The Journal of Fourier Analysis and Applications
Volume 7, Issue 5, 2001

The Construction of r-Regular Wavelets for Arbitrary Dilations

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Communicated by Y. Wang

ABSTRACT. *Given a dilation matrix A and a natural number r we construct an associated r-regular multiresolution analysis with r-regular wavelet basis. Here a dilation is an $n \times n$ expansive matrix A (all eigenvalues λ of A satisfy $|\lambda| > 1$) with integer entries. This extends a theorem of Strichartz which assumes the existence of a self-affine tiling associated with the dilation A. We also prove that regular wavelets have vanishing moments.*

1. Introduction

The main aim of this article is the construction of r-regular wavelet family with an associated r-regular multiresolution analysis for an arbitrary dilation matrix A preserving some lattice Γ. Strichartz [31] achieved this goal for a wide class of dilations having a Haar type wavelet basis, or equivalently a self-affine tiling, see [17].

Theorem 1 (Strichartz).
Assume the existence of a self-affine tiling. For every r there exists an r-regular multiresolution analysis and an associated wavelet basis.

We extend this result by removing the assumption of the existence of a self-affine tiling. This assumption is highly non-trivial and it was studied by many authors, see [13, 16], [19]–[25], [32]. Using methods of algebraic number theory classes of dilations without self-affine tilings have been recently found in dimensions $n \geq 4$, see [22, 30]. The simplest example is a 4×4 dilation matrix

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 \\ -1 & 0 & -1 & 1 \end{pmatrix}$$

that does not have a self-affine tiling and thus does not have a Haar-type wavelet basis, see [22]. Therefore it is of interest to extend Strichartz's construction to such cases.

Math Subject Classifications. 42C15.
Keywords and Phrases. wavelet, multiresolution analysis, r-regular function, Strang–Fix condition.

ISSN 1069-5869

The wavelets constructed in this article are not compactly supported nor of exponential decay, as is the case with Strichartz's. Nothing is known about the existence of compactly supported or exponential-type regular wavelets for a general dilation. In fact, it is not even known whether there are *any* compactly supported orthogonal wavelets associated with the above 4×4 dilation matrix. The compactly supported regular wavelets have been constructed only for very specific dilations in higher dimensions. One method is to use a tensoring technique to obtain separable wavelets from Daubechies wavelets. There are other methods for the construction of nonseparable wavelets, see [3, 4]. However, r-regular wavelets are a sufficient tool for the study of function spaces as is evidenced by the book of Meyer [28] in the isotropic setting and [8] in the case of anisotropic Hardy spaces. Furthermore, as a special benefit, all the moments of the constructed wavelets vanish.

In Section 3 it is shown that all r-regular wavelets must necessarily have a certain number of vanishing moments depending on r and the spectral properties of a dilation A. This was previously observed for the dilation $A = 2Id$ in [5, 15, 28].

Wavelet preliminaries. We are going to assume that we have a lattice Γ ($\Gamma = P\mathbb{Z}^n$ for some nondegenerate $n \times n$ matrix P) and an *expansive* matrix A preserving Γ, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$, and $A\Gamma \subset \Gamma$. Without loss of generality, we will assume that $\Gamma = \mathbb{Z}^n$.

Definition 1. Let Ψ be a finite family of functions $\Psi = \{\psi^1, \dots, \psi^L\} \subset L^2(\mathbb{R}^n)$. We say that Ψ is a *wavelet family* (or a *multiwavelet*) if $\{\psi^l_{j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^n, l = 1, \dots, L\}$ is an orthonormal basis for $L^2(\mathbb{R}^n)$. Here, for $\psi \in L^2(\mathbb{R}^n)$ we use the convention

$$\psi_{j,k}(x) = D_{A^j}\tau_k\psi(x) = |\det A|^{j/2}\psi\left(A^j x - k\right) \qquad j \in \mathbb{Z}, k \in \mathbb{Z}^n ,$$

where $\tau_y f(x) = f(x-y)$ is a translation operator by the vector $y \in \mathbb{R}^n$, and $D_A f(x) = \sqrt{|\det A|} f(Ax)$ is a dilation by the matrix A.

Definition 2. By a *multiresolution analysis* (MRA) we mean a sequence of closed subspaces $(V_i)_{i\in\mathbb{Z}} \subset L^2(\mathbb{R}^n)$ associated with a *scaling function* φ, satisfying:

(i) $V_i \subset V_{i+1}$ for $i \in \mathbb{Z}$,
(ii) $V_i = D_{A^i} V_0$ for $i \in \mathbb{Z}$,
(iii) $\overline{\bigcup_{i\in\mathbb{Z}} V_i} = L^2(\mathbb{R}^n)$,
(iv) $\bigcap_{i\in\mathbb{Z}} V_i = \{0\}$,
(v) $\{\tau_k\varphi\}_{k\in\mathbb{Z}^n}$ is an orthonormal basis of V_0.

A wavelet family $\Psi = \{\psi^1, \dots, \psi^L\}$ "generates" MRA, if

$$V_i = \bigoplus_{j<i} W_j, \qquad \text{where } W_j = \overline{\operatorname{span}}\left\{\psi^l_{j,k} : k \in \mathbb{Z}^n, l = 1, \dots, L\right\} . \tag{1.1}$$

This happens precisely when V_0 as a shift invariant subspace of $L^2(\mathbb{R}^n)$ has dimension function $\dim_{V_0}(\xi) = 1$ for a.e. $\xi \in \mathbb{R}^n$. For the definition of the dimension function for general shift invariant spaces, see [7]. However, the dimension function of V_0 is given by the explicit formula,

$$\dim_{V_0}(\xi) = \sum_{l=1}^{L}\sum_{j=1}^{\infty}\sum_{k\in\mathbb{Z}^n}\left|\hat{\psi}^l\left(B^j(\xi+k)\right)\right|^2 , \tag{1.2}$$

where $B = A^T$, see [9]. Since

$$\int_{(-1/2,1/2)^n} \dim_{V_0}(\xi)\, d\xi = 1/(b-1)\sum_{l=1}^{L}\left\|\psi^l\right\|^2 , \tag{1.3}$$

a wavelet family $\Psi = \{\psi^1, \dots, \psi^L\}$ can be associated with MRA only if $L = b - 1$, where $b = |\det A|$.

Definition 3. We say that a function f on $\mathbb{R}^n$ is r-regular, if f is of class C^r, $r = 0, 1, \dots, \infty$ and

$$\left|\partial^\alpha f(x)\right| \le c_{\alpha,k}(1 + |x|)^{-k}\,, \tag{1.4}$$

for each $k \in \mathbb{N}$, and each multi-index α, with $|\alpha| \le r$. A wavelet family $\Psi = \{\psi^1, \dots, \psi^L\}$ is r-regular, if $\psi^1, \dots, \psi^L$ are r-regular functions. An *MRA* is r-regular if the subspace V_0 given by (1.1) has an orthonormal basis of the form $\{\tau_k\varphi : k \in \mathbb{Z}^n\}$ for some r-regular scaling function φ.

If a wavelet family Ψ is r-regular for r sufficiently large, or more precisely $|\hat{\psi}^l|$ are continuous and $|\hat{\psi}^l(\xi)| \le C(1+|\xi|)^{-n/2-\varepsilon}$ for some $\varepsilon > 0$, then the sum (1.2) converges uniformly on compact subsets of $\mathbb{R}^n \setminus \mathbb{Z}^n$ to the continuous function $\dim_{V_0}(\xi)$ on $\mathbb{R}^n \setminus \mathbb{Z}^n$. Since $\dim_{V_0}(\xi)$ is $\mathbb{Z}^n$-periodic and integer-valued therefore $\dim_{V_0}(\xi)$ is constantly equal to d for some $d \in \mathbb{N}$. If $d = 1$ then r-regular wavelet family Ψ comes from some MRA (more generally, Ψ comes from MRA with multiplicity d). This was essentially shown by Auscher [1, 2]. In general, we can not expect that this *MRA* is also r-regular; for a counterexample see [29, Proposition 2, p. 88]. Conversely, having an r-regular MRA we cannot, in general, deduce the existence of r-regular wavelet family associated with it, see [34, Theorem 5.10 and Remark 5.6]. Nevertheless, we can deduce the existence of r-regular wavelet family by using the following result [34, Corollary 5.17].

Proposition 1 (Wojtaszczyk).
Assume that we have a multiresolution analysis on $\mathbb{R}^n$ associated with an integral dilation A, $|\det A| = b$. Assume that this MRA has an r-regular scaling function $\varphi(x)$ such that $\hat{\varphi}(\xi)$ is real for some integer $r \ge 0$. Then there exists a wavelet family associated with this MRA consisting of $(b - 1)$ r-regular function.

2. Regular Wavelets

Our goal is to construct r-regular wavelets for an arbitrary expansive matrix A with integer entries. Strichartz [31] has shown how to achieve this under the assumption of the existence of a self-affine tiling of $\mathbb{R}^n$ given by A.

Self-affine tilings. Suppose A is an $n \times n$ dilation matrix and $\mathcal{D} = \{d_1, \dots, d_b\}$ is any set of representatives of different cosets of $\mathbb{Z}^n / A\mathbb{Z}^n$, where $b = |\det A|$. Any such set $\mathcal{D}$ is called a *standard digit set* in [24, 25]. It follows from [17, 23] that for any choice of $(A, \mathcal{D})$ there exists a unique (non-empty) compact set $Q = Q(A, \mathcal{D})$ satisfying the set-valued functional equation

$$A(Q) = \bigcup_{d \in \mathcal{D}} (d + Q)\,,$$

which is given explicitly by

$$Q = Q(A, \mathcal{D}) = \left\{ x \in \mathbb{R}^n : x = \sum_{j=1}^{\infty} A^{-j}\varepsilon_j, \quad \text{where } \varepsilon_j \in \mathcal{D} \right\}.$$

The set $Q(A, \mathcal{D})$ is called a *self-affine tile.* For properties of self-affine tiles we refer the reader to [19, 23, 32]. Here we only mention a fundamental Lattice Tiling Theorem due to Lagarias and Wang [25].

Theorem 2 (Lagarias–Wang).

Every self-affine tile $Q(A, \mathcal{D})$ gives a Γ-tiling of $\mathbb{R}^n$ for some lattice $\Gamma \subset \mathbb{Z}^n$, i.e., $\{\gamma + Q(A, \mathcal{D}) : \gamma \in \Gamma\}$ is a partition of $\mathbb{R}^n$ (modulo null sets).

If the lattice $\Gamma = \mathbb{Z}^n$ in Theorem 2, which is easily seen to be equivalent with $|Q(A, \mathcal{D})| = 1$, then we simply say that $Q(A, \mathcal{D})$ gives a *self-affine tiling* of $\mathbb{R}^n$. Gröchenig and Madych have shown that there exist *self-affine tilings* of $\mathbb{R}^n$ given by the dilation A if and only if there exist Haar type wavelets associated with A, see [17].

The natural question is whether every dilation matrix admits a self-affine tiling. Lagarias and Wang have shown that all dilations in the dimensions ≤ 3 possess this property, see Proposition 2 below, whereas in the dimensions ≥ 4 there are dilations for which every choice of $\mathcal{D}$ yields the corresponding tile $Q(A, \mathcal{D})$ with measure bigger than 1, see [21, 22]. The simplest example is 4×4 matrix presented in the Introduction. It requires some knowledge of algebraic number theory to understand why this matrix does not have a self-affine tiling. Even though Proposition 2 has not been explicitly stated by Lagarias and Wang for $n = 3$, it does follow directly from their work [21, 22, 25].

Proposition 2 (Lagarias–Wang).

Suppose $n \leq 3$. If A is an $n \times n$ dilation matrix then there exists a digit set $\mathcal{D} \subset \mathbb{Z}^n$ such that $Q(A, \mathcal{D})$ gives a self-affine tiling of $\mathbb{R}^n$.

Proof. The 1-dimensional case is trivial, see [16]. The case $n = 2$ was shown by Lagarias and Wang, see [20, Theorem 1.1]. Finally, the 3-dimensional case is outlined below.

Suppose that A is a 3×3 dilation matrix. Assume first that A is *reducible,* i.e., A is integrally similar to a matrix $\tilde{A}$ of the form

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ C & A_2 \end{pmatrix},$$

where A_1, A_2 are $r \times r$ and $(3 - r) \times (3 - r)$ expansive integer matrices, respectively, and C is an $(3 - r) \times r$ integer matrix, $r = 1$ or 2. Since Proposition 2 holds for $n = 1, 2$, we can find a standard digit set $\mathcal{D}_i$ for the dilation matrix A_i satisfying $|Q(A_i, \mathcal{D}_i)| = 1$ for $i = 1, 2$. It is easy to verify that

$$\tilde{\mathcal{D}} = \left\{ \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} : d_1 \in \mathcal{D}_1, d_2 \in \mathcal{D}_2 \right\} \subset \mathbb{Z}^3$$

is a standard digit set for $\tilde{A}$. Furthermore, by an argument involving Fubini's theorem, see the proof of [25, Theorem 5.1],

$$\left|Q\left(\tilde{A}, \tilde{\mathcal{D}}\right)\right| = \left|Q\left(\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \tilde{\mathcal{D}}\right)\right| = \left|Q\left(A_1, \mathcal{D}_1\right) \times Q\left(A_2, \mathcal{D}_2\right)\right| = 1 .$$

Since, $A = P\tilde{A}P^{-1}$ for some invertible integer matrix P we have $Q(A, P\tilde{\mathcal{D}}) = P(Q(\tilde{A}, \tilde{\mathcal{D}}))$. Therefore, $|Q(A, \mathcal{D})| = 1$ for the digit set $\mathcal{D} = P\tilde{\mathcal{D}}$.

Assume next that A is *irreducible,* i.e., A is not integrally similar to a matrix of the form $\tilde{A}$ as above. Notice that every 3×3 dilation matrix A has a primitive standard digit $\mathcal{D}$, for the proof see [21, Theorem 1.5] with corrections in [22]. We say that a digit set $\mathcal{D}$ is primitive (for A) if $\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}^n$, where $\mathbb{Z}[A, \mathcal{D}]$ is the smallest A-invariant lattice containing $\mathcal{D} - \mathcal{D}$, i.e.,

$$\mathbb{Z}[A, \mathcal{D}] = \mathbb{Z}\left[\mathcal{D} - \mathcal{D}, A(\mathcal{D} - \mathcal{D}), \dots, A^{n-1}(\mathcal{D} - \mathcal{D})\right] .$$

By [25, Corollary 6.2] for every irreducible dilation A with primitive digit set $\mathcal{D}$ the corresponding self-affine tile satisfies $|Q(A, \mathcal{D})| = 1$. This completes the proof of Proposition 2. □

The construction. Lack of a self-affine tile Q with $|Q| = 1$ is a major obstacle in the construction of an r-regular MRA and a wavelet family. In the case when such a Q exists, Strichartz constructs r-regular scaling function by convolving the indicator function of Q, $\mathbf{1}_Q$, with itself and appropriately normalizing it. This works because of the minimal decay of $\hat{\mathbf{1}}_Q$. Instead, our construction is based on the careful construction of a smooth low-pass filter m and a scaling function φ satisfying $\hat{\varphi}(\xi) = m(B^{-1}\xi)\hat{\varphi}(B^{-1}\xi)$, $B = A^T$, such that m is equal to 1 on some prespecified closed neighborhood of the origin. This guarantees that all partial derivatives of $\hat{\varphi}$ have some minimal decay at infinity, of order $O(|\xi|^{-\varepsilon})$ for some $\varepsilon > 0$, independent of the order of the differentiation. This in turn yields another scaling function with desired order of regularity.

A set $\Delta \subset \mathbb{R}^n$ is said to be a (closed) *ellipsoid* if

$$\Delta = \left\{\xi \in \mathbb{R}^n : |P\xi| \leq 1\right\}, \tag{2.1}$$

for some nondegenerate $n \times n$ matrix P, where $|\cdot|$ denotes the standard norm in $\mathbb{R}^n$. By [32, Lemma 1.5.1], for any dilation matrix B there exists an ellipsoid Δ, and $r > 1$ such that

$$\Delta \subset r\Delta \subset B\Delta . \tag{2.2}$$

By a scaling we can additionally assume that

$$\Delta \subset (-1/2, 1/2)^n , \tag{2.3}$$

$$\left(\bigcup_{j=0}^{\infty} B^{-j}[-1/2, 1/2]^n\right) \cap \left((\mathbb{Z}^n \setminus \{0\}) + \Delta\right) = \emptyset . \tag{2.4}$$

Indeed, since $B^{-j}[-1/2, 1/2]^n \subset [-1/2, 1/2]^n$ for sufficiently large j, the set $\tilde{K} = \bigcup_{j=0}^{\infty} B^{-j}[-1/2, 1/2]^n$ is compact and $\tilde{K} \cap \mathbb{Z}^n = \{0\}$, because the matrix B preserves the lattice $\mathbb{Z}^n$. Therefore $\delta = \operatorname{dist}(\tilde{K}, \mathbb{Z}^n \setminus \{0\}) > 0$, and it suffices to take $\Delta \subset \mathbf{B}(0, \delta) := \{\xi \in \mathbb{R}^n : |\xi| < \delta\}$. The interior of Δ is denoted by Δ°.

The following result extends Strichartz's Theorem [31], see also [34, Theorem 5.25].

Theorem 3.

Suppose A is an integral dilation matrix on $\mathbb{R}^n$ with $b = |\det A|$. For every $r \in \mathbb{N}$ there exists an r-regular multiresolution analysis and an associated r-regular wavelet family of $(b-1)$ functions.

Proof. Choose an ellipsoid Δ satisfying (2.2)–(2.4). Pick a function $h : \mathbb{R}^n \to \mathbb{R}_+ = [0, \infty)$ in the class C^∞, such that

$$\left\{\xi \in \mathbb{R}^n : h(\xi) = 0\right\} = \Delta . \tag{2.5}$$

To see that such a function exists, consider $\tilde{h} : \mathbb{R}^n \to \mathbb{R}_+$ in the class C^∞ such that

$$\left\{\xi \in \mathbb{R}^n : h(\xi) = 0\right\} = \overline{\mathbf{B}(0, 1)} .$$

It suffices to take $h(\xi) = \tilde{h}(P^{-1}\xi)$, where the matrix P defines the ellipsoid Δ in (2.1).

For a given $R > \sqrt{n}\|B\|$, choose a C^∞ function $g : \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\operatorname{supp} g := \left\{\xi \in \mathbb{R}^n : g(\xi) \neq 0\right\} = \left\{\xi \in \mathbb{R}^n : |\xi| < R\right\} = \mathbf{B}(0, R) . \tag{2.6}$$

Define

$$\tilde{g}(\xi) := g(\xi) \prod_{k \in Z} h(\xi - k) , \tag{2.7}$$

where $Z := \{k \in \mathbb{Z}^n \setminus \{0\} : (k + \Delta) \cap \mathbf{B}(0, R) \neq \emptyset\}$. Clearly $\tilde{g}$ is C^∞, and

$$\left\{\xi \in \mathbf{B}(0, R) : \tilde{g}(\xi) = 0\right\} = \mathbf{B}(0, R) \cap \left(\mathbb{Z}^n \setminus \{0\} + \Delta\right) . \tag{2.8}$$

The function f given by

$$f(\xi) = \frac{\tilde{g}(\xi)}{\sum_{k \in \mathbb{Z}^n} \tilde{g}(\xi + k)} \tag{2.9}$$

is well defined by (2.3) and (2.8) It satisfies the following properties:

(i) $f : \mathbb{R}^n \to [0, 1]$ is in the class C^∞ and $\operatorname{supp} f \subset \mathbf{B}(0, R)$,

(ii) $\sum_{k \in \mathbb{Z}^n} f(\xi + k) = 1$,

(iii) $f(\xi) = 1 \iff \xi \in \Delta$,

(iv) if $|\xi| < R$ and $f(\xi) = 0$ then $\xi \in \mathbb{Z}^n + \Delta$.

Indeed, (i) follows from $\operatorname{supp} \tilde{g} \subset \mathbf{B}(0, R)$, (ii) is a consequence of (2.9). (iii), and (iv) follow from (2.8) and (2.9).

Finally, define function $m : \mathbb{R}^n \to [0, 1]$ by

$$m(\xi) = \sqrt{\sum_{k \in \mathbb{Z}^n} f(B(\xi + k))} . \tag{2.10}$$

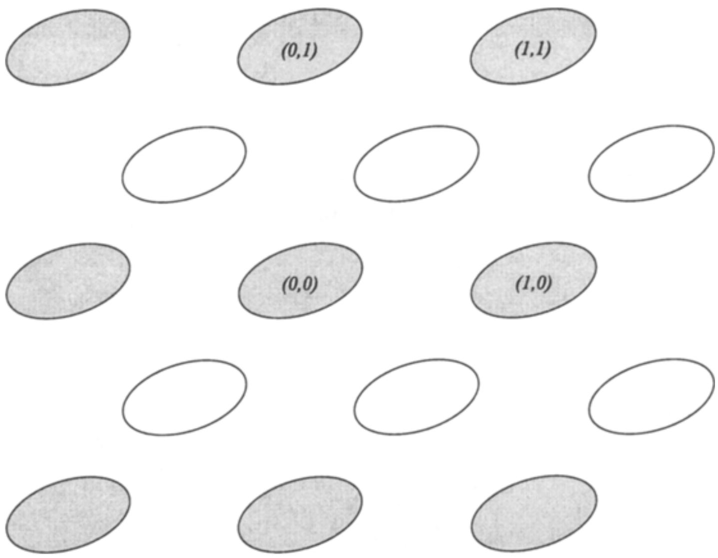

FIGURE 1 The low-pass filter m is constantly 1 in the shaded ellipses, constantly 0 in the unshaded ellipses and strictly between 0 and 1 elsewhere.

Claim 1.

The function m given by (2.10) *is C^∞, $\mathbb{Z}^n$-periodic, and*

$$\sum_{d \in \mathcal{D}} \left| m\left(\xi + B^{-1} d\right)\right|^2 = 1 \qquad \textit{for all } \xi \in \mathbb{R}^n , \tag{2.11}$$

$$m(\xi) = 0 \iff \xi \in B^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n + B^{-1}\Delta , \tag{2.12}$$

$$m(\xi) = 1 \iff \xi \in \mathbb{Z}^n + B^{-1}\Delta , \tag{2.13}$$

where the matrix $B := A^T$, and $\mathcal{D} = \{d_1, \dots, d_b\}$ is the set of representatives of different cosets of $\mathbb{Z}^n/B\mathbb{Z}^n$, where $b = |\det A|$.

Proof of Claim 1. To guarantee that m is C^∞, the function f must "vanish strongly," i.e., if $f(\xi_0) = 0$ for some ξ_0 then $\partial^\alpha f(\xi_0) = 0$ for any multi-index α. Since h, g and $\tilde{g}$ have this property thus f has it, too. It is clear that if some nonnegative function in C^∞ "vanishes strongly" then its square root is also C^∞.

For simplicity assume that $d_1 = 0$, i.e., 0 represents the coset $B\mathbb{Z}^n$. (2.11) is a consequence of

$$\sum_{d\in\mathcal{D}} \left|m\left(\xi + B^{-1}d\right)\right|^2 = \sum_{k\in\mathbb{Z}^n}\sum_{d\in\mathcal{D}} f\left(B\left(\xi + B^{-1}d + k\right)\right) = \sum_{k\in\mathbb{Z}^n}\sum_{d\in\mathcal{D}} f(\xi + d + Bk) = 1\,,$$

by property (ii). To see (2.12), take $\xi \in \mathbb{R}^n$ such that $m(\xi)^2 = \sum_{k\in\mathbb{Z}^n} f(B(\xi + k)) = 0$. Choose $k \in \mathbb{Z}^n$, so that $\xi + k \in [-1/2, 1/2)^n$. Since

$$|B(\xi + k)| \le ||B|||\xi + k| \le ||B||\sqrt{n}/2\,,$$

and $R > ||B||\sqrt{n}$, then by property (iv) $B(\xi+k) \in \mathbb{Z}^n+\Delta$, hence $\xi \in B^{-1}\mathbb{Z}^n+B^{-1}\Delta$. Furthermore, if $\xi \in \mathbb{Z}^n + B^{-1}\Delta$, i.e., $\xi \in k + B^{-1}\Delta$ for some $k \in \mathbb{Z}^n$, then

$$1 = f(B(\xi - k)) \le \sum_{l\in\mathbb{Z}^n} f(B(\xi + l)) \le m(\xi)^2 \le 1\,,$$

therefore $m(\xi) = 1$ for $\xi \in \mathbb{Z}^n + B^{-1}\Delta$. By (2.11) m vanishes precisely on $B^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n + B^{-1}\Delta$. This shows (2.12) and one implication of (2.13). To see other implication of (2.13) suppose $m(\xi) = 1$ for some $\xi \in \mathbb{R}^n$. By (2.11) we have $m(\xi + B^{-1}d) = 0$ for $d \in \mathcal{D} \setminus \{0\}$. By (2.12) $\xi + B^{-1}d \in B^{-1}\mathbb{Z}^n \setminus \mathbb{Z}^n + B^{-1}\Delta$, and hence $\xi \in B^{-1}\mathbb{Z}^n + B^{-1}\Delta$. Therefore by (2.12) we have $\xi \in \mathbb{Z}^n + B^{-1}\Delta$. This ends the proof of the claim. □

We can write m in the Fourier expansion as

$$m(\xi) = \frac{1}{\sqrt{|\det A|}} \sum_{k\in\mathbb{Z}^n} h_k e^{-2\pi i\langle k,\xi\rangle}\,, \tag{2.14}$$

where we include the factor $|\det A|^{-1/2}$ outside the summation as in [6]. Since m is C^∞, the coefficients h_k decay polynomially at infinity, that is for all $N > 0$ there is $C_N > 0$ so that

$$|h_k| \le C_N|k|^{-N} \qquad \text{for } k \in \mathbb{Z}^n \setminus \{0\}\,.$$

Since m satisfies (2.11) and $m(0) = 1$, m is a low-pass filter which is regular in the sense of the definition following [6, Theorem 1]. By [6, Theorem 5] $\varphi \in L^2(\mathbb{R}^n)$ defined by

$$\hat{\varphi}(\xi) = \prod_{j=1}^{\infty} m\left(B^{-j}\xi\right)\,, \tag{2.15}$$

has orthogonal translates, i.e.,

$$\langle \varphi, \tau_l\varphi\rangle = \delta_{l,0} \qquad \text{for } l \in \mathbb{Z}^n\,,$$

if and only if m satisfies the Cohen condition, that is there exists a compact set $K \subset \mathbb{R}^n$ such that

- K contains a neighborhood of zero,
- $|K \cap (l + K)| = \delta_{l,0}$ for $l \in \mathbb{Z}^n$,
- $m(B^{-j}\xi) \ne 0$ for $\xi \in K$, $j > 0$.

It is clear that $K = [-1/2, 1/2]^n$ does the job. Indeed, consider $\tilde{K} = \bigcup_{j=0}^{\infty} B^{-j}K$. By (2.4) and (2.12), we have

$$\emptyset = \left(B^{-1}\tilde{K}\right) \cap \left(\left(B^{-1}\mathbb{Z}^n \setminus \{0\}\right) + B^{-1}\Delta\right) \supset \left(B^{-1}\tilde{K}\right) \cap \left\{\xi \in \mathbb{R}^n : m(\xi) = 0\right\} .$$

Therefore φ is a scaling function for the multiresolution analysis $(V_j)_{j\in\mathbb{Z}}$ defined by

$$V_j = \overline{\operatorname{span}}\left\{D_{A^j}\tau_l\varphi : l \in \mathbb{Z}^n\right\} \qquad \text{for } j \in \mathbb{Z} .$$

It follows from a modification of a lemma due to Strichartz in [31] that φ possesses some minimal smoothness. However, we also need to estimate the derivatives of $\hat{\varphi}$, because in general φ does not have compact support.

Lemma 1.

Suppose m is a real-valued, $\mathbb{Z}^n$-periodic function of class C^∞. Assume that m satisfies (2.11)–(2.13) *as in Claim 1. Then there exists $\varepsilon > 0$ such that $\hat{\varphi}$ given by* (2.15) *satisfies*

$$\left|\partial^\alpha\hat{\varphi}(\xi)\right| \le C(\alpha)(1+|\xi|)^{-\varepsilon} \qquad \textit{for } \xi \in \mathbb{R}^n , \tag{2.16}$$

for some constant $C(\alpha)$ depending on a multi-index α.

Proof of Lemma 1. Note that the product (2.15) in the definition of $\hat{\varphi}$ converges uniformly on compact sets.

For a given integer $k \ge 0$, let $\mathfrak{D}^k m(\xi)$ be the derivative of m of order k at the point ξ thought of as a symmetric, multilinear functional, i.e., $\mathfrak{D}^k m(\xi) : (\mathbb{R}^n)^k = \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$. That is,

$$\mathfrak{D}^k m(\xi)\left(e_{j(1)}, \dots, e_{j(k)}\right) = \partial^\alpha m(\xi), \qquad \text{where } \alpha = e_{j(1)} + \cdots + e_{j(k)} ,$$

and $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. The norm is given by

$$\left\|\mathfrak{D}^k m(\xi)\right\| = \sup_{\substack{v_i\in\mathbb{R}^n, |v_i|=1,\\ i=1,\dots,k}} \left|\mathfrak{D}^k m(\xi)\left(v_1, \dots, v_k\right)\right| . \tag{2.17}$$

We will first show that there exists $\varepsilon > 0$, so that for all integers $N \ge 0$ there is a constant $C = C(N) > 0$ so that

$$\prod_{j=1}^{\infty} \left\|\mathfrak{D}^{k_j} m\left(B^{-j}\xi\right)\right\| \le C(N)|\xi|^{-\varepsilon} \qquad \text{for } \xi \ne 0 , \tag{2.18}$$

for every sequence of nonnegative integers $(k_j)_{j=1}^{\infty}$ all of which except a finite number are zeroes, and $N = \sum_{j=1}^{\infty} k_j$.

Indeed, let

$$\rho = \max\left\{m(\xi) : \xi \notin \Delta + \mathbb{Z}^n\right\} = \max\left\{m(\xi) : \xi \in \mathbb{R}^n \setminus \left(\Delta^\circ + \mathbb{Z}^n\right)\right\} , \tag{2.19}$$

where Δ° denotes the interior of Δ. Since m is $\mathbb{Z}^n$-periodic, and by (2.2) and (2.13) $\{\xi \in \mathbb{R}^n : m(\xi) = 1\} = B^{-1}\Delta + \mathbb{Z}^n \subset \Delta^\circ + \mathbb{Z}^n$, we conclude $\rho < 1$.

Define $\varepsilon > 0$ so that $||B||^\varepsilon\rho = 1$, i.e., $\varepsilon = -\ln(\rho)/\ln(||B||)$. Finally, define

$$M = \max\left\{\left\|\mathfrak{D}^k m(\xi)\right\| : \xi \in \mathbb{R}^n, k = 0, \dots, N\right\} .$$

Since $0 \le m(\xi) \le 1$ for all $\xi \in \mathbb{R}^n$ we have

$$\prod_{j=1}^{\infty} \left\|\mathfrak{D}^{k_j} m\left(B^{-j}\xi\right)\right\| \le M^N \qquad \text{for } \xi \in \mathbb{R}^n . \tag{2.20}$$

Furthermore, the above product equals zero, if $B^{-j}\xi \in \mathbb{Z}^n \setminus \{0\} + \Delta^\circ$ for some integer $j \geq 1$. Indeed, if $B^{-j}\xi \in l + \Delta^\circ$, for some $l \in \mathbb{Z}^n \setminus \{0\}$, then $l = B^{j_0} l_0$, where $j_0 \geq 0$, and $l_0 \in \mathbb{Z}^n \setminus B\mathbb{Z}^n$. Clearly

$$B^{-j-j_0-1}\xi \in B^{-1}l_0 + B^{-j_0-1}\Delta^\circ \subset B^{-1}l_0 + B^{-1}\Delta^\circ ,$$

thus $||\mathfrak{D}^{k_{j+j_0+1}} m(B^{-j-j_0-1}\xi)|| = 0$ by the virtue of (2.12). Therefore the product in (2.18) equals zero for $\xi \in Z$, where

$$Z := \bigcup_{j=1}^{\infty} B^j \left((\mathbb{Z}^n \setminus \{0\}) + \Delta^\circ \right) .$$

For any $\xi \notin Z$, $|\xi| > r := \sqrt{n}/2$, find a minimal integer $j_0 \geq 1$, so that $|B^{-j_0}\xi| < r$. For such a ξ we have by (2.3)

$$J := \left\{ j \geq 1 : B^{-j}\xi \notin \mathbb{Z}^n + \Delta^\circ \right\} = \left\{ j \geq 1 : B^{-j}\xi \notin \Delta^0 \right\} \supset \left\{ j \geq 1 : \left| B^{-j}\xi \right| \geq r \right\} . \quad (2.21)$$

The cardinality of J is at least $j_0 - 1$. Note also that

$$|\xi| \leq \left| B^{j_0} B^{-j_0}\xi \right| \leq \left\| B^{j_0} \right\| \left| B^{-j_0}\xi \right| < ||B||^{j_0} r ,$$

therefore

$$|\xi|^{-\varepsilon} \geq ||B||^{-\varepsilon j_0} r^{-\varepsilon} = \rho^{j_0} r^{-\varepsilon} . \quad (2.22)$$

Hence by (2.19), (2.21), and (2.22) for $\xi \notin Z$ and $|\xi| > r$

$$\prod_{j=1}^{\infty} \left\| \mathfrak{D}^{k_j} m \left(B^{-j}\xi \right) \right\| \leq M^N \prod_{\substack{j \in J \\ k_j = 0}} m \left(B^{-j}\xi \right) \leq M^N \rho^{j_0 - 1 - N} \leq (M/\rho)^N \left(r^\varepsilon / \rho \right) |\xi|^{-\varepsilon} .$$

Combined with (2.20) and the property of the set Z this shows (2.18).

To finish the proof of Lemma 1 define, for $j \geq 1$, functions $m_j(\xi) = m(B^{-j}\xi)$. By the chain rule for any integer $k \geq 1$

$$\mathfrak{D}^k m_j(\xi) (v_1, \ldots, v_k) = \mathfrak{D}^k m \left(B^{-j}\xi \right) \left(B^{-j} v_1, \ldots, B^{-j} v_k \right) , \quad (2.23)$$

for any vectors $v_1, \ldots, v_k \in \mathbb{R}^n$. Let α be a fixed multi-index. Let $\Gamma = \Gamma(\alpha)$ consists of all sequences $\beta = (\beta_j)_{j=1}^{\infty}$ of multi-indices β_j, such that $\beta_j = 0$ for all but finitely many j's, and $\sum_{j=1}^{\infty} \beta_j = \alpha$. By the infinite product rule

$$\partial^\alpha \hat{\varphi}(\xi) = \alpha! \sum_{\beta \in \Gamma} \prod_{j=1}^{\infty} \frac{\partial^{\beta_j} m_j(\xi)}{\beta_j!} , \quad (2.24)$$

provided the above series converges uniformly and absolutely. Indeed, by (2.23) and (2.18)

$$\sum_{\beta \in \Gamma} \prod_{j=1}^{\infty} |\partial^{\beta_j} m_j(\xi)| \leq \alpha! \sum_{\beta \in \Gamma} \prod_{j=1}^{\infty} \left\| \mathfrak{D}^{|\beta_j|} m \left(B^{-j}\xi \right) \right\| \cdot \left\| B^{-j} \right\|^{|\beta_j|}$$

$$= C(|\alpha|) |\xi|^{-\varepsilon} \sum_{\beta \in \Gamma} \prod_{j=1}^{\infty} \left\| B^{-j} \right\|^{|\beta_j|} \leq C(|\alpha|) |\xi|^{-\varepsilon} n^{|\alpha|} \left(\sum_{j=1}^{\infty} \left\| B^{-j} \right\| \right)^{|\alpha|} .$$

Since $\hat{\varphi}(\xi) = 1$ for $\xi \in \Delta^\circ$, by the above estimate applied for $\xi \notin \Delta^\circ$ we obtain (2.16). □

For a given integer $N > 0$ define function φ_1, by $\hat{\varphi}_1(\xi) = (\hat{\varphi}(\xi))^N$. By Lemma 1 and the product rule we have

$$|\partial^\alpha \hat{\varphi}_1(\xi)| \le C(\alpha)(1+|\xi|)^{-N\varepsilon} \qquad \text{for } \xi \in \mathbb{R}^n , \tag{2.25}$$

for some constant $C(\alpha)$ depending on a multi-index α. Therefore, for sufficiently large N, φ_1 is r-regular.

Since φ is a scaling function, by the orthonormality of the system $\{\tau_k\varphi : k \in \mathbb{Z}^n\}$ we have

$$\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi+k)|^2 = 1 \qquad \text{for a.e. } \xi \in \mathbb{R}^n . \tag{2.26}$$

By [6, Corollary 1] the above series converges uniformly on compact sets and we have equality for all $\xi \in \mathbb{R}^n$. We remark that the uniform convergence on compact sets in (2.26) does not follow from the estimate (2.16). Rather, it is a consequence of $\hat{\varphi}$ being in the appropriate Sobolev space. Therefore we can find a finite subset $S \subset \mathbb{Z}^n$ so that

$$\sum_{k\in S} |\hat{\varphi}(\xi+k)|^2 > 1/2 \qquad \text{for } \xi \in [-1/2, 1/2]^n . \tag{2.27}$$

Since we sum over a finite set of indices there exists a constant $c > 0$ so that

$$\sum_{k\in S} |\hat{\varphi}_1(\xi+k)|^2 = \sum_{k\in S} |\hat{\varphi}(\xi+k)|^{2N} > c \qquad \text{for } \xi \in [-1/2, 1/2]^n .$$

Therefore

$$c \le \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}_1(\xi+k)|^2 = \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi+k)|^{2N} \le \sum_{k\in\mathbb{Z}^n} |\hat{\varphi}(\xi+k)|^2 = 1 , \tag{2.28}$$

which means that $\{\tau_k\varphi_1 : k \in \mathbb{Z}^n\}$ forms a Riesz sequence. φ_1 is also refinable, i.e., $\hat{\varphi}_1(\xi) = (m(B^{-1}\xi))^N \hat{\varphi}_1(B^{-1}\xi)$, therefore it defines an MRA with

$$V_j = \overline{\operatorname{span}} \left\{ D_{A^j} \tau_l \varphi_1 : l \in \mathbb{Z}^n \right\} . \tag{2.29}$$

Indeed, since φ_1 is refinable we have $V_j \subset V_{j+1}$. By a simple adaptation of [18, Theorem 1.6, Chapter 2], spaces $(V_j)_{j\in\mathbb{Z}}$ satisfy $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$. Since $\varphi_1(0) = 1$, and φ_1 is continuous we also have $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^n)$. Finally, we can pick a scaling function φ_0 for V_0 by

$$\hat{\varphi}_0(\xi) = \hat{\varphi}_1(\xi) \left(\sum_{k\in\mathbb{Z}^n} |\hat{\varphi}_1(\xi+k)|^2 \right)^{-1/2} .$$

By [34, Corollary 5.14] φ_0 is also r-regular. Since $\hat{\varphi}_0$ is a real-valued scaling function, we can apply Proposition 1 to obtain r-regular wavelets $\{\psi^1, \dots, \psi^{b-1}\}$ associated with the MRA $(V_j)_{j\in\mathbb{Z}}$ defined by (2.29). This finishes the proof of Theorem 3. □

Remarks.

(i) The constructed scaling function φ_0, and the multiwavelet $\{\psi^1, \dots, \psi^{b-1}\}$ satisfy scaling equations

$$\begin{aligned} \hat{\varphi}_0(\xi) &= m_0\left(B^{-1}\xi\right) \hat{\varphi}_0\left(B^{-1}\xi\right) , \\ \hat{\psi}^l(\xi) &= m_l\left(B^{-1}\xi\right) \hat{\varphi}_0\left(B^{-1}\xi\right) \qquad \text{for } l = 1, \dots, b-1 , \end{aligned} \tag{2.30}$$

for all $\xi \in \mathbb{R}^n$, where $m_0, \dots, m_{b-1}$ are $\mathbb{Z}^n$-periodic C^∞ functions. If we write

$$m_l(\xi) = \frac{1}{\sqrt{|\det A|}} \sum_{k \in \mathbb{Z}^n} h_k^l e^{-2\pi i \langle k, \xi \rangle} , \qquad \text{for } l = 0, \dots, b-1 ,$$

then the sequence of vectors $(h^0, \dots, h^{b-1})$ is a wavelet matrix, as it is defined in [6], where $h^l = (h_k^l)_{k \in \mathbb{Z}^n}$. The coefficients of this wavelet matrix decay polynomially fast at infinity because m_l's are C^∞. We also have

$$\sum_{l=0}^{b-1} m_l \left(\xi + B^{-1} d\right) \overline{m_l \left(\xi + B^{-1} d'\right)} = \delta_{d,d'} \qquad \text{for } d, d' \in \mathcal{D},\ \xi \in \mathbb{R}^n , \tag{2.31}$$

where $\mathcal{D}$ is the set of representatives of different cosets of $\mathbb{Z}^n / B\mathbb{Z}^n$.

(ii) Note that the scaling function φ_0 vanishes in the Fourier domain in many places. The zero set for $\hat{\varphi}_0$ contains the set Z, so $\hat{\varphi}_0$ vanishes in a neighborhood of every lattice point of $\mathbb{Z}^n$ except 0. One can see that as we move off the origin those neighborhoods may only become larger than the guaranteed ellipsoidal neighborhood Δ. However, in general there is no guarantee that $\hat{\varphi}_0$ is going to vanish outside some bounded set. Nevertheless, for some specific dilations we can choose an ellipsoid Δ, so that $\hat{\varphi}_0$ has a compact support. For example, if the dimension $n = 1$, and $A = 2$, it suffices to take $\Delta = [-\delta, \delta]$, $1/3 \le \delta < 1/2$ to obtain $\operatorname{supp} \hat{\varphi}_0 = (\delta - 1, 1 - \delta)$. Therefore we obtain a scaling function φ_0 (and a wavelet ψ) in the Schwartz class. Naturally, ψ is a Meyer wavelet, see [28]. One could hope that by taking more general sets than ellipsoids for Δ, one could produce a filter m having "optimized" zero set, so that both $\hat{\varphi}$ has compact support and m satisfies the Cohen condition. This is indeed the case in the dimension $n = 2$, where Speegle and the author [10] have recently shown the existence of wavelets in the Schwartz class for all 2×2 integral dilations. We do not know whether one can achieve this for general dilations in higher dimensions.

Question. Does there exist an ∞-regular multiresolution analysis and an associated ∞-regular wavelet family of $(|\det A| - 1)$ functions for any $n \times n$ integral dilation A, $n \ge 3$?

A partial positive answer is given in [8]. It is shown there, that for any integral dilation A there exists a natural number m and an ∞-regular multiresolution analysis with an ∞-regular wavelet family of $(|\det A|^m - 1)$ functions associated with A^m.

3. Vanishing Moments

In this section we are going to show that r-regular orthonormal multiwavelets must automatically have vanishing moments. This fact was previously observed for the dilation $A = 2Id$. Here we are going to show this for all dilations, A, preserving some lattice, i.e., for dilations with integer entries. Before proceeding with the main results two technical lemmas are presented.

Lemma 2.

Suppose we have a tempered distribution f on $\mathbb{R}^n$ such that its (distributional) Fourier transform $\hat{f}$ is a regular distribution. Let L be a partial differential operator with constant coefficients of order r, i.e.,

$$Lf = \sum_{|\alpha| \le r} a_\alpha \partial^\alpha f , \tag{3.1}$$

where not all coefficients a_α are zero. If $Lf = 0$ then $f = 0$.

Proof. Apply the Fourier transform to Lf

$$\sum_{|\alpha|\le r}(-2\pi i\xi)^\alpha a_\alpha \hat{f} = 0\,. \tag{3.2}$$

Since the distribution $\hat{f}$ is regular, there exists locally integrable function (also denoted by $\hat{f}$), such that $\hat{f}(\varphi) = \int_{\mathbb{R}^n} \hat{f}(\xi)\varphi(\xi)\,d\xi$ for $\varphi \in \mathcal{S}$ in the Schwartz class of test functions. Thus

$$\int_{\mathbb{R}^n} \hat{f}(\xi) \sum_{|\alpha|\le r}(-2\pi i\xi)^\alpha a_\alpha \varphi(\xi)\,d\xi = 0 \qquad \text{for } \varphi \in \mathcal{S}\,.$$

Therefore, $\hat{f}(\xi) = 0$ for a.e. ξ outside the zero set of the nonzero polynomial as in (3.2). Since the zero set of a nonzero polynomial has measure zero, $\hat{f} = 0$, and consequently $f = 0$. □

The second lemma appears in the work of Cabrelli, Heil, and Molter [11, Lemma 4.2], where the proof can be found. It is also a consequence of [27, Theorem 4.3 on p. 122]. In this lemma we can relax the assumption that A is a dilation.

Lemma 3.

Suppose $A : \mathbb{C}^n \to \mathbb{C}^n$ is a linear map with eigenvalues $\lambda_1, \dots, \lambda_n$ (taken according to multiplicity). Let $S^k(\mathbb{C}^n)$ denote the space of homogeneous polynomials of degree $k \ge 0$ in n variables with complex coefficients. Define a linear map

$$A_{[k]} : S^k\left(\mathbb{C}^n\right) \to S^k\left(\mathbb{C}^n\right), \qquad \left(A_{[k]}p\right)(x) = p(Ax),\ \text{for } x \in \mathbb{C}^n$$

for $p \in S^k(\mathbb{C}^n)$. Then all eigenvalues of $A_{[k]}$ are of the form $\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}$ for some sequence of indices $1 \le i_1 \le i_2 \le \dots \le i_k \le n$.

Let us enumerate eigenvalues $\lambda_1, \dots, \lambda_n$ of A according to multiplicity, so that $1 < |\lambda_1| \le \dots \le |\lambda_n|$.

Theorem 4.

Suppose A is a dilation and $\psi, \tilde{\psi} \in L^2(\mathbb{R}^n)$ are such that

$$\left\langle \tilde{\psi}_{j,k}, \psi_{j',k'} \right\rangle = 0 \qquad \text{for } j \ne j' \in \mathbb{Z}, k, k' \in \mathbb{Z}^n\,. \tag{3.3}$$

Suppose also that $\tilde{\psi}$ is nonconstant, bounded and of class C^r, and $|\psi(x)| \le C(1+|x|)^{-N}$ for some $N > n + r$. Then

$$\int_{\mathbb{R}^n} x^\alpha \psi(x)\,dx = 0 \qquad \text{for any multi-index } \alpha,\ |\alpha| < r \ln|\lambda_1| / \ln|\lambda_n|\ , \tag{3.4}$$

where λ_1 is the smallest and λ_n the greatest eigenvalue of A (in modulus). Moreover, if the matrix A is diagonalizable then (3.4) *holds for $|\alpha| \le r \ln|\lambda_1| / \ln|\lambda_n|$.*

Proof. For a fixed point $u \in \mathbb{R}^n$ consider the Taylor expansion of $\tilde{\psi}$ of order r, i.e.,

$$\tilde{\psi}(u+x) = \sum_{|\alpha|\le r} \frac{\partial^\alpha \tilde{\psi}(u)}{\alpha!} x^\alpha + R(u,x)\,, \tag{3.5}$$

where $R(u,x) = o(|x|^r)$ as $|x| \to 0$, and $|R(u,x)| \le C|x|^r$ for some $C > 0$, since $\tilde{\psi}$ is bounded.

Let $I = \{\alpha : \alpha \text{ is a multi-index}, |\alpha| \le r\}$, $U = \bigcup_{j\in\mathbb{Z}} A^j\mathbb{Z}^n$. Consider the finite dimensional, complex linear space $V := \mathbb{C}^I = \{(x_\alpha)_{\alpha\in I} : x_\alpha \in \mathbb{C}, \alpha \in I\}$. We claim that the vectors $(\partial^\alpha\tilde{\psi}(u))_{\alpha\in I} \in V$, where $u \in U$ span the whole space V. Indeed, if $V' := \operatorname{span}\{(\partial^\alpha\tilde{\psi}(u))_{\alpha\in I} :$

$u \in U\} \neq V$, then there is a linear functional $a : V \to \mathbb{C}$, such that a is not zero, and $V' \subset \ker a$. Since we can identify $a = (a_\alpha)_{\alpha \in I} \in V$ we would have $L\tilde{\psi}(u) = 0$ for $u \in U$, where L is given by (3.1). Since U is dense in $\mathbb{R}^n$, and $\tilde{\psi}$ is C^r we have $L\tilde{\psi}(u) = 0$ for all $u \in \mathbb{R}^n$. By Lemma 2, $\tilde{\psi} = 0$ which is a contradiction.

Therefore, for any $a = (a_\alpha)_{\alpha \in I} \in \mathbb{C}^I$ we can find points $u_1, \dots, u_m \in U$, and complex scalars $c_1, \dots, c_m$ such that

$$a_\alpha = \sum_{i=1}^{m} c_i \partial^\alpha \tilde{\psi}(u_i)/\alpha! \qquad \text{for } \alpha \in I . \tag{3.6}$$

Applying (3.5) to points $u_1, \dots, u_m$ and taking linear combination with scalars $c_1, \dots, c_m$ we have

$$\sum_{i=1}^{m} c_i \tilde{\psi}(u_i + x) = \sum_{|\alpha| \le r} a_\alpha x^\alpha + R(x) , \tag{3.7}$$

where $R(x) = o(|x|^r)$ as $|x| \to 0$, and $|R(x)| \le C|x|^r$ for some $C > 0$. Write $u_i = A^{j_i} k_i$, where $j_i \in \mathbb{Z}$, $k_i \in \mathbb{Z}^n$ for $i = 1, \dots, m$. By (3.3) for $j > \max(-j_1, \dots, -j_m)$

$$\begin{aligned} \int_{\mathbb{R}^n} \tilde{\psi}(u_i + x)\overline{\psi(A^j x)}\, dx &= \int_{\mathbb{R}^n} \tilde{\psi}(x)\overline{\psi(A^j x - A^j u_i)}\, dx \\ &= \int_{\mathbb{R}^n} \tilde{\psi}(x)\overline{\psi(A^j x - A^{j+j_i} k_i)}\, dx = |\det A|^{-j/2} \left\langle \tilde{\psi}_{0,0}, \psi_{j, A^{j+j_i} k_i} \right\rangle = 0 , \end{aligned}$$

since the dilation A preserves the lattice $\mathbb{Z}^n$. Therefore, if we multiply both sides of (3.7) by $\overline{\psi(A^j x)}$ and integrate we obtain

$$\int_{\mathbb{R}^n} \sum_{|\alpha| \le r} a_\alpha x^\alpha \overline{\psi(A^j x)}\, dx = -\int_{\mathbb{R}^n} R(x)\overline{\psi(A^j x)}\, dx .$$

After conjugation and a change of variables we have

$$\int_{\mathbb{R}^n} \sum_{|\alpha| \le r} \overline{a_\alpha} \left(A^{-j} x\right)^\alpha \psi(x)\, dx = -\int_{\mathbb{R}^n} R\left(A^{-j} x\right) \psi(x)\, dx , \tag{3.8}$$

where $R(x) = o(|x|^r)$ as $|x| \to 0$, and $|R(x)| \le C|x|^r$.

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of A (according to multiplicity), so that $1 < |\lambda_1| \le \cdots \le |\lambda_n|$. By basic linear algebra this implies that the eigenvalues of A^{-1} are $\lambda_1^{-1}, \dots, \lambda_n^{-1}$ and thus are non-zero. Hence for any $\gamma > 1$ there is a constant $c = c(\gamma) > 0$ such that

$$\left|A^{-j} x\right| \le c\, |\lambda_1|^{-j/\gamma} |x| \qquad \text{for } j > 0, x \in \mathbb{R}^n . \tag{3.9}$$

Moreover, if the matrix A is diagonalizable, or more precisely, there are no Jordan blocks associated with the eigenvalues of modulus $|\lambda_1|$, then γ can be chosen to be 1.

We claim that

$$\lim_{j \to \infty} |\lambda_1|^{jr/\gamma} \int_{\mathbb{R}^n} \left| R\left(A^{-j} x\right) \psi(x) \right| dx = 0 . \tag{3.10}$$

Indeed, for any $\varepsilon > 0$, choose $s > 0$, so that $\int_{|x|>s} |x|^r (1 + |x|)^{-N}\, dx < \varepsilon$, where $N > n + r$. For sufficiently large j, $|R(A^{-j}x)| \le \varepsilon |A^{-j}x|^r$ for $|x| \le s$. Hence by (3.9)

$$\begin{aligned}
\int_{\mathbb{R}^n} \left|R\left(A^{-j}x\right)\psi(x)\right| dx &= \int_{|x|\le s} \left|R\left(A^{-j}x\right)\psi(x)\right| dx + \int_{|x|>s} \left|R\left(A^{-j}x\right)\psi(x)\right| dx \\
&\le \int_{|x|\le s} \varepsilon\, |\lambda_1|^{-jr/\gamma}\, |x|^r |\psi(x)|\, dx + \int_{|x|>s} C\, |\lambda_1|^{-jr/\gamma}\, |x|^r |\psi(x)|\, dx \\
&\le |\lambda_1|^{-jr/\gamma}\, \varepsilon C' \int_{|x|\le s} |x|^r (1+|x|)^{-N}\, dx + |\lambda_1|^{-jr/\gamma}\, C' \int_{|x|>s} |x|^r (1+|x|)^{-N}\, dx \\
&\le |\lambda_1|^{-jr/\gamma}\, \varepsilon C' \left(\int_{\mathbb{R}^n} |x|^r (1+|x|)^{-N}\, dx + 1 \right) .
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we obtain (3.10). We are now ready to conclude the proof. Take any integer $0 \le k < r \ln|\lambda_1| / \ln|\lambda_n|$ (or $\le$ if $\gamma = 1$). Consider the linear map $(A^{-j})_{[k]} : S^k(\mathbb{C}^n) \to S^k(\mathbb{C}^n)$, mapping a polynomial $p(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha$ to the polynomial $((A^{-j})_{[k]}p)(x) = \sum_{|\alpha|=k} a_\alpha (A^{-j} x)^\alpha$. By (3.8) for any choice of complex coefficients $(a_\alpha)_{|\alpha|=k}$, we have

$$\int_{\mathbb{R}^n} \sum_{|\alpha|=k} a_\alpha \left(A^{-j}x\right)^\alpha \psi(x)\, dx = - \int_{\mathbb{R}^n} R\left(A^{-j}x\right)\psi(x)\, dx\,, \tag{3.11}$$

for some $R(x) = o(|x|^r)$ as $|x| \to 0$, and $|R(x)| \le C|x|^r$. Take any eigenvalue λ of $(A^{-1})_{[k]}$. By Lemma 3, λ is a product of k eigenvalues of A^{-1}, i.e., $\lambda_1^{-1}, \dots, \lambda_n^{-1}$. Therefore $|\lambda| \ge |\lambda_n|^{-k}$. We claim that for any $s \ge 1$

$$\int_{\mathbb{R}^n} p(x)\psi(x)\, dx = 0 \qquad \text{for } p \in \ker\left(\lambda - \left(A^{-1}\right)_{[k]}\right)^s . \tag{3.12}$$

Indeed, let us proceed by induction on s. Let $s = 1$, $p = p(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha \in \ker(\lambda - (A^{-1})_{[k]})$, i.e., p is an eigenvector with eigenvalue λ. Since $((A^{-1})_{[k]})^j = (A^{-j})_{[k]}$, we have by (3.11)

$$\lambda^j \int_{\mathbb{R}^n} p(x)\psi(x)\, dx = \int_{\mathbb{R}^n} \left(\left(A^{-j}\right)_{[k]} p\right)(x)\psi(x)\, dx = - \int_{\mathbb{R}^n} R\left(A^{-j}x\right)\psi(x)\, dx\,. \tag{3.13}$$

By our choice of k we can find $\gamma > 1$ (or $\gamma = 1$ if A is diagonalizable) so that $|\lambda_n|^k |\lambda_1|^{-r/\gamma} \le 1$. Therefore by (3.10)

$$\begin{aligned}
\left|\int_{\mathbb{R}^n} p(x)\psi(x)\, dx\right| &\le |\lambda_n|^{kj} \int_{\mathbb{R}^n} \left|R\left(A^{-j}x\right)\psi(x)\right| dx \\
&= \left(|\lambda_n|^k\, |\lambda_1|^{-r/\gamma}\right)^j |\lambda_1|^{jr/\gamma} \int_{\mathbb{R}^n} \left|R\left(A^{-j}x\right)\psi(x)\right| dx \to 0 \text{ as } j \to \infty\,.
\end{aligned} \tag{3.14}$$

Suppose (3.12) holds for some $s \ge 1$. Take $p \in \ker(\lambda - (A^{-1})_{[k]})^{s+1}$. Then $p = \lambda p + p'$, where $p' \in \ker(\lambda - (A^{-1})_{[k]})^s$. Hence for some $p'' \in \ker(\lambda - (A^{-1})_{[k]})^s$,

$$\int_{\mathbb{R}^n} \left(\left(A^{-j}\right)_{[k]} p\right)(x)\psi(x)\, dx = \int_{\mathbb{R}^n} \left(\lambda^j p(x) + p''(x)\right)\psi(x)\, dx = \lambda^j \int_{\mathbb{R}^n} p(x)\psi(x)\, dx\,,$$

by the induction hypothesis. Thus (3.13) holds, which in turn implies (3.14).

Therefore (3.12) holds for any $s \ge 1$. Since λ was an arbitrary eigenvalue of $(A^{-1})_{[k]}$, we have

$$\int_{\mathbb{R}^n} p(x)\psi(x)\, dx = 0 \qquad \text{for any } p \in S^k\left(\mathbb{C}^n\right) .$$

This completes the proof. □

As a special case of Theorem 4 we have the following.

Corollary 1.

Suppose $\Psi = \{\psi^1, \ldots, \psi^L\}$ is r-regular wavelet family associated with a dilation A. Then for $l = 1, \ldots, L$,

$$\int_{\mathbb{R}^n} x^\alpha \psi^l(x)\,dx = 0 \qquad \textit{for any multi-index } \alpha,\ |\alpha| < r \ln|\lambda_1| / \ln|\lambda_n| \,, \tag{3.15}$$

where λ_1 is the smallest and λ_n the greatest eigenvalue of A (in modulus). Moreover, if the matrix A is diagonalizable over $\mathbb{C}$ then (3.15) *holds for $|\alpha| \le r \ln|\lambda_1| / \ln|\lambda_n|$.*

Our next goal is to investigate the moments of regular scaling functions. The results are a consequence of anisotropic Strang–Fix conditions introduced and investigated by Cohen, Gröchenig, and Villemoes [12]. First, we need some definitions.

Definition 4. We say that f belongs to the *anisotropic Sobolev space* associated with A, and write $f \in H^s_A$, if

$$||f||_{H^s_A} = \left(\int_{\mathbb{R}^n} (1+\rho(\xi))^{2s} \left|\hat{f}(\xi)\right|^2 d\xi \right)^{1/2} < \infty \,. \tag{3.16}$$

Here, ρ is any *quasi-norm* associated with $B = A^T$, that is $\rho : \mathbb{R}^n \to [0,\infty)$ is C^∞ on $\mathbb{R}^n \setminus \{0\}$ and

$$\rho(\xi) = 0 \iff \xi = 0 \,,$$
$$\rho(B\xi) = b^{1/n}\rho(\xi) \qquad \text{for all } \xi \in \mathbb{R}^n \,.$$

Definition 5. Suppose $m : \mathbb{R}^n \to \mathbb{C}$ is $\mathbb{Z}^n$-periodic. We say that m satisfies the *anisotropic Strang–Fix conditions* of order $s+1$, if and only if

$$\left| m\left(\xi + B^{-1}d\right)\right| = o\left(\rho(\xi)^s\right) \quad \text{as } \xi \to 0 \quad \text{for all } d \in \mathcal{D} \setminus \{0\} \,. \tag{3.17}$$

Here $\mathcal{D}$ denotes the set of b representatives of different cosets of $\mathbb{Z}^n / B\mathbb{Z}^n$; 0 represents the coset $B\mathbb{Z}^n$.

The following theorem was shown in [12] in the case when m is a trigonometric polynomial. However, it can be easily extended to the case when m is C^∞ or merely C^k for sufficiently large k.

Theorem 5 (Cohen, Gröchenig, Villemoes).

Suppose $\varphi \in H^s_A$ for some $s \ge 0$, and

$$\hat{\varphi}(\xi) = m\left(B^{-1}\xi\right) \hat{\varphi}\left(B^{-1}\xi\right) \qquad \textit{for } \xi \in \mathbb{R}^n \,, \tag{3.18}$$

for some $\mathbb{Z}^n$-periodic C^∞ function m, with $m(0) = 1$. If $\{\tau_k\varphi : k \in \mathbb{Z}^n\}$ is a Riesz system then m satisfies the anisotropic Strang–Fix conditions of order $s+1$.

The following result shows that every r-regular MRA has a scaling function with zeroth moment equal to 1 and with vanishing moments of order twice as large as we can expect from r-regular wavelets, see Corollary 1.

Theorem 6.

Suppose $(V_i)_{i\in\mathbb{Z}}$ is an r-regular MRA. Then there is an r-regular scaling function $\varphi_0 \in V_0$ such that

$$\int_{\mathbb{R}^n} x^\alpha \varphi_0(x)\,dx = \delta_{|\alpha|,0} \qquad \textit{for any multi-index } \alpha,\ |\alpha| < 2r \ln|\lambda_1| / \ln|\lambda_n| \,, \tag{3.19}$$

where λ_1 and λ_n, respectively, are the smallest and the greatest eigenvalues of A (in modulus). Moreover, if the matrix A is diagonalizable over $\mathbb{C}$ then (3.19) *holds for* $|\alpha| \le 2r \ln|\lambda_1|/\ln|\lambda_n|$.

Proof. Let $\varphi \in V_0$ be an r-regular scaling function of the MRA $(V_i)_{i\in\mathbb{Z}}$. Clearly, $\int_{\mathbb{R}^n}(1 + |\xi|)^{2r}|\hat{\varphi}(\xi)|^2 < \infty$. By a lemma due to Lemarié-Rieusset [26], for any $\gamma > 1$ there exists a constant $c > 0$,

$$1/c\rho(\xi)^{n\ln|\lambda_1|/(\gamma\ln b)} \le |\xi| \le c\rho(\xi)^{n\gamma\ln|\lambda_n|/\ln b} \qquad \text{for } |\xi| \ge 1\,, \tag{3.20}$$

$$1/c\rho(\xi)^{n\gamma\ln|\lambda_n|/\ln b} \le |\xi| \le c\rho(\xi)^{n\ln|\lambda_1|/(\gamma\ln b)} \qquad \text{for } |\xi| \le 1\,. \tag{3.21}$$

Moreover, if the matrix A has no Jordan blocks associated with the eigenvalues equal in modulus to $|\lambda_1|$ or $|\lambda_n|$, then γ can be chosen to be 1. By (3.20), $\varphi \in H^s_A$, where $s = nr\ln|\lambda_1|/(\gamma\ln b)$.

Since φ is a scaling function (and thus refinable) we have (3.18). The low-pass filter $m(\xi)$ is C^∞. Indeed, by the Sobolev Embedding Theorem, $\hat{\varphi}(\xi)$ is C^∞. Moreover, for any $\xi \in \mathbb{R}^n$ there exists a $k \in \mathbb{Z}^n$ such that $\hat{\varphi}(\xi + k) \ne 0$ by [6, Corollary 1]. By Theorem 5, the low-pass filter m of φ satisfies the anisotropic Strang–Fix conditions of order $s + 1$. In particular,

$$|m(\xi)|^2 = 1 + o\left(\rho(\xi)^{2s}\right) \qquad \text{as } \xi \to 0\,.$$

Therefore,

$$\left|\hat{\varphi}(\xi)\right|^2 = 1 + o\left(\rho(\xi)^{2s}\right) = 1 + o\left(\rho(\xi)^{2nr\ln|\lambda_1|/(\gamma\ln b)}\right) \qquad \text{as } \xi \to 0\,.$$

By (3.21)

$$\left|\hat{\varphi}(\xi)\right|^2 = 1 + o\left(|\xi|^{2r\ln|\lambda_1|/(\gamma^2\ln|\lambda_n|)}\right) \qquad \text{as } \xi \to 0\,. \tag{3.22}$$

Any other scaling function φ_0 of the MRA $(V_i)_{i\in\mathbb{Z}}$ must be of the form $\hat{\varphi}_0(\xi) = \nu(\xi)\hat{\varphi}(\xi)$ for some $\mathbb{Z}^n$-periodic measurable function ν with $|\nu(\xi)| = 1$. Define ν by $\nu(\xi) = \overline{\hat{\varphi}(\xi)}/|\hat{\varphi}(\xi)|^2$ in a small neighborhood of 0 where $\hat{\varphi} \ne 0$. By a simple partition of unity argument we can extend ν to be $\mathbb{Z}^n$-periodic, C^∞, and unimodular. Define φ_0 by $\hat{\varphi}_0(\xi) = \nu(\xi)\hat{\varphi}(\xi)$. By [34, Corollary 5.14], $\varphi_0(x)$ is also r-regular. Since $\hat{\varphi}_0(\xi)$ is real in the small neighborhood of 0 and $\hat{\varphi}_0(0) = 1$ we have by (3.22),

$$\hat{\varphi}_0(\xi) = 1 + o\left(|\xi|^{2r\ln|\lambda_1|/(\gamma^2\ln|\lambda_n|)}\right) \qquad \text{as } \xi \to 0\,.$$

This immediately implies (3.19). □

Note that Theorem 6 implies Corollary 1 in the case when the r-regular wavelet family $\{\psi^1, \dots, \psi^{b-1}\}$ is associated with an r-regular MRA with a scaling function φ. Indeed, by (2.30) and (2.31) we have the identity

$$\left|\hat{\varphi}(\xi)\right|^2 + \left|\hat{\psi}^1(\xi)\right|^2 + \dots + \left|\hat{\psi}^{b-1}(\xi)\right|^2 = \left|\hat{\varphi}\left(B^{-1}\xi\right)\right|^2\,. \tag{3.23}$$

Hence by (3.22), $|\hat{\psi}^l(\xi)| = o(|\xi|^{r\ln|\lambda_1|/(\gamma^2\ln|\lambda_n|)})$ as $\xi \to 0$, which shows (3.15).

Remarks.

(i) Theorem 4 is an extension of the well-known result about vanishing moments for wavelets associated with the dilation $A = 2$ in the dimension $n = 1$. We used an argument in the direct domain as in [14, Theorem 5.5.1]. One could perform the proof in the Fourier domain as in [5] or in the multiresolution setup as in [28].

(ii) Careful examination of the proof of Theorem 4 implies that we have some extra vanishing moments beyond $r\ln|\lambda_1|/\ln|\lambda_n|$, e.g., those associated with eigenvalue λ_1. One might expect to

show vanishing moments up to the order r. However, the other argument in the Fourier domain used in the proof of Theorem 6 yields the same bounds which may be inevitable for general dilations. One possible remedy is to alter the definition of r-regular functions by requiring varied orders of differentiation which depend on the growth rates of the quasi-norm ρ in different directions.
(iii) A direct examination can give more vanishing moments than what is guaranteed by Theorem 4. For example, the Fourier transform of the r-regular scaling function φ_0 constructed in Theorem 3 is equal to 1 in a neighborhood of the origin, so we have

$$\int_{\mathbb{R}^n} x^\alpha \varphi_0(x)\, dx = \delta_{|\alpha|,0} \qquad \text{for all } \alpha\,.$$

By (3.23) the Fourier transforms of the associated r-regular wavelets $\psi^1, \dots, \psi^{b-1}$ vanish in a neighborhood of the origin. Hence,

$$\int_{\mathbb{R}^n} x^\alpha \psi^l(x)\, dx = 0 \qquad \text{for all } \alpha,\ l = 1, \dots, b-1\,.$$

Acknowledgments

The author wishes to thank Prof. Y. Wang for helpful comments regarding this article.

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Received June 9, 2000

Revision received January 10, 2001

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