, $T_{-\alpha}(f)(\xi) \geq \hat{f}(\xi)\hat{f}(\xi - \alpha) > 0$ on a set of positive measure. Thus, any such f does not belong to $\overline{\mathcal{W}_A}$ by Theorem 2.1. □

One could ask whether the converse inclusion $\mathcal{Z}_A \subset \overline{\mathcal{W}_A}$ holds in Theorem 2.1 as well. As we will see in Theorem 2.3 this is in general false for arbitrary real dilations. However, it remains an open problem whether the inclusion $\mathcal{Z}_A \subset \overline{\mathcal{W}_A}$ holds for integer dilations A, that is when $A\mathbb{Z}^n \subset \mathbb{Z}^n$.

The following result provides a partial answer to this problem. Theorem 2.2 says that if a function $f \in \mathcal{Z}_A$ has sufficiently small support in the Fourier domain, then the answer is yes.

Theorem 2.2 *Suppose $f \in L^2(\mathbb{R}^n)$ has the property that*

$$|W \cap (k + W)| = 0 \quad \textit{for all } k \in \mathbb{Z}^n \setminus \{0\}, \textit{ where } W = \operatorname{supp} \hat{f}. \tag{2.7}$$

Then, $f \in \overline{\mathcal{W}_A}$.

Proof Note that the property (2.7) automatically implies that $f \in \mathcal{Z}_A$. Indeed, take any $\alpha \in \mathbb{Z}^n \setminus \{0\}$. Equation (2.7) implies that for a.e. $\xi \in \mathbb{R}^n$, either $\hat{f}(B^{-j}\xi) = 0$ or $\hat{f}(B^{-j}(\xi + \alpha)) = 0$ since $B^{-j}\alpha \in \mathbb{Z}^n \setminus \{0\}$. Thus, $T_\alpha f(\xi) = 0$.

For $N \in \mathbb{N}$ define the set

$$E_N = \{\xi \in \mathbb{R}^n : |\hat{f}(\xi)| \geq N\} \cap \{\xi \in \mathbb{R}^n : |\xi| \leq 1/N \text{ or } |\xi| \geq N\}. \tag{2.8}$$

By Chebyshev's inequality $|E_N| \to 0$ as $N \to \infty$. Thus, $\mathbf{1}_{E_N}(\xi) \to 0$ for a.e. ξ as $N \to \infty$. Since the dilation B is expansive we can find an ellipsoid Δ satisfying $\Delta \subset B(\Delta)$, e.g. see [3, Lemma 2.2]. Let $F_N = 2^{-N}\Delta$ be a dilated copy of Δ, and $F_N^P = \bigcup_{k \in \mathbb{Z}^n}(F_N + k)$ be the periodization of F_N. Since $|F_N| \to 0$ as $N \to \infty$, we have $\mathbf{1}_{F_N^P}(\xi) \to 0$ for a.e. ξ as $N \to \infty$.

Finally, define the function $f_N \in L^2(\mathbb{R}^n)$ by

$$\hat{f}_N(\xi) = \mathbf{1}_{(E_N \cup F_N^P)^c}(\xi)\hat{f}(\xi). \tag{2.9}$$

By the Lebesgue Dominated Convergence Theorem $f_N \to f$ in L^2-norm as $N \to \infty$. This is due to the fact that $\mathbf{1}_{(E_N \cup F_N^P)^c}(\xi) \to 1$ for a.e. ξ as $N \to \infty$. Furthermore, $f_N \in \mathcal{Z}_\alpha$ for any $\alpha \in \mathbb{Z}^n \setminus \{0\}$. This follows from the assumption that f satisfies (2.7) and $\operatorname{supp} \hat{f}_N \subset \operatorname{supp} \hat{f}$. Thus, it remains to modify f_N in such way that it becomes a tight frame wavelet.

Define the annulus $G_N = F_N \setminus (B^{-1}F_N)$. Since ellipsoid F_N expands under the action of the expansive dilation B, the family $\{B^j(G_N)\}_{j \in \mathbb{Z}}$ is a partition of $\mathbb{R}^n$ (modulo the origin). By [1, Lemma 2.3] we have,

$$\#\{j \in \mathbb{Z} : 1/N < |B^j\xi| < N\} \leq \log_\lambda(N^2/c) \quad \text{for all } \xi \in \mathbb{R}^n,$$

where the constants $\lambda > 1$ and $c > 0$ are chosen such that $|B^j\xi| \geq c\lambda^j|\xi|$ for all $j > 0$. Define the function $\varphi_N(\xi) = \sum_{j \in \mathbb{Z}} |\hat{f}_N(B^j\xi)|^2$. Thus, using (2.8) and (2.9) we have $\operatorname{supp} \hat{f}_N \subset \{\xi : 1/N < |\xi < N\}$, and

$$\|\varphi_N\|_{L^\infty} \leq N^2 \log_\lambda(N^2/c) \leq CN^3. \tag{2.10}$$

Define the function $g_N \in L^2(\mathbb{R}^n)$ by

$$\hat{g}_N(\xi) = \sqrt{\|\varphi_N\|_\infty - \varphi_N(\xi)}\mathbf{1}_{G_N}(\xi). \tag{2.11}$$

Finally, let $h_N = f_N + g_N$. Since the supports of $\hat{f}_N$ and $\hat{g}_N$ are disjoint

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |\hat{h}_N(B^j\xi)|^2 &= \sum_{j\in\mathbb{Z}} |\hat{f}_N(B^j\xi)|^2 + \sum_{j\in\mathbb{Z}} (\|\varphi_N\|_\infty - \varphi_N(B^j\xi))\mathbf{1}_{G_N}(B^j\xi) \\ &= \varphi_N(\xi) + (\|\varphi_N\|_\infty - \varphi_N(\xi)) \sum_{j\in\mathbb{Z}} \mathbf{1}_{G_N}(B^j\xi) = \|\varphi_N\|_\infty,\end{aligned}$$

where in the penultimate step we used the dilation periodicity of φ_N. Thus, h_N satisfies the Calderón formula (2.2). In addition, h_N satisfies (2.7) since $\operatorname{supp}\hat{h}_N \subset \operatorname{supp}\hat{g}_N \cup \operatorname{supp}\hat{f}_N \subset F_N \cup (\operatorname{supp}\hat{f} \setminus F_N^P)$. Thus, (2.3) holds and $h_N \in \mathcal{W}_A$. Furthermore, by (2.10) and (2.11)

$$\|g_N\|_2^2 \le \|\varphi_N\|_\infty |G_N| \le CN^3 |2^{-N}\Delta| = CN^3 2^{-nN} |\Delta| \to 0 \quad \text{as } N \to \infty.$$

Therefore, $h_N = f_N + g_N \to f$ in L^2-norm as $N \to \infty$, which completes the proof of Theorem 2.2. □

Despite that Theorem 2.2 is in general only a partial converse of Theorem 2.1, for certain dilations it is actually a full converse.

Theorem 2.3 *Assume that A is an expansive real dilation such that $B = A^T$ satisfies*

$$\mathbb{Z}^n \cap B^j(\mathbb{Z}^n) = \{0\} \quad \textit{for all } j \in \mathbb{Z} \setminus \{0\}. \tag{2.12}$$

Then, $\overline{\mathcal{W}_A} = \{f \in L^2(\mathbb{R}^n) : f$ satisfies (2.7)$\}$.

Proof Suppose that $\psi \in \mathcal{W}_A$. The assumption (2.12) implies that the sum in (2.3) consists of only one term corresponding to $j = 0$. Hence, (2.3) yields $\hat{\psi}(\xi)\overline{\hat{\psi}(\xi+\alpha)} = 0$ for a.e. ξ and $\alpha \in \mathbb{Z}^n \setminus \{0\}$. This shows that any $\psi \in \mathcal{W}_A$ satisfies (2.7).

Suppose now that we have a sequence of functions $\{f_N\}_{N\in\mathbb{N}}$ each satisfying (2.7) and converging in L^2-norm to some $f \in L^2(\mathbb{R}^n)$. By choosing a subsequence, $\hat{f}_N(\xi) \to \hat{f}(\xi)$ for a.e. ξ as $N \to \infty$. Thus, for any $k \in \mathbb{Z}^n \setminus \{0\}$,

$$0 = \hat{f}_N(\xi)\hat{f}_N(\xi - k) \to \hat{f}(\xi)\hat{f}(\xi - k) \quad \text{for a.e. } \xi \text{ as } N \to \infty.$$

Consequently, f must satisfy (2.7) as well. This shows that the space consisting of functions satisfying (2.7) is closed. Therefore, we have the inclusion $\subset$ in Theorem 2.3. The converse inclusion $\supset$ was established by Theorem 2.1. □

Remark 2.1 The dilations A satisfying (2.12) are quite pathological in the sense that they are farthest from preserving the lattice $\mathbb{Z}^n$, i.e., $A\mathbb{Z}^n \subset \mathbb{Z}^n$. Indeed, any dilation A in this class admits only minimally supported frequency (MSF) wavelets, see [2, 6]. Thus, the famous problem of connectivity of the set of orthogonal wavelets is reduced to the connectivity of MSF wavelets established by Speegle [19]. Likewise, the parallel problem of connectivity of $\mathcal{W}_A$ can be easily solved as shown below.

Theorem 2.4 *Assume that A is an expansive real dilation as in Theorem 2.3. Then, $\mathcal{W}_A$ is pathwise connected as a subset of $L^2(\mathbb{R}^n)$.*

Proof Take any $\psi_0 \in \mathcal{W}_A$, and let $W = \operatorname{supp} \hat{\psi}_0$. By a simple scaling, we can also assume that ψ_0 is a Parseval wavelet. By Theorem 2.3, W satisfies (2.7). Let W_1 be any subset of W such that $\{B^j(W_1)\}_{j\in\mathbb{Z}}$ is a partition of $\mathbb{R}^n$ (modulo null sets). Define $\psi_1 \in L^2(\mathbb{R}^n)$ by

$$\hat{\psi}_1(\xi) = i\frac{\hat{\psi}_0(\xi)}{|\hat{\psi}_0(\xi)|}\mathbf{1}_{W_1}(\xi).$$

Clearly, ψ_1 is an MSF (minimally supported frequency) Parseval wavelet. Finally define a continuous path $\{\psi_t\}_{0\le t\le 1}$ by $\psi_t = \sqrt{1-t}\,\psi_0 + \sqrt{t}\,\psi_1$. Since $\operatorname{supp} \hat{\psi}_t \subset W$ for all $0 \le t \le 1$, it suffices to verify that ψ_t satisfies the Calderón condition (2.2) to be a Parseval wavelet. Note that

$$\begin{aligned}|\hat{\psi}_t(\xi)|^2 &= (1-t)|\hat{\psi}_0(\xi)|^2 + t|\hat{\psi}_1(\xi)|^2 + 2\sqrt{(1-t)t}\,\Re(\hat{\psi}_0(\xi)\overline{\hat{\psi}_1(\xi)}) \\ &= (1-t)|\hat{\psi}_0(\xi)|^2 + t|\hat{\psi}_1(\xi)|^2,\end{aligned}$$

since $\Re(\hat{\psi}_0(\xi)\overline{\hat{\psi}_1(\xi)}) = \Re(-i|\hat{\psi}_0(\xi)|) = 0$ for $\xi \in W_1$. Thus,

$$\sum_{j\in\mathbb{Z}}|\hat{\psi}_t(B^j\xi)|^2 = (1-t)\sum_{j\in\mathbb{Z}}|\hat{\psi}_0(B^j\xi)|^2 + t\sum_{j\in\mathbb{Z}}|\hat{\psi}_1(B^j\xi)|^2 = 1 \quad \text{for a.e. } \xi \in \mathbb{R}^n.$$

This construction gives a continuous path joining any $\psi \in \mathcal{W}_A$ with an MSF Parseval wavelet ψ_1. Finally, the collection of all MSF Parseval wavelets is pathwise connected due to result of Paluszyński, Šikić, Weiss, and Xiao [18]. The result in [18] is stated for dyadic wavelets in one dimension, but the same argument generalizes to higher dimensions as well. This is due to the fact the original result of Speegle [19] on the connectivity of MSF orthogonal wavelets works in the setting of real expansive dilations and the corresponding result for MSF Parseval wavelets requires only relatively simple modifications. □

Remark 2.2 Note that the fact that the dilation A satisfies (2.12) is used only once in the beginning of the proof of Theorem 2.4. Hence, the above argument shows that the set of Parseval wavelets satisfying (2.7) is pathwise connected for any dilation A. In the classical case of $A = 2$, this easily follows from a result of Garrigós, Hernández, Šikić, Soria, Weiss, and Wilson [10] asserting connectivity for the subset of Parseval wavelets with sparse enough $\operatorname{supp} \hat{\psi}$.

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