

## ON RESTRICTIONS OF MODULAR SPIN REPRESENTATIONS OF SYMMETRIC AND ALTERNATING GROUPS

ALEXANDER S. KLESHCHEV AND PHAM HUU TIEP

ABSTRACT. Let  $\mathbb{F}$  be an algebraically closed field of characteristic  $p$  and  $H$  be an almost simple group or a central extension of an almost simple group. An important problem in representation theory is to classify the subgroups  $G$  of  $H$  and  $\mathbb{F}H$ -modules  $V$  such that the restriction  $V \downarrow_G$  is irreducible. For example, this problem is a natural part of the program of describing maximal subgroups in finite classical groups. In this paper we investigate the case of the problem where  $H$  is the Schur's double cover  $\hat{A}_n$  or  $\hat{S}_n$ .

### 1. INTRODUCTION

Let  $\mathbb{F}$  be an algebraically closed field of characteristic  $p$  and  $H$  be an almost simple group or a central extension of an almost simple group. An important problem in representation theory is to classify the subgroups  $G$  of  $H$  and  $\mathbb{F}H$ -modules  $V$  such that the restriction  $V \downarrow_G$  is irreducible. For example, this problem is a natural part of the program of describing maximal subgroups in finite classical groups, see [1, 23, 27].

In this paper we investigate the case of the problem where  $\text{soc}(H/Z(H))$  is the alternating group  $A_n$ . Assume first that the center  $Z(H)$  is trivial, i.e.  $H = A_n$  or  $S_n$ . In this case Saxl [30] has classified all pairs  $(G, V)$  as above, providing the ground field  $\mathbb{F}$  has characteristic 0. In positive characteristic the same has been achieved in [25, 7, 26], at least if  $p \neq 2, 3$  (and we have obtained a lot of partial information even in the exceptional cases  $p = 2, 3$ ).

Assume from now on that  $Z(H)$  is non-trivial. If  $n \neq 6, 7$  then the only non-trivial central extensions are the Schur's double covers  $\hat{A}_n$ ,  $\hat{S}_n$  and  $\tilde{S}_n$ . So we may and will assume *from now on* that  $p \neq 2$ , as otherwise the center acts trivially and so the problem reduces to the case  $Z(H) = 1$ . Moreover, the group algebras  $\mathbb{F}\hat{S}_n$  and  $\mathbb{F}\tilde{S}_n$  are canonically isomorphic, so we only have to deal with  $\hat{A}_n$  and  $\hat{S}_n$ .

To be more precise,  $\hat{S}_n$  is the double cover of the symmetric group  $S_n$ , in which transpositions lift to involutions. It can be described as the group generated by

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$t_1, t_2, \dots, t_{n-1}, z$  subject to the following relations:

$$\begin{aligned} z^2 &= 1 = t_i^2 \quad (1 \leq i \leq n-1), \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1} \quad (1 \leq i \leq n-2), \\ t_i t_j &= z t_j t_i \quad (1 \leq i, j \leq n-1 \text{ and } |i-j| > 1). \end{aligned}$$

Then  $z \in \hat{S}_n$  is a central element of order 2, and we have the following exact sequence:

$$1 \longrightarrow \langle z \rangle \longrightarrow \hat{S}_n \xrightarrow{\pi} S_n \longrightarrow 1.$$

For any subgroup  $G < S_n$  we denote  $\hat{G} := \pi^{-1}(G)$ , where  $\pi : \hat{S}_n \rightarrow S_n$  is the natural projection. For example, we have  $\hat{A}_n$  for the alternating group  $A_n < S_n$ .

The results of Kleidman and Wales [22] deal with the problem of irreducible restrictions from  $\hat{S}_n$  and  $\hat{A}_n$  to subgroups under the assumption that the ground field has characteristic 0. In this paper we study the situation in characteristic  $p$ .

In §3 we get our first main result: a lower boundary for dimensions of faithful  $\mathbb{F}\hat{S}_n$  and  $\mathbb{F}\hat{A}_n$ -modules. This turns out to be an effective tool for studying irreducible restrictions. In positive characteristic our result refines the Wagner's lower bound [31]. Set

$$\begin{aligned} \delta(\hat{S}_n) &= \begin{cases} 2^{k-1} & \text{if } n = 2k, \\ 2^{k-1} & \text{if } n = 2k+1 \text{ and } p|n, \\ 2^k & \text{if } n = 2k+1 \text{ and } p \nmid n; \end{cases} \\ \delta(\hat{A}_n) &= \begin{cases} 2^{k-2} & \text{if } n = 2k \text{ and } p|n, \\ 2^{k-1} & \text{if } n = 2k \text{ and } p \nmid n, \\ 2^{k-1} & \text{if } n = 2k+1. \end{cases} \end{aligned}$$

Setting  $\kappa_n = 1$  if  $p|n$  and 0 otherwise, we have

$$(1.1) \quad \delta(\hat{S}_n) = 2^{\lfloor (n-1-\kappa_n)/2 \rfloor} \quad \text{and} \quad \delta(\hat{A}_n) = 2^{\lfloor (n-2-\kappa_n)/2 \rfloor}.$$

Let  $H = \hat{S}_n$  or  $\hat{A}_n$ . It is known (see Lemma 2.1 below) that  $\delta(H)$  is the dimension of a basic spin  $\mathbb{F}H$ -module. The following theorem shows that  $\delta(H)$  is actually the *minimal* possible dimension of a faithful  $\mathbb{F}H$ -module, and there are no other faithful modules with dimensions in the interval  $[\delta(H), 2\delta(H))$ .

**Theorem A.** *Let  $n \geq 8$ ,  $H = \hat{S}_n$  or  $\hat{A}_n$ , and let  $V$  be an irreducible faithful  $\mathbb{F}H$ -module of dimension less than  $2\delta(H)$ . Then  $V$  is a basic spin module and  $\dim V = \delta(H)$ .*

Our next main theorem classifies the irreducible restrictions from  $\hat{S}_n$  and  $\hat{A}_n$  to subgroups  $G$  such that  $\pi(G)$  is a primitive subgroup of  $S_n$ . We refer the reader to §2 for the definition of the second basic module.

**Theorem B.** *Let  $H = \hat{S}_n$  or  $\hat{A}_n$  with  $n \geq 5$ ,  $D$  be a faithful irreducible  $\mathbb{F}H$ -module, and let  $G$  be a subgroup of  $H$  such that  $\pi(G) < S_n$  is a primitive subgroup which does not contain  $A_n$ . Then  $D \downarrow_G$  is irreducible if and only if one of the following holds:*

- (i)  $H = \hat{S}_n$ ,  $D$  is a basic spin representation, and one of the following holds:
  - (a)  $n = 5$ ,  $p \neq 5$ , and  $\pi(G) = \mathbb{Z}_5 : \mathbb{Z}_4$ ;
  - (b)  $n = 6$ , and  $\pi(G) = S_5$ ;
  - (c)  $n = 6$ ,  $p \neq 3$ , and  $\pi(G) = A_5$ ;
  - (d)  $n = 8$ , and  $\pi(G) = \text{AGL}_3(2)$ ;

- (e)  $n = 10$ ,  $p \neq 3, 5$ , and  $\pi(G) = S_6, M_{10}$  or  $\text{Aut}(A_6)$ ;
- (f)  $n = 11$ ,  $p = 11$ , and  $\pi(G) = M_{11}$  (two classes);
- (g)  $n = 12$ ,  $p \neq 3$ , and  $\pi(G) = M_{12}$ .
- (ii)  $H = \hat{A}_n$ ,  $D$  is a basic spin representation, and one of the following holds:
  - (a)  $n = 5$ ,  $p \neq 5$ , and  $\pi(G) = \mathbb{Z}_5 : \mathbb{Z}_2$ ;
  - (b)  $n = 6$ , and  $\pi(G) = A_5$ ;
  - (c)  $n = 7$ , and  $\pi(G) = L_2(7)$  (two classes);
  - (d)  $n = 8$ , and  $\pi(G) = \text{AGL}_3(2)$  (two classes);
  - (e)  $n = 9$ ,  $p \neq 3$ , and  $3^2 : Q_8 \leq \pi(G) \leq 3^2 : \text{SL}_2(3)$  or  $L_2(8) \leq \pi(G) \leq \text{Aut}(L_2(8))$ ;
  - (f)  $n = 10$ ,  $p \neq 3$ , and  $\pi(G) = M_{10}$ ;
  - (g)  $n = 10$ ,  $p = 5$ , and  $\pi(G) = A_6$ ;
  - (h)  $n = 11$ ,  $p \neq 3$ , and  $\pi(G) = M_{11}$  (two classes);
  - (i)  $n = 12$ ,  $p \neq 3$ , and  $\pi(G) = M_{12}$  (two classes).
- (iii)  $H = \hat{A}_n$ ,  $D$  is a second basic spin representation, and one of the following holds:
  - (a)  $n = 6$ ,  $p = 3$ , and  $\pi(G) = A_5$ ;
  - (b)  $n = 8$ ,  $p \neq 7$ , and  $\pi(G) = \text{AGL}_3(2)$  (two classes);
  - (c)  $n = 12$ ,  $p \neq 3, 11$ , and  $\pi(G) = M_{12}$  (two classes).
- (iv)  $D$  is neither a basic nor a second basic spin representation, and one of the following holds:
  - (a)  $n = 5$ ,  $p \neq 3, 5$ ,  $H = \hat{S}_5$ ,  $\pi(G) = \mathbb{Z}_5 : \mathbb{Z}_4$ , and  $\dim(D) = 4$ ;
  - (b)  $n = 6$ ,  $p \neq 3, 5$ ,  $H = \hat{S}_6$ ,  $\pi(G) = S_5$ , and  $\dim(D) = 4$ ;
  - (c)  $n = 7$ ,  $p = 3$ ,  $H = \hat{A}_7$ ,  $\pi(G) = L_2(7)$  (two classes), and  $\dim(D) = 6$ .

Theorem B is proved in §5. In §6, we treat the nearly simple case, that is we classify the triples  $(G, H, D)$  such that  $H = \hat{S}_n$  or  $\hat{A}_n$ ,  $G < H$  with  $\text{soc}(G/Z(G))$  simple,  $D$  a faithful  $\mathbb{F}H$ -module, and  $D \downarrow_G$  is irreducible. This is the most interesting case for applications to maximal subgroups in finite classical groups. We note that the case  $p = 0$  has not been treated completely in [22]: Theorem 1.3 of [22] assumes that  $G/Z(G)$  is simple. The missing information can be recovered from Theorem C below by looking at the cases where  $p > n$ .

To state Theorem C, we need to introduce a delicate combinatorics and recall some results from [9]. For any  $n \geq 0$ , let  $\lambda = (\lambda_1, \lambda_2, \dots)$  be a partition of  $n$ . We call  $\lambda$  a  $p$ -strict partition if  $p$  divides  $\lambda_r$  whenever  $\lambda_r = \lambda_{r+1}$ . Let  $\mathcal{P}_p(n)$  denote the set of all  $p$ -strict partitions of  $n$ . We say that  $\lambda \in \mathcal{P}_p(n)$  is *restricted* if

$$\begin{cases} \lambda_r - \lambda_{r+1} < p & \text{if } p \mid \lambda_r, \\ \lambda_r - \lambda_{r+1} \leq p & \text{if } p \nmid \lambda_r \end{cases}$$

for each  $r \geq 1$ . Let  $\mathcal{R}\mathcal{P}_p(n)$  denote the set of all restricted  $p$ -strict partitions of  $n$ .

Let  $\lambda \in \mathcal{P}_p(n)$ . We identify  $\lambda$  with its *Young diagram*

$$\lambda = \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda_r\}.$$

Elements  $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$  are called *nodes*. Define  $\ell := (p-1)/2$ , and label the nodes of  $\lambda$  with *residues*, which are the elements of the set  $I = \{0, 1, \dots, \ell\}$ . The labelling depends only on the column and follows the repeating pattern  $0, 1, \dots, \ell - 1, \ell, \ell - 1, \dots, 1, 0$ , starting from the first column and going to the right, see the example below. The residue of the node  $A$  is denoted  $\text{res } A$ . Denote by  $a(\lambda)$  the number of nodes in the Young diagram  $\lambda$  of residue different from 0.



pair  $(\lambda, \sigma)$  is denoted by  $D(\lambda, \sigma)$ . Moreover, we have  $D(\lambda, 0) \otimes \mathbf{sgn} \cong D(\lambda, 0)$  and  $D(\lambda, \pm) \otimes \mathbf{sgn} \cong D(\lambda, \mp)$ , so Clifford theory yields the following. If  $\lambda \in \mathcal{RP}_p(n)$  has  $a(\lambda)$  even, then  $D(\lambda, 0) \downarrow_{\hat{A}_n}$  decomposes as a direct sum  $E(\lambda, +) \oplus E(\lambda, -)$  of two non-isomorphic irreducible  $\mathbb{F}\hat{A}_n$ -modules. If  $a(\lambda)$  is odd, then  $D(\lambda, +) \downarrow_{\hat{A}_n} \cong D(\lambda, -) \downarrow_{\hat{A}_n}$  is irreducible denoted  $E(\lambda, 0)$ . Now,

$$\{E(\lambda, 0) \mid \lambda \in \mathcal{RP}_p(n), a(\lambda) \text{ odd}\} \sqcup \{E(\lambda, +), E(\lambda, -) \mid \lambda \in \mathcal{RP}_p(n), a(\lambda) \text{ even}\}$$

gives a complete set of pairwise non-isomorphic irreducible  $\mathbb{F}\hat{A}_n$ -modules.

The partition  $\omega_n$ , for which  $D(\omega_n, \sigma)$  and  $E(\omega_n, \sigma)$  are basic spin modules, can be described as follows, see [9, 9.12]. Write  $n = ap + b$  with  $0 \leq b < p$ . Then

$$(1.2) \quad \omega_n := \begin{cases} (p^a, b) & \text{if } b \neq 0, \\ (p^{a-1}, p-1, 1) & \text{if } b = 0. \end{cases}$$

Let  $\lambda \in \mathcal{RP}_p(n)$ . We write  $\lambda \in JS$  if the bottom removable node of  $\lambda$  is its only normal node ( $JS$  stands for Jantzen-Seitz, cf. [18, 24]). Let  $i \in I$ . We write  $\lambda \in JS(i)$  if  $\lambda \in JS$  and the bottom removable node of  $\lambda$  has residue  $i$ .

We set  $\Omega := \{1, 2, \dots, n\}$  and say that a subgroup  $X < S_n$  is of *type*

$$(n_1, n_2, \dots, n_t)$$

if the  $X$ -orbits on  $\Omega$  have lengths  $n_1, n_2, \dots, n_t$ .

The following result includes Theorems 9.17, 9.18 from [9] and some results from [29] (all of which are used in the proof).

**Theorem C.** *Let  $H = \hat{S}_n$  or  $\hat{A}_n$ ,  $D$  be a faithful irreducible  $\mathbb{F}H$ -module, and  $G$  be a subgroup of  $H$  such that  $G/Z(G)$  is almost simple and  $\pi(G)$  does not contain  $A_n$ . Set  $S := \text{soc}(\pi(G))$ . Then  $D \downarrow_G$  is irreducible if and only if one of the following holds:*

- (i)  $S = A_m$  ( $m \geq 5$ ),  $n = m+1$ , and one of the following holds (all embeddings being of type  $(m, 1)$ ):
  - (a)  $H = \hat{S}_{m+1}$ ,  $G = \hat{S}_m$ , and either  $D = D(\lambda, 0)$  for some  $\lambda \in JS(0)$  with  $a(\lambda)$  even or  $D = D(\lambda, \pm)$  for some  $\lambda \in JS$  with  $a(\lambda)$  odd.
  - (b)  $H = \hat{S}_{m+1}$ ,  $G = \hat{A}_m$ , and  $D = D(\lambda, \pm)$  for some  $\lambda \in JS(0)$  with  $a(\lambda)$  odd.
  - (c)  $H = \hat{A}_{m+1}$ ,  $G = \hat{A}_m$ , and either  $D = E(\lambda, 0)$  for some  $\lambda \in JS(0)$  with  $a(\lambda)$  odd or  $D = E(\lambda, \pm)$  for some  $\lambda \in JS$  with  $a(\lambda)$  even.
- (ii)  $S = A_m$  ( $m \geq 5$ ),  $n = m+2$ , and one of the following holds:
  - (a)  $H = \hat{S}_{m+2}$ ,  $G = \hat{S}_m$  (embedding of type  $(m, 1, 1)$ ), and  $D = D(\lambda, \pm)$  for some  $\lambda \in JS(0)$  with  $a(\lambda)$  odd, such that  $\lambda_{[0]} \in JS(1)$ .
  - (b)  $H = \hat{A}_{m+2}$ ,  $\pi(G) = (S_m \times S_2) \cap A_{m+2} \cong S_m$  (embedding of type  $(m, 2)$ ), and one of the following holds:
    - (1)  $D = E(\lambda, \sigma)$  for some  $\lambda \in JS(0)$  such that  $\lambda_{[0]} \in JS(1)$ .
    - (2)  $D$  is basic spin,  $m$  is odd, and  $m \not\equiv -2, 0 \pmod{p}$ .
  - (c)  $H = \hat{S}_{m+2}$ ,  $\pi(G) = (S_m \times S_2) \cap A_{m+2} \cong S_m$  (embedding of type  $(m, 2)$ ), and  $D = D(\lambda, \pm)$  for some  $\lambda \in JS(0)$  with  $a(\lambda)$  odd such that  $\lambda_{[0]} \in JS(1)$ .
  - (d)  $H = \hat{A}_{m+2}$ ,  $G = \hat{A}_m$  (embedding of type  $(m, 1, 1)$ ), and  $D = E(\lambda, \pm)$  for some  $\lambda \in JS(0)$  with  $a(\lambda)$  even such that  $\lambda_{[0]} \in JS(1)$ .

- (iii)  $S = A_m$  ( $m \geq 5$ ),  $n = m + 3$ ,  $3 \neq p|n$ ,  $m$  is odd,  $H = \hat{A}_{m+3}$ ,  $\pi(G) = (S_m \times S_2 \times S_1) \cap A_{m+3} \cong S_m$  of type  $(m, 2, 1)$ , and  $D$  is basic spin.
- (iv)  $\pi(G)$  is primitive on  $\Omega$ ,  $S \in \{A_5, A_6, L_2(7), L_2(8), M_{11}, M_{12}\}$ , and the triple  $(G, H, D)$  is as listed in Theorem B.
- (v)  $S = A_5$ ,  $n = 7$ ,  $\pi(G)$  is of type  $(6, 1)$ , and one of the following holds:
  - (a)  $H = \hat{A}_7$ ,  $\pi(G) = A_5$ , and either  $p = 3$  and  $D$  is second basic (of dimension 6), or  $p \neq 3$  and  $D$  is basic spin.
  - (b)  $H = \hat{S}_7$ ,  $p = 7$ ,  $\pi(G) = A_5$  or  $S_5$ , and  $D$  is basic spin.
- (vi)  $S = A_6$ ,  $n = 11$ ,  $D$  is basic spin, and  $\pi(G)$  is of type  $(10, 1)$ . If  $H = \hat{A}_{11}$  then  $p \neq 3, 5$  and  $\pi(G) = M_{10}$ . If  $H = \hat{S}_{11}$  then  $p = 11$  and  $\pi(G) = M_{10}$ ,  $S_6$ , or  $\text{Aut}(A_6)$ .
- (vii)  $S = L_2(8)$ ,  $H = \hat{A}_{10}$ ,  $p = 5$ ,  $\pi(G) = S$  or  $\text{Aut}(S)$ ,  $\pi(G)$  is of type  $(9, 1)$ , and  $D$  is basic spin.
- (viii)  $S = M_{12}$ ,  $D$  is basic spin, and one of the following holds:
  - (a)  $n = 13$ ,  $\pi(G) = S$  is of type  $(12, 1)$ , and  $(H, p) = (\hat{S}_{13}, 13)$  or  $(\hat{A}_{13}, \neq 3)$ ;
  - (b)  $n = 14$ ,  $\pi(G) = S$  is of type  $(12, 1, 1)$ , and  $(H, p) = (\hat{A}_{14}, 7)$ .

Finally, we consider subgroups  $G$  of  $\hat{S}_n$  and  $\hat{A}_n$  with  $\pi(G)$  imprimitive and not almost simple. Here we have the following results.

**Theorem D.** *Let  $p > 3$ ,  $n \geq 7$  and  $H = \hat{S}_n$  or  $\hat{A}_n$ . Suppose that  $G < H$  is a subgroup with  $\pi(G)$  imprimitive on  $\Omega$ , and such that  $G/Z(G)$  is not almost simple. Assume further that  $D$  is a faithful non-basic  $\mathbb{F}H$ -module such that  $D \downarrow_G$  is irreducible. Then  $\pi(G)$  is transitive,  $n$  is even, and one of the following happens:*

- (i)  $\pi(G)$  has 2 blocks of imprimitivity of size  $n/2$ .
- (ii)  $\pi(G)$  has  $n/2$  blocks of imprimitivity of size 2.

We believe that even in the exceptional cases of Theorem D the restriction is usually reducible (with very few exceptions, like the one in [22, Theorem 1.1(6)]), and that the restriction  $p > 3$  is unnecessary, but we can not prove this at the moment. Note also that we have excluded basic modules in Theorem D. The following result describes the irreducible restrictions of basic modules to maximal imprimitive subgroups. We refer the reader to §4 for more results on basic modules.

**Theorem E.** *Let  $H = \hat{S}_n$  or  $\hat{A}_n$ ,  $D$  be a basic spin  $\mathbb{F}H$ -module, and let  $G < H$  be a subgroup with  $\pi(G)$  maximal imprimitive. Then  $D \downarrow_G$  is irreducible if and only if one of the following holds:*

- (i)  $H = \hat{S}_n$  and one of the following holds:
  - (a)  $\pi(G) = S_{n-a} \times S_a$ ,  $a < n/2$ ,  $p \nmid a$ ,  $p \nmid (n-a)$ , and either  $n$  is even, or  $n$  is odd and  $p | n$ .
  - (b)  $\pi(G) = S_a \wr S_b$  for some  $a, b \geq 2$  with  $n = ab$  and  $p \nmid a$ .
- (ii)  $H = \hat{A}_n$  and one of the following holds:
  - (a)  $\pi(G) = A_n \cap (S_{n-a} \times S_a)$ ,  $a < n/2$ ,  $p \nmid a$ ,  $p \nmid (n-a)$ , and either  $n$  is odd, or  $n$  is even and  $p | n$ .
  - (b)  $\pi(G) = A_n \cap (S_a \wr S_b)$  for some  $a, b \geq 2$  with  $n = ab$  and  $p \nmid a$ .

Finally we mention a result from §8 on irreducible tensor products, which is also relevant to the problem on maximal subgroups, and follows from the results of §7.

Similar results for symmetric and alternating groups in arbitrary characteristic or Schur's double covers in characteristic 0 were obtained in [4, 5, 6, 3, 13]. In the next theorem  $h(\lambda)$  denotes the number of non-zero parts of a partition  $\lambda$  and  $\lambda^{\mathbf{M}}$  is the Mullineux image of  $\lambda$ .

**Theorem F.** *Let  $p > 3$  and  $n \geq 7$ .*

- (i) *Let  $H = \hat{S}_n$  or  $\hat{A}_n$ , and  $V, W$  be non-basic faithful  $\mathbb{F}H$ -modules. Then  $V \otimes W$  is reducible.*
- (ii) *Let  $V$  be a non-basic faithful  $\mathbb{F}\hat{S}_n$ -module, and  $W = D^\lambda$  be an irreducible  $\mathbb{F}S_n$ -module with  $h(\lambda) > 2$  and  $h(\lambda^{\mathbf{M}}) > 2$ . Then  $V \otimes W$  is reducible.*

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## 2. PRELIMINARIES

Recall that  $\mathbb{F}$  is an algebraically closed field of characteristic  $p \neq 2$ . For any group  $G$  we denote by  $\mathbf{1}_G$  or just  $\mathbf{1}$  the trivial  $\mathbb{F}G$ -module. Also denote by  $\mathbf{sgn}$  the 1-dimensional sign representation of the symmetric group  $S_n$ .

If  $g \in S_n$  is any element, we denote by  $\hat{g}$  any element in  $\pi^{-1}(g)$ . Thus,  $\pi^{-1}(g) = \{\hat{g}, z\hat{g}\}$ . Also, if  $g \in S_n$  is an element of odd order then we can write  $\pi^{-1}(g) = \{g^+, g^-\}$ , where  $g^+$  has the same order as  $g$  and  $g^- = zg^+$ .

For any composition  $\lambda$  of  $n$  we denote the corresponding Young subgroup of  $S_n$  by  $S_\lambda$  and  $A_\lambda := S_\lambda \cap A_n$ . Set

$$M^\lambda = (\mathbf{1}_{\hat{S}_\lambda}) \uparrow^{\hat{S}_n}.$$

Note that  $z$  acts trivially on  $M^\lambda$ . On restriction,  $M^\lambda$  is also an  $\mathbb{F}\hat{A}_n$ -module, isomorphic to  $(\mathbf{1}_{\hat{A}_\lambda}) \uparrow^{\hat{A}_n}$ .

If  $G$  is a finite group and  $V$  is a finite dimensional  $\mathbb{C}G$ -module, we denote by  $\bar{V}$  its reduction modulo  $p$ , considered as an element of the Grothendieck group of  $\mathbb{F}G$ -modules. If  $L_1, \dots, L_k$  are irreducible  $\mathbb{F}G$ -modules, the notation

$$\bar{V} = a_1 L_1 + \dots + a_k L_k$$

means that  $\bar{V} = a_1[L_1] + \dots + a_k[L_k]$  in the Grothendieck group. We will often abuse the notion and speak of  $\bar{V}$  as a module—for example if  $\bar{V}$  is irreducible then we will speak of the irreducible module  $\bar{V}$ , which is defined up to isomorphism.

Let  $G = \hat{S}_n$  or  $\hat{A}_n$ . Then  $G$  possesses (one or two) *basic (spin) modules* over  $\mathbb{C}$ —these are the modules whose characters correspond to the partition  $(n)$  in Schur's classification. Composition factors of their reductions modulo  $p$  will be called *basic (spin) modules* over  $\mathbb{F}G$ . Moreover, the irreducible  $\mathbb{C}G$ -modules with characters corresponding to the partition  $(n-1, 1)$  will be referred to as *second basic (spin) modules*. Composition factors of their reductions modulo  $p$ , different from basic spin modules, will be referred to as *second basic (spin) modules* over  $\mathbb{F}G$ . Basic and second basic modules over  $\mathbb{C}G$  and  $\mathbb{F}G$  were studied in detail by Wales, see [32, Tables III, IV]. We reproduce this information for basic modules:

**Lemma 2.1.** [32, Table III] *Suppose  $n \geq 4$ .*

- (i) *Let  $n$  be even. There are two basic spin modules  $Y_{n,\mathbb{C}}^+$  and  $Y_{n,\mathbb{C}}^-$  over  $\mathbb{C}\hat{S}_n$  and one basic spin module  $U_{n,\mathbb{C}}$  over  $\mathbb{C}\hat{A}_n$ , all of dimension  $2^{\binom{n-2}{2}}$ , and  $Y_{n,\mathbb{C}}^+ \downarrow_{\hat{A}_n} \cong Y_{n,\mathbb{C}}^- \downarrow_{\hat{A}_n} \cong U_{n,\mathbb{C}}$ . Moreover,*

- (a) If  $p \nmid n$ , then  $Y_n^\pm := \bar{Y}_{n,\mathbb{C}}^\pm$  and  $U_n := \bar{U}_{n,\mathbb{C}}$  are irreducible. Moreover,  $Y_n^+ \not\cong Y_n^-$ , and  $Y_n^+ \downarrow_{\hat{A}_n} \cong Y_n^- \downarrow_{\hat{A}_n} \cong U_n$ .
- (b) If  $p \mid n$ , then  $Y_n := \bar{Y}_{n,\mathbb{C}}^+ \cong \bar{Y}_{n,\mathbb{C}}^-$  is irreducible. Moreover,  $Y_n \downarrow_{\hat{A}_n} \cong U_n^+ \oplus U_n^-$  for irreducible  $\hat{A}_n$ -modules  $U_n^+ \not\cong U_n^-$ , and  $\bar{U}_{n,\mathbb{C}} = U_n^+ + U_n^-$ .
- (ii) Let  $n$  be odd. There is one basic spin modules  $Y_{n,\mathbb{C}}$  over  $\mathbb{C}\hat{S}_n$  and two basic spin modules  $U_{n,\mathbb{C}}^+$  and  $U_{n,\mathbb{C}}^-$  over  $\mathbb{C}\hat{A}_n$ , with  $\dim Y_{n,\mathbb{C}} = 2 \dim U_{n,\mathbb{C}}^\pm = 2^{(n-1)/2}$ , and  $Y_{n,\mathbb{C}} \downarrow_{\hat{A}_n} \cong U_{n,\mathbb{C}}^+ \oplus U_{n,\mathbb{C}}^-$ . Moreover,
- (a) If  $p \nmid n$ , then  $Y_n := \bar{Y}_{n,\mathbb{C}}$  and  $U_n^\pm := \bar{U}_{n,\mathbb{C}}^\pm$  are irreducible. Moreover,  $U_n^+ \not\cong U_n^-$ , and  $Y_n \downarrow_{\hat{A}_n} \cong U_n^+ \oplus U_n^-$ .
- (b) If  $p \mid n$ , then  $\bar{Y}_{n,\mathbb{C}}$  has two composition factors  $Y_n^+ \not\cong Y_n^-$ , and  $U_n := \bar{U}_{n,\mathbb{C}}^- \cong \bar{U}_{n,\mathbb{C}}^+$  is irreducible. Moreover,  $Y_n^+ \downarrow_{\hat{A}_n} \cong Y_n^- \downarrow_{\hat{A}_n} \cong U_n$ .

Occasionally, while speaking of the basic spin representations  $Y_{n,\mathbb{C}}^{(\pm)}$ ,  $Y_n^{(\pm)}$ , etc., we will allow for  $n = 2$  and  $3$ . For such  $n$ , projective representations of  $S_n$  and  $A_n$  are linear, and we use the following interpretations:  $Y_{2,\mathbb{C}}^+$  (resp.  $Y_2^+$ ) is the trivial  $\mathbb{C}S_3$ - (resp.  $\mathbb{F}S_3$ -) module,  $Y_{2,\mathbb{C}}^-$  (resp.  $Y_2^-$ ) is the sign  $\mathbb{C}S_3$ -module,  $Y_{3,\mathbb{C}}$  is the natural 2-dimensional  $\mathbb{C}S_3$ -module,  $Y_3$  is the natural 2-dimensional  $\mathbb{F}S_3$ -module if  $p > 3$ , and  $Y_3^+$  and  $Y_3^-$  are, respectively, the trivial and the sign modules over  $\mathbb{F}S_3$  if  $p = 3$ . Observe that under these interpretations the formulas for dimensions in Lemma 2.1, and the facts about reductions modulo  $p$  and restrictions to  $A_n$  still hold.

We recall some other known results. The first one is due to Wagner.

**Lemma 2.2.** [31] *Let  $n \geq 9$ ,  $s$  be the number of terms in the 2-adic expansion of  $n$ , and let  $H = \hat{S}_n$  or  $\hat{A}_n$ . Then  $2^{\lfloor (n-s-1)/2 \rfloor}$  divides  $\dim D$  for any faithful  $\mathbb{F}H$ -module  $D$ .  $\square$*

Lemma 2.2 implies that  $\dim D \geq 2^{\lfloor (n-s-1)/2 \rfloor}$  for any faithful  $\mathbb{F}H$ -module  $D$ . In §3 we prove Theorem A, which improves this lower bound. Next, we cite a result of Wales. If  $p \neq 3$ , it shows that certain elements of order 3 in  $\hat{S}_n$  have three different eigenvalues on irreducible modules, except possibly basic spin modules.

**Lemma 2.3.** [32, 8.1] *Let  $n \geq 5$ ,  $c = (123)^+ \in \hat{S}_n$ , and let  $V$  be an irreducible  $\mathbb{F}\hat{S}_n$ -module on which  $c$  has a quadratic minimal polynomial. Then  $V$  is basic spin.  $\square$*

The following ‘recognition’ result for basic spin representations was also essentially demonstrated by Wales in the proof of [32, 8.1].

**Lemma 2.4.** *Let  $n \geq 6$ , and  $V$  be an irreducible  $\mathbb{F}\hat{S}_n$ -module (resp.  $\mathbb{F}\hat{A}_n$ -module). Then  $V$  is basic spin if and only if all composition factors of the restriction  $V \downarrow_{\hat{S}_{n-1}}$  (resp.  $V \downarrow_{\hat{A}_{n-1}}$ ) are basic spin.*

*Proof.* We prove the result for  $\hat{S}_n$ , the proof for  $\hat{A}_n$  being similar. By Frobenius reciprocity we may assume that  $V$  is a constituent of a reduction modulo  $p$  of  $(Y_{n-1,\mathbb{C}}^{(\pm)}) \uparrow \hat{S}_n$ . By the branching theorem [15, 10.2], this induced module is a sum of basic modules  $Y_n^{(\pm)}$  and second basic modules  $Z_n^{(\pm)}$ . Now if  $V$  is not basic, then  $V$  has to be a non-basic constituent of reduction  $\bar{Z}_n^{(\pm)}$ . But then it follows from [32, Table IV] that the restriction  $V \downarrow_{\hat{S}_{n-1}}$  contains a non-basic constituent, contrary to our assumption.  $\square$



**Lemma 2.5.** *Let  $n \geq 7$ ,  $p \neq 2, 3$ , and  $\varphi$  be an irreducible faithful Brauer character of  $\hat{S}_n$ . Define the elements  $c, d \in \hat{S}_n$  by setting  $c := (123)^+$  and  $d := ((123)(456))^+$ . Then  $\varphi(c)$  and  $\varphi(d)$  are rational integers with*

- (i)  $\varphi(1)/7 \geq \varphi(c) \geq -\varphi(1)/2$ , and  $\varphi(c) = -\varphi(1)/2$  if and only if  $\varphi$  is basic spin.
- (ii)  $\varphi(1)/4 \geq \varphi(d) \geq -\varphi(1)/8$ , and  $\varphi(d) = \varphi(1)/4$  if and only if  $\varphi$  is basic spin. Moreover, if  $n \geq 11$  then  $\varphi(d) \geq -\varphi(1)/56$ .

*Proof.* We prove (i), the proof of (ii) being similar. Apply induction on  $n$ . The statement is true for small  $n$  by [19]. For inductive step, assume (i) holds for  $n-1$ . Then  $\varphi \downarrow_{\hat{S}_{n-1}} = \sum_{i=1}^t \varphi_i$  for some irreducible faithful Brauer characters  $\varphi_i$  of  $\hat{S}_{n-1}$ . By induction hypothesis,  $\varphi_i(1)/7 \geq \varphi_i(c) \geq -\varphi_i(1)/2$  for each  $i$ , whence  $\varphi(1)/7 \geq \varphi(c) \geq -\varphi(1)/2$ . Moreover, if  $\varphi(c) = -\varphi(1)/2$  then  $\varphi_i(c) = -\varphi_i(1)/2$  for each  $i$ , whence  $\varphi_i$  is basic by induction hypothesis, and so  $\varphi$  itself is basic by Lemma 2.4.  $\square$

### 3. MINIMAL DIMENSIONS (THEOREM A)

In this section we prove Theorem A. Apply induction on  $n$ . For small  $n$  the statement holds by [19]. Let  $n \geq 12$ . Assume  $V$  is not basic. We have to show that  $\dim V \geq 2\delta(H)$ . Let  $\varphi$  be the Brauer character of  $V$ . We will often use the fact that the minimal polynomial of  $c$  on  $V$  has degree 3, see Lemma 2.3. We consider several cases.

*Case S1:*  $H = \hat{S}_n$ ,  $n = 2k$ , and  $p \nmid (n-2)$ . Note that  $C_H(t_1) \geq \langle t_1 \rangle \times \hat{A}_{n-2}$ . Since  $t_1$  is not central, it has both eigenvalues 1 and  $-1$  on  $V$ . Let  $V_+$  and  $V_-$  be the corresponding eigenspaces. We have  $t_3 t_1 t_3 = z t_1$  and  $z = -1$  on  $V$ , so  $t_3$  swaps  $V_+$  and  $V_-$ . Observe that  $V_{\pm}$  are  $\hat{A}_{n-2}$ -modules, and  $t_3$  normalizes  $\hat{A}_{n-2}$ . In particular, all composition factors of  $V_+$  are basic for  $\hat{A}_{n-2}$  if and only if all composition factors of  $V_-$  are basic for  $\hat{A}_{n-2}$ . But all composition factors of both  $V_+$  and  $V_-$  could not be basic as then  $V$  itself would be basic by Lemma 2.4. Thus both  $V_{\pm}$  have non-basic composition factors. By induction hypothesis,  $\dim V_{\pm} \geq 2\delta(\hat{A}_{n-2}) = 2^{k-1}$ , hence  $\dim V \geq 2^k = 2\delta(\hat{S}_n)$ .

*Case S2:*  $H = \hat{S}_n$ ,  $p \mid n = 2k+1$ . Argue as in S1.

*Case S3:*  $H = \hat{S}_n$ ,  $n = 2k$ ,  $p \mid (n-2)$ , and  $p \neq 3$ . Clearly,  $C_H(c) \geq \langle c \rangle \times \hat{S}_{n-3}$ . By Lemma 2.3,  $c$  has all three eigenvalues  $\omega^i$ ,  $i = 0, 1, 2$ , on  $V$ , where  $\omega \in \mathbb{F}$  is a primitive third root of unity. Let  $V_i$ ,  $i = 1, 2, 3$ , be the corresponding eigenspaces, considered as modules over  $\hat{S}_{n-3}$ . Then at least one of them has a non-basic composition factor by Lemma 2.4. Thus

$$\dim V \geq 4\delta(\hat{S}_{n-3}) = 4 \cdot 2^{k-2} = 2\delta(\hat{S}_n).$$

*Case S4:*  $H = \hat{S}_n$ ,  $n = 2k+1$ ,  $p \nmid n$ ,  $p \nmid (n-3)$ , and  $p \neq 3$ . We restrict  $V$  to  $K := S_3 \times \hat{A}_{n-3}$ , where  $S_3$  is generated by  $t_1$  and  $t_2$ . Since  $p \neq 2, 3$ , there are three irreducible characters of  $S_3$ : the trivial character  $\chi$ , the sign character  $\psi$ , and a character  $\alpha$  of degree 2. By Lemma 2.3,  $\alpha$  and at least one of  $\chi, \psi$  appear in  $V \downarrow_{S_3}$ . Next,  $t_4$  centralizes  $c$  and  $t_4 t_1 t_4 = z t_1$ , hence  $t_4$  swaps the characters  $\chi$  and  $\psi$ . Thus, all three characters  $\chi, \psi$ , and  $\alpha$  appear in  $\varphi \downarrow_{S_3}$ , and we can write

$$\varphi \downarrow_K \cong \alpha \otimes \beta_1 + \chi \otimes \beta_2 + \psi \otimes \beta_3,$$

for some Brauer characters  $\beta_1, \beta_2, \beta_3$  of  $\hat{A}_{n-3}$ . Moreover, the characters  $\beta_2$  and  $\beta_3$  are  $t_4$ -conjugate. We have  $\alpha(c) = -1$  and  $\beta_3(g) = \beta_2(t_4gt_4)$  for any  $g \in \hat{A}_{n-3}$ . In particular,  $\beta_2(1) = \beta_3(1)$ . So  $\varphi(c) = -\beta_1(1) + 2\beta_2(1)$ . Let  $c' = (6, 7, 8)^+ \in \hat{A}_{n-3}$ . As  $c'$  is conjugate to  $c$  and is centralized by  $t_4$ , we get  $\varphi(c) = \varphi(c') = 2\beta_1(c') + 2\beta_2(c')$ . Therefore

$$(3.1) \quad \beta_1(1) + 2\beta_1(c') = 2(\beta_2(1) - \beta_2(c')) > 0.$$

Hence  $\beta_1$  contains a non-basic irreducible Brauer character by Lemma 2.5. If  $\beta_1$  is not irreducible, then

$$\varphi(1) \geq \alpha(1)(2\delta(\hat{A}_{n-3}) + \delta(\hat{A}_{n-3})) + 2\delta(\hat{A}_{n-3}) = 2^{k+1} = 2\delta(\hat{S}_n).$$

Now, let  $\beta_1$  be irreducible. If either  $\beta_2$  is not irreducible or  $\beta_2$  is not basic, then

$$\varphi(1) \geq 2\alpha(1)\delta(\hat{A}_{n-3}) + 4\delta(\hat{A}_{n-3}) \geq 2\delta(\hat{S}_n).$$

We are left with the case where all  $\beta_i$  are irreducible and  $\beta_2, \beta_3$  are basic. Moreover, we may assume that  $\beta_1(1) < 3 \cdot 2^{k-2}$ . By Lemma 2.5, the right-hand side of (3.1) equals  $3 \cdot 2^{k-2}$ , so  $\beta_1(c') > 0$ . As  $cc'$  is conjugate to the element  $d$  from Lemma 2.5, we now have by that lemma:

$$-\varphi(1)/8 \leq \varphi(d) = -\beta_1(c') + 2\beta_2(c') < 2\beta_2(c') = -\beta_2(1) = -2^{k-2}.$$

Hence  $\varphi(1) > 2^{k+1}$ .

*Case S5:*  $H = \hat{S}_n$ ,  $n = 2k + 1$ ,  $p \nmid n$ ,  $p \mid (n - 3)$ , and  $p \neq 3$ . We restrict  $V$  to the central product  $K := \hat{A}_4 * \hat{S}_{n-4}$ , where  $\hat{S}_{n-4}$  is generated by  $t_j$ ,  $5 \leq j \leq n - 1$ , and  $\hat{A}_4$  is generated by  $c = t_1t_2$  and  $t_2t_3$ . Observe that  $\hat{A}_4 \cong Q_8 \cdot \langle c \rangle$  has exactly 3 faithful irreducible characters, say  $\alpha_i$ ,  $i = 0, 1, 2$ , all of degree 2, with  $\alpha_i(c) = -\omega^i$  (see also [15, 4.8]). So we can write

$$\varphi \downarrow_K = x\alpha_1 \otimes \beta_1 + y\alpha_2 \otimes \beta_2 + z\alpha_3 \otimes \beta_3$$

for some faithful Brauer characters  $\beta_i$  of  $\hat{A}_{n-3}$  and  $x, y, z \in \{0, 1\}$ . Note that  $u := t_1t_5$  normalizes both  $\hat{A}_4$  and  $\hat{S}_{n-4}$ . Also  $ucu^{-1} = c^{-1}$ , so the characters  $\alpha_2$  and  $\alpha_3$  are  $u$ -conjugate. It follows that the components  $y\alpha_2 \otimes \beta_2$  and  $z\alpha_3 \otimes \beta_3$  are  $u$ -conjugate. Now Lemma 2.3 implies that  $y = z = 1$ .

If  $\beta_2$  is not basic or is not irreducible, then

$$\dim V \geq 8\delta(\hat{S}_{n-4}) = 8 \cdot 2^{k-2} = 2\delta(\hat{S}_n).$$

Otherwise  $x = 1$  and  $\beta_1$  has to be non-basic by Lemma 2.4. Then  $\beta_1(1) \geq 2\delta(\hat{S}_{n-4})$  and again  $\dim V \geq 8\delta(\hat{S}_{n-4}) = 2\delta(\hat{S}_n)$ .

*Case S6:*  $H = \hat{S}_n$ ,  $n = 2k$ , and  $3 = p \mid (n - 2)$ . Set  $V^i = \text{Ker}((c - 1)^i)$  for  $i = 1, 2, 3$ ,  $V^0 = 0$ , and  $W^i = V^i/V^{i-1}$ . By Lemma 2.3, each  $W^i$  is a non-zero faithful module for  $\hat{S}_{n-3} \leq C_H(c)$ . Also, at least one of the modules  $W^i$  is non-basic by Lemma 2.4. Therefore

$$\dim V \geq 4\delta(\hat{S}_{n-3}) = 4 \cdot 2^{k-2} = 2\delta(\hat{S}_n).$$

*Case S7:*  $H = \hat{S}_n$ ,  $n = 2k + 1$ , and  $3 = p \nmid n$ . We restrict  $V$  to  $K := \langle t_1 \rangle \times \hat{A}_{n-3}$  and use the notation of S6. Since  $t_1$  inverts  $c$  and  $\hat{A}_{n-3}$  centralizes  $c$ , each  $W^i$  is a (nonzero)  $K$ -module. Let  $V \downarrow_{\hat{A}_{n-3}}$  consist of  $a + b$  (not necessarily distinct) composition factors, with  $a$  of them non-basic and  $b$  basic. The argument with  $t_4$  given in S1 shows that both  $a$  and  $b$  are even. Also,  $a > 0$  by Lemma 2.4. Moreover,

the multiplication by  $c - 1$  yields injective homomorphisms  $W^3 \hookrightarrow W^2 \hookrightarrow W^1$  as  $\hat{A}_{n-3}$ -modules, whence  $a \geq 3$  or  $b \geq 3$ . If  $a \geq 3$ , then  $a \geq 4$  as  $a$  is even, whence

$$\dim V \geq 4 \cdot 2\delta(\hat{A}_{n-3}) = 8 \cdot 2^{k-2} = 2\delta(\hat{S}_n).$$

If  $a = 2$ , then  $b \geq 4$ , whence

$$\dim V \geq (2a + b)\delta(\hat{A}_{n-3}) \geq 8\delta(\hat{A}_{n-3}) = 2\delta(\hat{S}_n).$$

The induction step for  $\hat{S}_n$  is complete. So we now assume that the result holds for  $\hat{S}_n$ .

*Case A1:*  $H = \hat{A}_n$ , and either  $p|n = 2k$  or  $p \nmid n = 2k + 1$ . Using the result for  $\hat{S}_n$ , we get

$$\dim V \geq (1/2) \cdot 2 \cdot \delta(\hat{S}_n) = 2\delta(\hat{A}_n).$$

*Case A2:*  $H = \hat{A}_n$ ,  $n = 2k + 1$ , and  $p|n$ . Since  $V$  is non-basic, at least one of the composition factors of  $V \downarrow_{\hat{A}_{n-1}}$  is non-basic. Hence  $\dim V \geq 2\delta(\hat{A}_{n-1}) = 2\delta(\hat{A}_n)$ .

*Case A3:*  $H = \hat{A}_n$ ,  $n = 2k$ ,  $p \nmid n$ ,  $p \nmid (n - 4)$ , and  $p \neq 3$ . The same argument as in S5 but applied to  $K = \hat{A}_4 * \hat{A}_{n-4}$  yields

$$\dim V \geq 8\delta(\hat{A}_{n-4}) = 8 \cdot 2^{k-3} = 2\delta(\hat{A}_n).$$

*Case A4:*  $H = \hat{A}_n$ ,  $n = 2k$ ,  $p \nmid n$ ,  $p \nmid (n - 3)$ , and  $p \neq 3$ . In this and the next cases we restrict  $V$  to the subgroup  $G = \hat{A}_{n-3} \cdot \langle u \rangle$  inside  $N_H(\langle c \rangle)$ , where  $u = t_1 t_4$ . First, we prove:

(\*) *Let  $p \geq 3$ ,  $n = 2k$ ,  $p \nmid (n - 3)$ , and  $W$  be a faithful  $\mathbb{F}G$ -module of dimension less than  $2^{k-1}$ . Then  $W$  is irreducible of dimension  $2^{k-2}$  and  $W \downarrow_{\hat{A}_{n-3}}$  involves only basic spin modules.*

Indeed, let us identify all the groups under consideration with subgroups of  $GL(W)$ . Let  $\mathbf{i} \in GL(W)$  be the multiplication by a primitive fourth root of unity (in  $\mathbb{F}$ ) on  $W$ . Then  $(\mathbf{i}u)^2 = \mathbf{i}^2 z = 1 = (t_4)^2$ . Next,  $\mathbf{i}$  and  $t_1$  centralize  $\hat{A}_{n-3}$ , so the (conjugation) action of  $\mathbf{i}u$  on  $\hat{A}_{n-3}$  is exactly the same as that of  $t_4$ . It follows that  $B := \hat{A}_{n-3} \cdot \langle \mathbf{i}u \rangle \cong \hat{S}_{n-3}$ . Now  $W$  is also a faithful irreducible  $B$ -module of dimension less than  $2^{k-1} = 2\delta(\hat{S}_{n-3})$ , whence  $W$  is basic for  $B$ , of dimension  $2^{k-2}$ , and so  $W \downarrow_{\hat{A}_{n-3}}$  involves only basic modules.

Abusing the terminology, we will call  $W$  as in (\*) *basic module for  $G$* .

As in S3,  $V_0$  and  $V' := V_1 \oplus V_2$  are nonzero faithful  $G$ -modules. By Lemma 2.4, at least one of  $V_0$  and  $V'$  have to involve non-basic  $G$ -modules. If both of them involve non-basic modules, or only one of them involves a non-basic module and one of them is not irreducible, then by (\*),  $\dim V \geq 2 \cdot 2^{k-1} = 2\delta(\hat{A}_n)$ . Hence we may assume that both of them are irreducible and exactly one of them is non-basic, and that  $\dim V < 2^k$ .

Assume first that  $V_0$  is non-basic and  $V'$  is basic. Then  $\dim V' \leq (\dim V_0)/2$  by (\*), and so  $\dim V \leq 3(\dim V_0)/2$ . By the definition of  $V'$ , the trace of  $c$  on  $V'$  is  $-(\dim V')/2$ . Hence

$$\varphi(c) = \dim V_0 - (\dim V')/2 \geq 3(\dim V_0)/4 \geq (\dim V)/2 = \varphi(1)/2,$$

contrary to Lemma 2.5.

Assume next that  $V_0$  is basic, with Brauer  $G$ -character  $\varphi_0$ , and  $V'$  is non-basic, with Brauer  $G$ -character  $\varphi'$ . Then

$$\varphi(c) = \dim V_0 - (\dim V')/2 = 2^{k-2} - \varphi'(1)/2.$$

Choosing a conjugate  $c'$  of  $c$  lying in  $\hat{A}_{n-3}$  we have

$$\varphi(c') = \varphi_0(c') + \varphi'(c') = -2^{k-3} + \varphi'(c').$$

Hence

$$(3.2) \quad \varphi'(c') = 3 \cdot 2^{k-3} - \varphi'(1)/2 > 0,$$

as  $\varphi'(1) = \dim V - \dim V_0 < 3 \cdot 2^{k-2}$  by our assumption. Next, we look at the element  $d = cc'$ . Decompose  $V'$  as the sum of two nontrivial  $c$ -eigenspaces and let  $\alpha, \beta$  be the characters of  $\hat{A}_{n-3}$  on them. Since  $t_1$  inverts  $c$  and centralizes  $\hat{A}_{n-3}$ , we have  $\beta(g) = \alpha(t_1 g t_1) = \alpha(g)$  for any  $g \in \hat{A}_{n-3}$ . Thus

$$\varphi'(d) = \omega\alpha(c') + \omega^2\beta(c') = (\omega + \omega^2)\alpha(c') = -\alpha(c') = -\varphi'(c')/2.$$

Using (3.2), the assumption that  $V_0$  is basic, and  $\dim V < 2^k$ , we now obtain

$$\begin{aligned} \varphi(d) &= \varphi_0(d) + \varphi'(d) = \varphi_0(c') - \varphi'(c')/2 \\ &= -\varphi_0(1)/2 - \varphi'(c')/2 < -2^{k-3} < -\varphi(1)/8, \end{aligned}$$

contrary to Lemma 2.5.

*Case A5:*  $H = \hat{A}_n$ ,  $n = 2k$ ,  $p \nmid n$ ,  $p \nmid (n-3)$ , and  $p = 3$ . We use the notation of A4 and S6. Arguing as in S6, we see that each  $W^i$  is a nonzero faithful  $G$ -module, and by Lemma 2.4, at least one of them is non-basic. According to the claim proved in A5,  $\dim V \geq 2^{k-1} + 2 \cdot 2^{k-2} = 2\delta(\hat{A}_n)$ , as desired.

The proof of Theorem A is complete.  $\square$

The following easy corollary of Theorem A will be often used in what follows.

**Corollary 3.1.** *Let  $n \geq 5$ ,  $H = \hat{S}_n$  or  $\hat{A}_n$ , and  $D$  be an irreducible faithful FH-module. If  $G$  is a subgroup of  $H$  containing  $Z(H)$  and such that  $D \downarrow_G$  is irreducible, then*

$$|\pi(G)| \geq (\dim D)^2 \geq 2^{(n-4)}.$$

*Proof.* Indeed, the second inequality comes from Theorem A if  $n \geq 8$  and [12, 19] if  $n = 5, 6, 7$ . As  $D \downarrow_G$  is irreducible and  $D$  is faithful,  $G$  has a faithful irreducible complex representation of degree  $\geq \dim D$ . The sum of squared degrees of irreducible complex representations of  $G$  that are trivial on  $Z$  equals  $|\pi(G)|$ . Hence  $|G| - |\pi(G)| = |\pi(G)|$  is at least  $(\dim D)^2$ .  $\square$

#### 4. BASIC SPIN MODULES

Basic modules play a very special role in the representation theory of  $\hat{S}_n$  and  $\hat{A}_n$ . In this section we will study restrictions of these modules to subgroups  $G$  such that  $\pi(G)$  is either a Young subgroup or a wreath product subgroup. These subgroups are important because among them we find maximal imprimitive subgroups.

We will have to use the terminology of superalgebra and some result from [8, 9]. We review what is needed referring the reader to [8, 9] for details. Let  $\mathcal{S}_n$  be the twisted group algebra of  $S_n$ . This may be described as the superalgebra with degree  $\bar{1}$  generators  $t_1, \dots, t_{n-1}$  and relations  $t_i^2 = 1$ ,  $t_i t_j = -t_j t_i$ ,  $t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}$  for all admissible  $i, j$  with  $|i-j| > 1$ . Any subgroup  $H$  of  $S_n$  yields a sub(super)algebra  $\mathcal{H} \subset \mathcal{S}_n$ . We identify spin modules over  $\hat{S}_n$  with modules over  $\mathcal{S}_n$  (there is an isomorphism of categories) and do the same for  $\hat{A}$  and  $\mathcal{H}$ . Thus we are interested in irreducible restrictions from  $\mathcal{S}_n$  to  $\mathcal{H}$ . If  $H$  is a standard Young subgroup  $S_{a_1} \times \dots \times S_{a_b} < S_n$  then we have  $\mathcal{H} \cong \mathcal{S}_{a_1} \otimes \dots \otimes \mathcal{S}_{a_b}$  where the tensor product

is the *tensor product of superalgebras*, i.e. to multiply the tensors we use the usual sign rule.

We recall also that irreducible supermodules over superalgebras can be of two types: type **M** when the supermodule is irreducible considered as a usual module, and type **Q** when the supermodule splits as a direct sum of two non-isomorphic irreducible modules considered as a usual module. If  $A$  and  $B$  are superalgebras,  $V$  is an  $A$ -supermodule, and  $W$  is a  $B$ -supermodule, we denote by  $V \boxtimes W$  the outer tensor product of  $V$  and  $W$ , which is an  $A \otimes B$ -supermodule. Note that that we use the sign rule to define the action of  $A \otimes B$  on  $V \otimes W$ , namely

$$(a \otimes b)(v \otimes w) = (-1)^{\deg b \deg v} av \otimes bw$$

for homogeneous elements  $a \in A$ ,  $b \in B$ ,  $v \in V$ ,  $w \in W$ . If  $V$  and  $W$  are both irreducible of type **M** then  $V \boxtimes W$  is also irreducible of type **M**. If one of  $V$  and  $W$  is of type **M** and another one is of type **Q** then  $V \boxtimes W$  is irreducible of type **Q**. Finally if both  $V$  and  $W$  are irreducible of type **Q** then  $V \boxtimes W$  splits as a direct sum of two copies of the same irreducible supermodule of type **M**, which is denoted by  $V \circledast W$ . To have a unified notation we also denote  $V \boxtimes W$  by  $V \circledast W$  in the first two cases. Thus,  $\circledast$  is an operation which always takes a pair of irreducible supermodules to an irreducible supermodule.

An example of a superalgebra is obtained as follows. Let  $G$  be a finite group and  $H < G$  be a subgroup of index two. Let  $\mathcal{A}$  be a group algebra of  $G$  with canonical basis  $\{t_g \mid g \in G\}$ . Define the  $\mathbb{Z}_2$ -grading on  $\mathcal{A}$  by setting  $\mathcal{A}_0 := \text{span}\{t_g \mid g \in H\}$  and  $\mathcal{A}_1 := \text{span}\{t_g \mid g \notin H\}$ . Let  $\mathbf{sgn}$  denote the 1-dimensional non-trivial  $\mathbb{F}G$ -module with trivial action of  $H$ . Let  $L_1^\pm, \dots, L_k^\pm, L_{k+1}, \dots, L_m$  be a complete list of irreducible non-isomorphic  $\mathbb{F}G$ -modules, such that  $L_i^\pm \otimes \mathbf{sgn} \cong L_i^\mp$  for  $1 \leq i \leq k$  and  $L_i \otimes \mathbf{sgn} \cong L_i$  for  $k < i \leq m$ . An application of the Clifford theory implies that there are  $m$  irreducible  $\mathcal{A}$ -supermodules  $\mathcal{L}_1, \dots, \mathcal{L}_m$  such that, considered as usual modules,  $\mathcal{L}_i = L_i^+ \oplus L_i^-$  for  $1 \leq i \leq k$  and  $\mathcal{L}_i = L_i$  for  $k < i \leq m$ , and these are all irreducible  $\mathcal{A}$ -supermodules up to isomorphism.

Recall the basic modules  $Y_n^{(\pm)}$  from Lemma 2.1. By general facts described in the previous paragraph and Lemma 2.1, there is an irreducible supermodule  $\mathcal{Y}_n$ , called the *basic supermodule*, which is determined uniquely up to an isomorphism by the following property. Considered as a usual module,  $\mathcal{Y}_n$  is  $Y_n$  if  $n$  is even and  $p|n$  or  $n$  is odd and  $p \nmid n$ , and it is  $Y_n^+ \oplus Y_n^-$  otherwise, see Lemma 2.1. Correspondingly, the supermodule  $\mathcal{Y}_n$  is of type **M** if and only if  $n$  is even and  $p|n$ , or  $n$  is odd and  $p \nmid n$ . By Lemma 2.1, we have

$$\dim \mathcal{Y}_n = \begin{cases} 2^{n/2} & \text{if } n \text{ is even and } p \nmid n \\ 2^{(n-1)/2} & \text{if } n \text{ is odd} \\ 2^{(n-2)/2} & \text{if } n \text{ is even and } p | n \end{cases}.$$

Similarly, if the ground field is  $\mathbb{C}$ , we get the basic spin supermodule  $\mathcal{Y}_{n,\mathbb{C}}$  which is of type **M** if and only if  $n$  is odd. We have

$$\dim \mathcal{Y}_{n,\mathbb{C}} = \begin{cases} 2^{n/2} & \text{if } n \text{ is even} \\ 2^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}.$$

**Lemma 4.1.** *Let  $\lambda = (a_1, a_2, \dots, a_b)$  be a composition of  $n$ , and*

$$\hat{S}_\lambda = \pi^{-1}(S_{a_1} \times S_{a_2} \times \dots \times S_{a_b}) < \hat{S}_n, \quad \hat{A}_\lambda = \hat{A}_n \cap S_\lambda.$$

Suppose that  $V$  (resp.  $V'$ ) is an irreducible  $\mathbb{F}\hat{S}_\lambda$ - (resp.  $\mathbb{F}\hat{A}_\lambda$ -) module such that composition factors of every restriction  $V \downarrow_{\hat{S}_{a_i}}$  (resp.  $V' \downarrow_{\hat{A}_{a_i}}$ ),  $i = 1, 2, \dots, b$ , are basic spin. Then

$$\dim V = 2^{(n-s-u)/2-t-\lfloor(\tau+s+1)/2\rfloor} \quad (\text{resp. } \dim V' = 2^{(n-s-u)/2-t-\lfloor(\tau+s+2)/2\rfloor}),$$

where

$$\begin{aligned} r &= \#\{i \mid a_i \text{ is even and } p \nmid a_i\}, \\ s &= \#\{i \mid a_i \text{ is odd and } p \mid a_i\}, \\ t &= \#\{i \mid a_i \text{ is even and } p \mid a_i\}, \\ u &= \#\{i \mid a_i \text{ is odd and } p \nmid a_i\}. \end{aligned}$$

*Proof.* Let  $\mathcal{V}$  be an irreducible supermodule over  $\mathcal{S}(a_1) \otimes \cdots \otimes \mathcal{S}(a_b)$  such that all composition factors of its restriction to any  $\mathcal{S}(a_i)$  are basic. Then  $\mathcal{V} = \mathcal{Y}_{a_1} \otimes \cdots \otimes \mathcal{Y}_{a_b}$ . Moreover,  $\dim V = \dim V' = (\dim \mathcal{V})/2$  if  $\mathcal{V}$  is of type Q and  $\dim V = 2 \dim V' = \dim \mathcal{V}$  if  $\mathcal{V}$  is of type M. Therefore

$$\dim V = 2^{-\lfloor(\tau+s+1)/2\rfloor} \prod_{i=1}^b \dim \mathcal{Y}_{a_i}, \quad \dim V' = 2^{-\lfloor(\tau+s+2)/2\rfloor} \prod_{i=1}^b \dim \mathcal{Y}_{a_i},$$

which implies the result.  $\square$

**Corollary 4.2.** *Let  $H = \hat{S}_n$  or  $\hat{A}_n$ ,  $D$  be a basic spin  $\mathbb{F}H$ -module, and  $\lambda = (\lambda_1, \dots, \lambda_h)$  be a composition of  $n$  with  $h > 1$  non-zero parts. Let  $G$  be  $\hat{S}_\lambda$  or  $\hat{A}_\lambda$ . Then the restriction  $D \downarrow_G$  is irreducible if and only if  $p \nmid \lambda_i$  for all  $i = 1, 2, \dots, h$  and one of the following happens:*

- (i)  $H = \hat{S}_n$ ,  $G = \hat{S}_\lambda$ , and either  $n$  is even and  $h = 2$ , or  $n$  is odd,  $p \mid n$ , and  $h \leq 3$ .
- (ii)  $H = \hat{S}_n$ ,  $G = \hat{A}_\lambda$ ,  $n$  is odd,  $p \mid n$ , and  $h = 2$ .
- (iii)  $H = \hat{A}_n$ ,  $G = \hat{A}_\lambda$ , and either  $n$  is odd and  $h = 2$ , or  $n$  is even,  $p \mid n$ , and  $h \leq 3$ .

**Theorem 4.3.** *Let  $D$  be a basic spin  $\mathbb{F}\hat{S}_n$ -module, and let  $G$  be a maximal subgroup of  $\hat{S}_n$  with  $\pi(G)$  imprimitive. Then  $D \downarrow_G$  is irreducible if and only if one of the following holds:*

- (i)  $\pi(G) = S_{n-a} \times S_a$ ,  $a < n/2$ ,  $p \nmid a$ ,  $p \nmid (n-a)$ , and either  $n$  is even, or  $n$  is odd and  $p \mid n$ .
- (ii)  $\pi(G) = S_a \wr S_b$  for some  $a, b \geq 2$  with  $n = ab$  and  $p \nmid a$ .

*Proof.* By assumption we only need to consider subgroups  $G$  with  $\pi(G) = S_{n-a} \times S_a$ ,  $a < n/2$ , or  $\pi(G) = S_a \wr S_b$ ,  $a, b \geq 2$ ,  $ab = n$ . In the former case the result follows from Corollary 4.2. So let  $\pi(G) = S_a \wr S_b$ . We will sometimes write  $G_{a,b}$  for  $G$ . Denote

$$K_{a,b} := \pi^{-1}(S_a \times \cdots \times S_a) \triangleleft G.$$

Then  $G_{a,b}/K_{a,b} \cong S_b$ . We will also need the subgroup

$$B := \pi^{-1}(S_a \wr S_b) \triangleleft G_{a,b}.$$

It is proved in [22] that the restriction to  $G$  of a basic module  $D_{\mathbb{C}}$  over the field of complex numbers to  $G$  is irreducible.

Consider the irreducible  $\mathbb{F}K_{a,b}$ -supermodule  $\mathcal{Y}_a \otimes \cdots \otimes \mathcal{Y}_a$ , where  $\mathcal{Y}_a$  appears  $b$  times. This supermodule is of type Q if and only if  $b$  is odd and either  $a$  is even and

$p \nmid a$ , or  $a$  is odd and  $p|a$ . Now consider  $\mathcal{Y}_a \otimes \cdots \otimes \mathcal{Y}_a$  as a usual module. Then it has two composition factors,  $M_{a,b}^+$  and  $M_{a,b}^-$  say, when the supermodule is of type  $\mathbb{Q}$ , and only one composition factor,  $M_{a,b}$  say, otherwise. The dimensions of  $M_{a,b}^{(\pm)}$  can be found using Lemma 4.1. Note that  $M_{a,b}^{(\pm)}$  are the only possible composition factors of the restriction  $D \downarrow_{K_{a,b}}$ .

Now we apply Clifford theory to identify possible composition factors of the restriction  $D \downarrow_G$ . From the previous paragraph, we conclude that these factors may be of two forms:

(a)  $V \otimes W$ , where  $V$  and  $W$  are (possibly projective) irreducible representations of  $\mathbb{F}G$  such that  $V \downarrow_{K_{a,b}} \cong M_{a,b}^{(\pm)}$  and  $K_{a,b}$  acts trivially on  $W$ . Here the inertia group is  $G$ .

(b)  $\text{ind}_B^G(V \otimes W)$ , where  $V$  and  $W$  are (possibly projective) irreducible  $\mathbb{F}B$ -representations such that  $V \downarrow_{K_{a,b}} \cong M_{a,b}^\pm$  and  $K_{a,b}$  acts trivially on  $W$ . Here the inertia group is  $B$ .

In case (a) (resp. (b)),  $W$  may be considered as a (possibly projective) representation of  $\mathbb{F}S_b$  (resp.  $\mathbb{F}A_b$ ). We claim that this representation must be basic spin. By a reduction modulo  $p$  argument it suffices to prove this over  $\mathbb{C}$ . As  $D_{\mathbb{C}} \downarrow_G$  is irreducible, it must be isomorphic to a module described in (a) or (b), namely  $V_{\mathbb{C}} \otimes W_{\mathbb{C}}$  or  $\text{ind}_B^G(V_{\mathbb{C}} \otimes W_{\mathbb{C}})$  with  $V_{\mathbb{C}} \downarrow_{K_{a,b}} \cong M_{a,b,\mathbb{C}}^{(\pm)}$ . Using known dimensions of  $D_{\mathbb{C}}$  and  $M_{a,b,\mathbb{C}}^{(\pm)}$ , we find that  $\dim W_{\mathbb{C}}$  is equal to the dimension of the basic module  $Y_{b,\mathbb{C}}^{(\pm)}$  (resp.  $U_{b,\mathbb{C}}^{(\pm)}$ ) if the restriction is of type (a) (resp. type (b)). Using [2, 2.4], we deduce that  $W_{\mathbb{C}}$  is not a linear representation if  $b \geq 10$ . In this case Theorem A implies that  $W_{\mathbb{C}}$  is basic spin. Finally, for small values of  $b$ , pick sufficiently large  $c > b$ , and note that composition factors of  $Y_{ab,\mathbb{C}}^{(\pm)} \downarrow_{G_{a,b}}$  are among the composition factors of the restrictions  $Y_{ac,\mathbb{C}}^{(\pm)} \downarrow_{G_{a,c}} \downarrow_{G_{a,b}}$ . Now the required result follows from the one for  $c \geq 10$ , as all composition factors of the restriction  $Y_{c,\mathbb{C}}^{(\pm)} \downarrow_{\hat{S}_b}$  are basic spin.

We also note that the restriction  $Y_{n,\mathbb{C}}^{(\pm)} \downarrow_G$  is of type (b) if and only if  $a$  is even and  $b$  is odd. Indeed, in all other cases there is only one irreducible  $\mathbb{C}K_{a,b}$ -module  $M_{a,b,\mathbb{C}}$ , which can appear as a composition factor, so  $Y_{n,\mathbb{C}}^{(\pm)} \downarrow_G$  could only be of type (a). On the other hand, if  $a$  is even and  $b$  is odd, there are two possible composition factors  $M_{a,b,\mathbb{C}}^+$  and  $M_{a,b,\mathbb{C}}^-$ . To see that in this case the restrictions  $Y_{n,\mathbb{C}}^\pm \downarrow_G$  are of type (b), it suffices to note that  $Y_{n,\mathbb{C}}^\pm \downarrow_{K_{a,b}} \cong M_{a,b,\mathbb{C}}^+ \oplus M_{a,b,\mathbb{C}}^-$ . If the last equality does not hold then (up to a choice of signs) we have  $Y_{n,\mathbb{C}}^\pm \downarrow_{K_{a,b}} \cong 2M_{a,b,\mathbb{C}}^\pm$ , and in particular  $Y_{n,\mathbb{C}}^+ \downarrow_{K_{a,b}} \cong Y_{n,\mathbb{C}}^- \downarrow_{K_{a,b}}$ . But the characters of  $Y_{n,\mathbb{C}}^+$  and  $Y_{n,\mathbb{C}}^-$  only differ on conjugacy classes corresponding to the  $n$ -cycle [22, 2A], and there are no such elements in  $K_{a,b}$ , which shows that the characters of the restrictions  $Y_{n,\mathbb{C}}^+ \downarrow_{K_{a,b}}$  and  $Y_{n,\mathbb{C}}^- \downarrow_{K_{a,b}}$  are the same, giving a contradiction.

Now we consider various cases.

*Case 1.*  $a$  is odd and  $p \nmid a$ . Here  $\dim M_{a,b} = 2^{(ab-b)/2}$ , and all composition factors of the restriction  $D \downarrow_G$  must be of type (a) above. Now one checks that in all cases we have  $\dim D = \dim M_{a,b} \dim Y_b^{(\pm)}$ , so  $D \downarrow_G$  must be irreducible.

*Case 2.*  $a$  is odd and  $p|a$ . If  $b$  is even, then we have  $\dim M_{a,b} = 2^{(ab-2b)/2}$ , and all composition factors of the restriction  $D \downarrow_G$  must be of type (a) above. Now one checks that  $\dim D > \dim M_{a,b} \dim Y_b^{(\pm)}$ , so  $D \downarrow_G$  could not be irreducible. If  $b$  is

odd, then we have  $\dim M_{a,b}^\pm = 2^{(ab-2b-1)/2}$ , and so whether we have a composition factor of type (a) or (b), its dimension is strictly less than that of  $D$ , and so the restriction is reducible again.

*Case 3.*  $a$  is even and  $p|a$ . We have  $\dim M_{a,b} = 2^{(ab-2b)/2}$ , and we can use dimensions as in Case 2 to see that the restriction is reducible.

*Case 4.*  $a$  is even,  $p \nmid a$ ,  $b$  is even. We have  $\dim M_{a,b} = 2^{(ab-b)/2}$ , and the restriction is irreducible by dimensions.

*Case 5.*  $a$  is even,  $p \nmid a$ ,  $b$  is odd. We have  $\dim M_{a,b}^\pm = 2^{(ab-b-1)/2}$ . Assume first that  $p \nmid b$ . Then  $\dim Y_b = 2^{(b-1)/2}$ , and  $\dim U_b^\pm = 2^{(b-3)/2}$ , see Lemma 2.1. Now a composition factor of  $D \downarrow_G$  of type (a) has dimension

$$2^{(ab-b-1)/2} \cdot 2^{(b-1)/2} = 2^{(n-2)/2} = \dim D,$$

so  $D \downarrow_G$  is irreducible if such composition factor occurs. On the other hand, composition factor of type (b) has dimension  $2 \cdot 2^{(ab-b-1)/2} \cdot 2^{(b-3)/2} = \dim D$ , so  $D \downarrow_G$  is irreducible anyway. We finally assume that  $p|b$ . Then  $\dim Y_b^\pm = \dim U_b = 2^{(b-3)/2}$ , and the dimension argument as above shows that the restriction is irreducible if there is a composition factor of type (b), but it is reducible otherwise. So in this final case we do have to figure out what composition factors occur. We proved above that  $D_{\mathbb{C}} \downarrow_G \cong \text{ind}_B^G(V_{\mathbb{C}} \otimes U_{b,\mathbb{C}}^\pm)$ . Now, we reduce both sides of this equality modulo  $p$  to see that composition factors of the restriction  $D \downarrow_G$  are of type (b), which completes the proof.  $\square$

**Theorem 4.4.** *Let  $D$  be a basic spin  $\mathbb{F}\hat{A}_n$ -module, and let  $H$  be a maximal subgroup of  $\hat{A}_n$  with  $\pi(H)$  imprimitive. Then  $D \downarrow_H$  is irreducible if and only if one of the following holds:*

- (i)  $\pi(H) = A_n \cap (S_{n-a} \times S_a)$ ,  $a < n/2$ ,  $p \nmid a$ ,  $p \nmid (n-a)$ , and either  $n$  is odd, or  $n$  is even and  $p|n$ .
- (ii)  $\pi(H) = A_n \cap (S_a \wr S_b)$  for some  $a, b \geq 2$  with  $n = ab$  and  $p \nmid a$ .

*Proof.* (i) is proved using Corollary 4.2. We now prove (ii). Let  $G = \pi^{-1}(S_a \wr S_b)$ . Then  $H$  is a subgroup of  $G$  of index 2. We will use notation introduced in the proof of Theorem 4.3.

Assume first that there is only one basic spin module  $Y_n$  over  $\hat{S}_n$ , and the restriction  $Y_n \downarrow_G$  is irreducible. Then the module

$$U_n^+ \downarrow_H \oplus U_n^- \downarrow_H \cong Y_n \downarrow_G \downarrow_H$$

has at most two composition factors. This proves that  $U_n^\pm \downarrow_H$  is irreducible. Thus  $D \downarrow_H$  is irreducible in the cases where either  $n$  is odd and  $p \nmid n$  or  $n$  is even,  $p \nmid a$ , and  $p|b$ . Now we consider the remaining cases.

*Case 1.*  $n$  is even and  $p \nmid n$ . In this case  $p \nmid a$  and  $p \nmid b$ , so reductions modulo  $p$  of (the complex analogues of) all the modules involved are irreducible, and the result follows from the corresponding result over  $\mathbb{C}$ , see [22].

*Case 2.*  $n$  is odd and  $p|n$ . Assume first that  $p|a$ . Composition factors of  $U_n \downarrow_H$  are restrictions from  $G$  to  $H$  of composition factors of  $Y_n^\pm \downarrow_G$ , whose dimension is at most  $2^{(ab-b-2)/2}$  (see the proof of Theorem 4.3), while  $\dim U_n = 2^{(ab-3)/2}$ . Therefore  $U_n \downarrow_H$  must be reducible. Now, let  $p \nmid a$ . By Lemma 2.1 we have  $Y_n^\pm \downarrow_{\hat{A}_n} = U_n$ , and by Theorem 4.3, the restrictions  $Y_n^\pm \downarrow_G$  is irreducible. We have  $Y_{n,\mathbb{C}} \downarrow_G = V_{\mathbb{C}} \otimes Y_{b,\mathbb{C}}$ . Reducing modulo  $p$  we get  $(Y_n^+ + Y_n^-) \downarrow_G = V \otimes (Y_b^+ + Y_b^-)$ , so up to a choice of signs we must have  $Y_n^\pm \downarrow_G = V \otimes Y_b^\pm$ . By uniqueness in Clifford theory,



we have  $V \otimes Y_b^+ \not\cong V \otimes Y_b^-$ . It follows that  $V \otimes Y_b^+ \cong (V \otimes Y_b^-) \otimes \mathbf{sgn}$ , so the restrictions  $V \otimes Y_b^\pm \downarrow_H$  are irreducible, whence  $D \downarrow_H$  is also irreducible.

*Case 3.*  $a$  is even,  $b$  is odd, and  $p|a$ . We introduce the new subgroup

$$K'_{a,b} := K_{a,b} \cap \hat{A}_n.$$

Note that  $K'_{a,b} \triangleleft H$ , and  $H/K'_{a,b} \cong S_b$ . We apply Clifford theory to this situation. By Lemma 4.1,  $\dim M_{a,b} = 2^{(ab-2b)/2}$ . Also,  $M_{a,b}$  comes from the corresponding supermodule of type  $\mathbb{M}$ , so on restriction to  $K'_{a,b}$  it splits as a direct sum of two irreducible modules of dimension  $2^{(ab-2b-2)/2}$ . If the inertia group of a composition factor  $W$  of  $D \downarrow_H$  is  $H$  then we get from Clifford theory that

$$\dim W = 2^{(ab-2b-2)/2} \dim Y_b < 2^{(ab-4)/2} = \dim D,$$

so the restriction  $D \downarrow_H$  is reducible. Otherwise the inertia group is  $K'_{a,b} A_b < H$ , in which case we still have

$$\dim W = 2 \cdot 2^{(ab-2b-2)/2} \dim U_b^{(\pm)} = 2^{(ab-b-3)/2} < \dim D$$

as  $b \geq 3$ .

*Case 4.*  $b$  is even, and  $p|a$ . Here  $\dim U_n^\pm = 2^{(ab-4)/2}$ , and composition factors of  $Y_n \downarrow_G$  have dimensions  $2^{(ab-b-2)/2}$ , so we may assume that  $b = 2$ . Set

$$J := \pi^{-1}(A_a \times A_a).$$

Also, pick an element  $y$  in  $\pi^{-1}((1, a+1)(2, a+2) \dots (a, 2a))$  if  $a$  is even and  $\pi^{-1}((1, a+1)(2, a+2) \dots (a, 2a)(1, 2))$  if  $a$  is odd. Finally, set  $x = t_1 t_a$  where  $t_i$  are the standard generators of  $\hat{S}_n$ . Then  $J \triangleleft H$ ,  $K' := K'_{a,2} = \langle J, x \rangle$ , and  $H = \langle J, x, y \rangle$ .

Assume now that  $a$  is odd. Then  $\pi(y^2) = \pi(x)$ , whence  $H/J \cong \mathbb{Z}_4$ . Let  $W$  be any composition factor of  $D \downarrow_H$ . Every composition factor of  $W \downarrow_J$  must be isomorphic to the module  $M := U_a \otimes U_a$  of dimension  $2^{a-3}$ , so  $M$  must be  $H$ -stable. Since  $H/J$  is cyclic,  $M$  extends to  $H$ , so  $\dim W = 2^{a-3} < 2^{a-2} = \dim D$ .

Finally, let  $a$  be even. Now, there are two basic modules  $U_a^\pm$  for  $\hat{A}_a$ , and  $\dim U_a^\pm = 2^{(a-4)/2}$ . It follows that there can be two different composition factors  $V_1$  and  $V_2$  in the restriction  $D \downarrow_{K'}$ , characterized by  $V_1 \downarrow_J = U_a^+ \otimes U_a^+ \oplus U_a^- \otimes U_a^-$  and  $V_2 \downarrow_J = U_a^+ \otimes U_a^- \oplus U_a^- \otimes U_a^+$ . Observe that the Brauer characters of  $V_1$  and  $V_2$  are zero on  $K' \setminus J$  and  $y$  stabilizes  $V_1 \downarrow_J$  and  $V_2 \downarrow_J$ . It follows that  $y$  stabilizes the Brauer characters of  $V_1$  and  $V_2$ , so  $V_1$  and  $V_2$  extend to  $H$ , whence the dimension of every composition factor of  $D \downarrow_H$  equals  $\dim V_i$ , which is less than  $\dim D$ . Thus in both subcases  $D \downarrow_H$  is reducible.  $\square$

## 5. PRIMITIVE SUBGROUPS (THEOREM B)

In this section we prove Theorem B. Assume  $D \downarrow_G$  is irreducible. Without loss of generality we may assume that  $G$  contains  $Z := Z(H)$  (however  $Z$  may split out in  $G$ ). The cases  $n = 5, 6, 7$  can be checked easily using [12] and [19], so we will assume that  $n \geq 8$ .

In view of Corollary 3.1, we may assume that  $\pi(G)$  is a primitive subgroup of  $S_n$  of order  $\geq 2^{n-4}$ . Such subgroups are classified in [22, 6.2]. We will proceed case by case according to that classification. If  $\pi(G)$  is almost simple, we set  $S := \text{soc}(\pi(G))$ , and let  $\varphi$  be the Brauer character of  $D$ . We will use results from [12, 19] without special reference.

**A.** First we assume that the  $FH$ -module  $D$  does not lift to characteristic 0, that is it is not a reduction modulo  $p$  of an ordinary irreducible module. In particular,  $p$  divides  $|H|$ .

*Case  $n = 8$ .* Here we have either (a)  $S = L_2(7)$  or (b)  $\mathbb{Z}_3^3 \triangleleft \pi(G) \leq AGL_3(2)$ . By Lemma 2.2, 8 divides  $\dim D$ . In the case (a)  $G$  has no irreducible characters of degree  $> 8$ , so  $\dim D = 8$ . But in this case  $D$  lifts to characteristic 0, a contradiction. In the case (b), using Corollary 3.1, we conclude that  $\dim D \leq 32$ . But  $D$  does not lift to characteristic 0, so either  $p = 5$  and  $\dim D = 32$ , or  $p = 7$  and  $\dim D = 16$ . In particular  $|\pi(G)| \geq 16^2$ , whence  $\pi(G) = AGL_3(2) = \mathbb{Z}_3^3 : GL_3(2)$ . As  $A := GL_3(2)$  acts irreducibly on  $\mathbb{Z}_3^3$ , we have that  $K := \pi^{-1}(\mathbb{Z}_3^3) \cong \mathbb{Z}_2^4$ . If  $p = 5$  then one can view  $D \downarrow_G$  as an irreducible complex representation of  $G$ , whence  $(\dim D) | [G : K] | = 168$  by Ito's Theorem [16, 6.15], a contradiction. So  $p = 7$  and  $\dim D = 16$ . Decompose  $\varphi \downarrow_K = s \sum_{i=1}^t \lambda_i$ , where  $t \geq 2$  and  $\{\lambda_1, \dots, \lambda_t\}$  is an  $A$ -orbit on the set of 8 linear characters of  $K$  that are faithful on  $Z$ . As  $t | \dim D$ ,  $t$  is a 2-power. Inspecting the subgroups of  $A$  we see that  $A_0 := \text{Stab}_A(\lambda_1)$  is isomorphic to  $\mathbb{Z}_7 : \mathbb{Z}_3$  and  $t = 8$ . Thus  $A$  acts transitively on the eight linear characters of  $K$  that are faithful on  $Z$ . Now  $\varphi \downarrow_G = \theta \uparrow^G$ , where  $\theta$  is an irreducible Brauer character of degree 2 of  $G_0 := K : A_0$  and  $\theta \downarrow_K = 2\lambda_1$ . The quotient of  $G_0$  by  $\text{Ker}(\lambda_1)$  is  $(\mathbb{Z}_2 \times \mathbb{Z}_7) : \mathbb{Z}_3$ . But  $p = 7$  so  $\mathbb{Z}_7$  acts trivially. Thus  $\theta$  yields an irreducible character of degree 2 of  $\mathbb{Z}_2 : \mathbb{Z}_3 \cong \mathbb{Z}_6$ , a contradiction.

*Case  $n = 9$ .* Here either (a)  $S = L_2(8)$  or (b)  $\mathbb{Z}_3^2 \triangleleft \pi(G) \leq AGL_2(3)$ . As  $D$  does not lift to characteristic 0, either  $\dim D \geq 48$  or  $p = 3$ ,  $H = \hat{S}_9$ , and  $\dim D = 8$ . But the former case is impossible by Corollary 3.1. In the latter case we must be in (b), as in (a)  $G$  has no irreducible 3-Brauer character of degree 8. But then  $G$  has a normal subgroup  $\mathbb{Z}_3^2$ , which must act trivially on  $D$ . So  $D$  factors through to give an irreducible module of dimension 8 over  $G/\mathbb{Z}_3^2$ , which is impossible by Corollary 3.1.

*Case  $n = 10$ .* Here  $S = A_5$  or  $S = A_6$ . As  $|\text{Out}(S)| \leq 4$  and  $\pi^{-1}(S)$  has no irreducible character of degree  $> 10$ , we have  $\dim D \leq 40$ . On the other hand,  $8 | \dim D$  by Lemma 2.2 and  $D$  does not lift to characteristic 0. Hence  $p = 5$ ,  $H = \hat{A}_{10}$  and  $\dim D = 8$ . Then  $S \neq A_5$ . The analysis in [22, p. 463] shows that  $\pi(G) = A_6$  or  $M_{10}$  and  $\pi^{-1}(S) = \mathbb{Z}_2 \times A_6$ . Now one checks that  $D \downarrow_{\pi^{-1}(S)}$  is in fact irreducible, giving exceptions in (ii)(g) and (ii)(f).

*Case  $n = 11$ .* Here  $S$  is  $L_2(11)$  or  $M_{11}$ . Checking the irreducible characters of  $G$ , we conclude that  $\dim D \leq 55$ . Also,  $D$  does not lift to characteristic 0. Therefore  $p = 11$ ,  $H = \hat{S}_{11}$ ,  $\dim D = 16$ , and  $S = M_{11}$ , in which case  $D \downarrow_G$  is irreducible, giving (i)(f).

*Case  $n = 12$ .* Here  $S$  is  $L_2(11)$ ,  $M_{11}$ , or  $M_{12}$ . Checking the irreducible characters of  $G$ , we conclude that  $\dim D \leq 320$ . As  $D$  does not lift to characteristic 0, one of the following holds: (a)  $p = 3$  and  $\dim D = 16, 144, 288$  or  $320$ ; (b)  $p = 11$  and  $\dim D = 128$ . However,  $G$  has no irreducible  $p$ -Brauer characters of those degrees.

*Case  $n = 13$ .* Here  $S = L_3(3)$ , whence  $\dim D \leq 52$ . As  $D$  does not lift to characteristic 0, we have  $p = 13$ ,  $H = \hat{S}_{13}$ , and  $\dim D = 32$ . But  $G$  does not have irreducible 13-Brauer characters of such degree.

*Case  $n = 14$ .* Here  $S = L_2(13)$ , whence  $\dim D \leq 14$ , which contradicts Corollary 3.1.

*Case  $n = 15$ .* Here  $S$  is  $A_7$  or  $A_8$ . By Lemma 2.2,  $32 | \dim D$ . By checking the Brauer characters of  $G$  we see that  $\dim D = 32, 64$ ,  $S = A_8$ , and  $p \neq 3, 7$ .

But  $D$  does not lift to characteristic 0. By Theorem A and [32, Table III], this implies  $p = 5$ ,  $H = \hat{S}_{15}$ , and  $\dim D = 64$ . Now if  $c \in G$  is an element of order 3 corresponding to a 3-cycle in  $S$  then  $\varphi(c) = 4$ . On the other hand,  $D \downarrow_{\hat{A}_{15}}$  is just the reduction modulo 5 of a complex basic spin module of  $\hat{A}_{15}$ , and so  $\varphi(c) = -2$  (see e.g. [22, p. 465]), a contradiction.

*Case  $n = 16$ .* Here  $\pi(G) = \mathbb{Z}_2^4 : A$  for some  $A \leq GL_4(2)$ . By Lemma 2.2,  $2^7 \mid \dim D$ , so  $|A| \geq 2^{10}$  using Corollary 3.1. Moreover,  $A$  acts irreducibly on  $\mathbb{Z}_2^4$  by an argument similar to the one in [22, p. 461]. It follows that  $A = A_7$  or  $A_8$ , and  $A$  acts transitively on the non-trivial elements of  $\mathbb{Z}_2^4$ . This in turn implies that  $K := \pi^{-1}(\mathbb{Z}_2^4) \cong \mathbb{Z}_2^5$ . Writing  $\dim D = 2^7 \cdot a$  for an integer  $a$ , we have  $1 \leq a \leq 4$  because of Corollary 3.1. Decompose  $\varphi \downarrow_K = s \sum_{i=1}^t \lambda_i$ , where  $16 \geq t \geq 2$  and  $\{\lambda_1, \dots, \lambda_t\}$  is an  $A$ -orbit on the set of 16 linear characters of  $K$  that are faithful on  $Z$ . Set  $A_0 := \text{Stab}_A(\lambda_1)$ . Then  $\varphi \downarrow_G = \theta \uparrow^G$ , where  $\theta$  is an irreducible Brauer character of degree  $s$  of  $K : A_0$  and  $\theta \downarrow_K = s\lambda_1$ . Analyzing possible subgroups  $A_0$  of  $A$  and dimensions of modules over them, we see that  $\theta \uparrow^G$  can not have dimension of the form  $2^7 \cdot a$  as above.

*Cases  $n = 17$  and  $n = 21$ .* Here  $S = L_2(16)$  and  $L_3(4)$ , respectively, and we argue as in the case  $n = 14$ .

*Case  $n = 22$ .* Here  $S = M_{22}$ . By Lemma 2.2,  $2^9 \mid \dim D$ . If  $U$  is a composition factor of  $D \downarrow_G$  then, since  $\text{Out}(S) = \mathbb{Z}_2$ , we see that  $2^8 \mid \dim U$ . But  $G$  has no such irreducible representation  $U$ .

Similarly we can deal with the cases  $n = 23$  (where  $S = M_{23}$ ),  $n = 24$  (where  $S = M_{24}$ ), and  $n = 32$  (where  $\mathbb{Z}_2^5 \triangleleft H \leq AGL_5(2)$ ).

**B.** Now we assume that the  $\mathbb{F}H$ -module  $D$  lifts to characteristic 0. So  $D$  is a reduction modulo  $p$  of a complex irreducible spin representation  $V$  of  $H$ . As  $D \downarrow_G$  is irreducible so is  $V \downarrow_G$ , and we can apply Theorems 6.3 and 6.4 of [22] which describe all such pairs  $(V, G)$ . It remains to check whether reduction modulo  $p$  of  $V \downarrow_G$  is irreducible for each such pair. Denote this reduction by  $V_p$ .

**B1.** Suppose  $H = \hat{S}_n$ . Here  $V$  is always a basic spin representation.

*Case  $n = 8$ .* Then  $\pi(G) = AGL_3(2)$ . Using analysis of the case  $n = 8$  in **A**, we conclude that  $K \cong \mathbb{Z}_2^4$  and  $G$  permutes transitively the 8 linear characters of  $K$  that occur on  $V$ . This implies that  $V_p$  is irreducible (as  $p \neq 2$ ), giving the exception (i)(d).

*Case  $n = 10$ .* By [22, p. 463], we have  $\pi^{-1}(S) = \mathbb{Z}_2 \times A_6$ . If  $p > 5$  then  $p \nmid |G|$ , and we get the exceptions in (i)(e). On the other hand, if  $p = 3$  or  $5$  then  $G$  does not have irreducible modules of dimension 16.

*Case  $n = 12$ .* This leads to the exception (i)(g).

**B2.** Suppose  $H = \hat{A}_n$ .

*Case  $n = 8$ .* Then  $\pi(G) = AGL_3(2)$  and  $V$  is either basic or second basic. The basic module lifts to  $\hat{S}_n$ , so (ii)(d) follows from the case  $n = 8$  in **B1**. Let  $V$  be second basic. Then  $p \neq 7$ , as  $V$  reduces modulo 7. Adopt the notation of the case  $n = 8$  in **A**. Then  $K \cong \mathbb{Z}_2^4$  and  $G$  acts transitively on the 8 linear characters of  $K$  that occur on  $V$ . Note that  $A_0$  acts on  $V_0$ , the  $\lambda_1$ -homogeneous component of dimension 3, as  $\mathbb{Z}_2 \times (\mathbb{Z}_7 : \mathbb{Z}_3)$ , with  $\mathbb{Z}_3$  cyclically permuting 3 nontrivial linear characters of  $\mathbb{Z}_7$  on  $V_0$ . Hence  $V_p$  is irreducible for  $p \neq 7$ , giving (iii)(b).

*Case  $n = 9$ .* Here  $V$  is basic and either  $\mathbb{Z}_2^3 : Q_8 \leq \pi(G) \leq \mathbb{Z}_2^3 : SL_2(3)$  or  $S = L_2(8)$ . In the former case  $V_3$  is reducible, as otherwise  $\mathbb{Z}_3^2$  would act trivially

on  $V_3$ , contrary to the faithfulness. If  $p > 3$  then  $p$  is coprime to  $|G|$ , leading to part of (ii)(e). The rest of (ii)(e) comes from checking the characters of  $L_2(8)$  and its automorphism group.

Case  $n = 10$  leads to the exception (ii)(f) (except for the case  $p \neq 5$ , which has been covered in the case **A**).

Case  $n = 11$  leads to the exception (ii)(h).

Case  $n = 12$  leads to the exceptions (ii)(i) and (iii)(c).

The proof of Theorem B is complete.

## 6. NEARLY SIMPLE SUBGROUPS (THEOREM C)

We rely on the following result of Kleidman and Wales:

**Proposition 6.1.** [22, 6.1] *Assume that  $S$  is a non-abelian proper simple subgroup of  $A_n$  and that  $|\text{Aut}(S)| \geq 2^{n-4}$ . Then one of the following holds:*

- (a)  $S \cong A_m$  for some  $m \geq 9$  and each orbit of  $S$  on  $\Omega$  has length 1 or  $m$ .
- (b)  $S \cong A_m$  with  $m = 5, 6, 7$  or  $8$ , and  $n \leq 10, 14, 16$ , or  $19$ , respectively.
- (c)  $S \cong L_2(q)$  with  $q = 7, 8, 11, 13$ , or  $16$ , and  $n \leq 12, 14, 14, 15$ , or  $17$ , respectively.
- (d)  $S \cong L_3(3)$  or  $L_3(4)$ , and  $n \leq 17$  or  $21$ , respectively.
- (e)  $S \cong M_t$  with  $t = 11, 12, 22, 23$ , or  $24$ , and  $n \leq 16, 21, 23, 27$ , or  $31$ , respectively.

Let  $H = \hat{S}_n$  or  $\hat{A}_n$ , and  $D$  be a faithful  $\mathbb{F}H$ -module which remains irreducible on restriction to a nearly simple subgroup  $G$ . Then  $\pi(G)$  is almost simple, because  $Z(G)$  acts as scalar matrices on  $D$ . Moreover, by Corollary 3.1, we may assume that the socle  $S$  of  $\pi(G)$  is among the subgroups listed in Proposition 6.1. Finally, in view of Theorem B, we may assume that  $\pi(G)$  is imprimitive on  $\Omega$  (cf. Theorem C(iv)). The following result of Phillips will help us to deal with intransitive subgroups. The proposition treats the irreducible restrictions of non-basic modules over  $\hat{S}_n$  and  $\hat{A}_n$  to Young type subgroups. The basic modules are treated in Corollary 4.2. The terminology used in Proposition 6.2 is explained in the introduction.

**Proposition 6.2.** [29] *Let  $H = \hat{S}_n$  or  $\hat{A}_n$ ,  $D$  be a non-basic faithful irreducible  $\mathbb{F}H$ -module. Suppose  $\mu = (\mu_1, \mu_2, \dots)$  is a non-trivial composition of  $n$ , and  $G$  is a subgroup in  $H$  satisfying*

$$A_{\mu_1} \times A_{\mu_2} \times \dots \leq \pi(G) \leq S_{\mu_1} \times S_{\mu_2} \times \dots$$

*Then the restriction  $D \downarrow_G$  is irreducible if and only if one of the following happens:*

- (i)  $H = \hat{S}_n$ ,  $D = D(\lambda, \sigma)$ , and one of the following happens:
  - (a)  $G = \hat{S}_{n-1}$ ,  $\lambda \in JS$ , and  $\lambda \in JS(0)$  if  $a(\lambda)$  is even.
  - (b)  $G = \hat{A}_{n-1}$ ,  $\lambda \in JS(0)$ , and  $a(\lambda)$  is odd.
  - (c)  $\pi(G) = S_{n-2} \times S_2$ ,  $\lambda \in JS(0)$ , and  $\lambda_{[0]} \in JS(1)$ .
  - (d)  $\pi(G) = (S_{n-2} \times S_2) \cap A_n \cong S_{n-2}$ ,  $\lambda \in JS(0)$ ,  $a(\lambda)$  is odd, and  $\lambda_{[0]} \in JS(1)$ .
  - (e)  $G = \hat{S}_{n-2}$ ,  $\lambda \in JS(0)$ ,  $a(\lambda)$  is odd, and  $\lambda_{[0]} \in JS(1)$ .
- (ii)  $H = \hat{A}_n$ ,  $D = E(\lambda, \sigma)$ , and one of the following happens:
  - (a)  $G = \hat{A}_{n-1}$ ,  $\lambda \in JS$ , and  $\lambda \in JS(0)$  if  $a(\lambda)$  is odd.
  - (b)  $\pi(G) = (S_{n-2} \times S_2) \cap A_n \cong S_{n-2}$ ,  $\lambda \in JS(0)$ , and  $\lambda_{[0]} \in JS(1)$ .
  - (c)  $G = \hat{A}_{n-2}$ ,  $\lambda \in JS(0)$ ,  $a(\lambda)$  is even, and  $\lambda_{[0]} \in JS(1)$ .

Now, we can deal with a major part of Theorem C.

**Proposition 6.3.** *Let  $H = \hat{S}_n$  or  $\hat{A}_n$ ,  $D$  be a faithful irreducible  $\mathbb{F}H$ -module, and  $G$  be a subgroup of  $H$  such that  $\pi(G) < S_n$  is an almost simple subgroup with  $S := \text{soc}(\pi(G)) \cong A_m$  for some  $5 \leq m < n$ . Assume that each orbit of  $S$  on  $\Omega$  has length 1 or  $m$ . Then  $D \downarrow_G$  is irreducible if and only if one of the one of the cases (i)-(iii) of Theorem C occurs.*

*Proof.* If  $H, G, D$  are as in (i)-(iii) of Theorem C then the restriction is irreducible by Proposition 6.2 and Corollary 4.2, using the description of the partition  $\omega_n$ , which labels the basic spin module over  $\mathbb{F}H$ , given in (1.2). Conversely, let  $D \downarrow_G$  be irreducible for  $G$  as in the assumption.

We claim that there is only one  $S$ -orbit of length  $m$  and other orbits are of length 1. Indeed, otherwise  $\pi(G)$  is contained in a Young subgroup of the form  $S_m \times S_m \times \dots$ . If  $D$  is non-basic, it follows from Proposition 6.2 that the restriction is reducible. If  $D$  is basic then using Corollary 4.2 we conclude that  $H = \hat{S}_n$  and one of the following happens:

- (1)  $p \nmid n = 2m$ ,  $S < S_m \times S_m$ ;
- (2)  $p \mid n = 2m + 1$ ,  $S < S_m \times S_m \times S_1$ .

However, in both cases  $\pi(G)$  belongs to the subgroup  $A_\mu := S_\mu \cap A_n$ , and the restriction to (the double cover of) such subgroup is reducible by Corollary 4.2 again.

So  $\pi(G)$  is actually itself a Young type subgroup, i.e.  $G$  satisfies the assumptions of Proposition 6.2. The result now follows by applying Proposition 6.2 and Corollary 4.2, and (1.2).  $\square$

Now we complete the proof of Theorem C by going through the remaining cases listed in Proposition 6.1(b)-(e). If  $(H, G, D)$  is as in Theorem C(iv)-(viii), then one readily checks using [12] and [19] that  $D \downarrow_G$  is irreducible. Conversely, let  $D \downarrow_G$  be irreducible, and  $G$  be as in Proposition 6.1(b)-(e) but not satisfy the assumptions of Proposition 6.3. Denote by  $\mu(G)$  the largest dimension of an irreducible  $\mathbb{F}G$ -representation. Then, of course, we have

$$(6.1) \quad \mu(G) \geq \delta(H) \geq \delta(\hat{A}_n).$$

We now claim that  $S$  has exactly one nontrivial orbit on  $\Omega$ , on which it acts primitively. For groups listed in Proposition 6.1(c)-(e) this is clear, because the minimal index  $P(S)$  of proper subgroups of  $S$  is bigger than  $n/2$  (see [12]). Let  $S$  be as in Proposition 6.1(b) and  $S$  have more than one nontrivial orbit on  $\Omega$ . Then the upper bound on  $n$  implies that  $S \cong A_m$  ( $5 \leq m \leq 8$ ) has exactly 2 orbits of length  $m$  and  $n - 2m$  fixed points on  $\Omega$ . So we can apply Proposition 6.3. Thus  $S$  has exactly one non-trivial orbit  $\Omega_1$ , and we can use [12] again to conclude that  $S$  must be primitive on this orbit. Let  $s = |\Omega_1|$ . Define

$$(6.2) \quad K = \pi^{-1} \left( \bigcap_{j \in \Omega - \Omega_1} \text{Stab}_{\pi(H)}(j) \right).$$

Now, we consider the groups appearing in (b)-(e) of Proposition 6.1 one by one. For the cases appearing in (b) we will assume that  $s > m$ , as otherwise the situation is covered by Proposition 6.3. In all cases we will assume that  $s < n$ , as otherwise  $\pi(G)$  is primitive on  $\Omega$ .

*Case 1:*  $S = A_5$ . Here  $s = 6$ . As  $\mu(G) \leq 6$ , we have  $n \leq 7$ , thanks to (6.1). So  $n = 7$  and  $G$  is of type (6,1). Using the fact that  $\pi(G)$  is primitive on  $\Omega_1$  and Theorem B, we arrive to the cases listed in Theorem C(v).

*Case 2:*  $S = A_6$ . As  $\mu(\hat{A}_{6.2_2}) \leq 10$ , we have  $\mu(G) \leq 20$ , so  $n \leq 12$  by (6.1). Suppose  $n = 12$ . Since  $\delta(\hat{A}_{12}) = 32$  for  $p \neq 3$ , we have  $p = 3$ . Moreover,  $\dim(D) \leq 20$  implies that  $H = \hat{A}_{12}$ ,  $D$  is basic spin, and  $\dim(D) = 16$ . If  $D \downarrow_G$  is irreducible and  $U$  is any composition factor of  $D \downarrow_{\hat{S}}$  then the dimension of  $U$  must divide 16. It follows that  $\dim(U) = 2$  or 4, whence  $U$  extends to  $\hat{S}_{2_2}$ . This in turn implies that  $\dim(D) \leq 8$ , a contradiction. Now, let  $n \leq 11$ . Then  $s = 10$ ,  $n = 11$ ,  $\pi(G)$  is of type (10,1) and  $\pi(G)$  is contained in  $\pi(K)$  as a primitive subgroup. Applying Theorem B and Corollary 4.2 leads to the cases in Theorem C(vi).

*Case 3:*  $S = A_7$ . Here  $\mu(G) \leq 36$ , so  $n \leq 14$  by (6.1). Hence  $s = 7$ , and so there are no cases to consider (as we have assumed that  $s > m$ ).

*Case 4:*  $S = A_8$ . Here  $\mu(G) \leq 112$ , hence  $n \leq 15$  by (6.1). As we have assumed  $m < s < n$ , there are no cases to consider.

*Case 5:*  $S = L_2(7)$ . Here  $\mu(G) \leq 8$ , hence  $n \leq 10$  by (6.1) and  $s = 7$  or 8. Moreover,  $\delta(\hat{A}_8) = 8$ , so  $\dim(D) = 8$  and  $D$  is basic spin. First suppose that  $s = 8$ . In this case  $\hat{S} < \hat{A}_8$ , and  $D \downarrow_{\hat{A}_8}$  lifts to a complex representation, say  $W$ . According to [22, p. 462],  $W \downarrow_{\hat{S}}$  has two irreducible constituents, of degree 1 and 7. This implies that  $D \downarrow_G$  is reducible. Next suppose that  $s = 7$ . Clearly,  $\pi(G)$  acts on the  $S$ -orbit  $\Omega_1$  of length 7, but  $L_2(7).2$  has no subgroup of index 7. Therefore  $\pi(G) = S$  and  $G$  is contained in  $K$  as a primitive subgroup. By Theorem B applied to  $(K, G, D)$ ,  $\dim(D) = 4$  or 6, a contradiction.

*Case 6:*  $S = L_2(8)$ . Here  $\mu(G) \leq 27$ , hence  $n \leq 12$  by (6.1) and  $s = 9$ . Suppose  $n = 11$  or 12. Then  $\dim(D) \geq \delta(\hat{A}_{11}) = 16$ , but  $\pi(G) = S$  or  $S.3$ . Therefore  $\dim(D) = 21$  or 27, which is a contradiction as  $H$  has no irreducible representations of such degree. Hence  $n = 10$ , and  $G$  is contained in  $K = \hat{A}_9$  as a primitive subgroup. By Theorem B applied to  $(K, G, D)$ ,  $D$  is basic spin of dimension 8. This leads to Theorem C(vii).

*Case 7:*  $S = L_2(11)$ , or  $L_2(13)$ . Here  $\mu(G) \leq 14$ , hence  $n \leq 10$  by (6.1). On the other hand,  $n \geq P(S) \geq 11$ , a contradiction.

*Case 8:*  $S = L_2(16)$ , or  $L_3(4)$ . In this case  $n \leq P(S)$ , whence  $\pi(G)$  is primitive on  $\Omega$ , a contradiction.

*Case 9:*  $S = L_3(3)$ . Here  $n \leq 17$  and so  $s = 13$ . Since  $L_3(3).2$  cannot have an orbit of length 13,  $\pi(G) = S$ . Thus  $G$  is contained in  $K = \hat{A}_{13}$  as a primitive subgroup. This is impossible by Theorem B applied to  $(K, G, D)$ .

*Case 10:*  $S = M_{11}$ . Here  $n \leq 16$  and so  $s = 11$  or 12. Moreover  $\pi(G) = S$ , therefore  $G$  is contained in  $K = \hat{A}_s$  as a primitive subgroup. By Theorem B applied to  $(K, G, D)$ ,  $s = 11$ ,  $p \neq 3$ , and  $\dim(D) = 16$ . But  $n > s$ , so  $\dim(D) \geq \delta(\hat{A}_{12}) = 32$ , a contradiction.

*Case 11:*  $S = M_{12}$ . Here  $n \leq 21$  and so  $s = 12$ . As  $M_{12}.2$  cannot have an orbit of length 12,  $\pi(G) = S$ . Thus  $G$  is contained in  $K = \hat{A}_{12}$  as a primitive subgroup. By Theorem B applied to  $(K, G, D)$ , either  $p \neq 3$  and  $\dim(D) = 32$ , or  $p \neq 3, 11$  or  $\dim(D) = 160$ . The first case leads to Theorem C(viii). The second case is impossible as  $K < \hat{A}_{13} \leq H$  and  $\hat{A}_{13}$  has no irreducible representation of degree 160.

*Case 12:*  $S = M_{22}, M_{23},$  or  $M_{24}$ . If  $S = M_{22}$  then  $n \leq 23 = P(S) + 1$ , whence  $s = 22$  and  $\pi(G)$  is of type  $(22, 1)$ . If  $S = M_{23}$  or  $M_{24}$ , then  $\pi(G) = S$  and  $s = 23$ , resp. 24. In all cases  $G$  is contained in  $K$  as a primitive subgroup. This is impossible by Theorem B applied to  $(K, G, D)$ .

The proof of Theorem C is complete.

## 7. ARBITRARY SUBGROUPS (THEOREM D)

In this section we consider restrictions of non-basic modules over  $\hat{S}_n$  and  $\hat{A}_n$  to arbitrary subgroups. For information on basic modules see §4. In most results of this section we have to assume that  $p > 3$ . Recall the notation  $\hat{g}$  from §2.

**Lemma 7.1.** *Let  $p > 3$ ,  $n \geq 6$ , and  $C$  be the elementary abelian subgroup of order 8 in  $S_n$  generated by the transpositions  $(1, 4), (2, 5)$ , and  $(3, 6)$ . Set*

$$x := \sum_{c \in C} (\text{sgn } c) \hat{c}(1 + t_1 t_2 + t_2 t_1) \hat{c}^{-1} \in \mathbb{F} \hat{A}_n.$$

*Then  $x D \neq 0$  for any non-basic faithful irreducible  $\mathbb{F} \hat{A}_n$ -module  $D$ .*

*Proof.* If we prove the lemma for  $n = 6$  then the general case will follow by induction on  $n$  using Lemma 2.4. So let  $n = 6$ . First, we consider the case  $\mathbb{F} = \mathbb{C}$ . Let  $V$  be an irreducible faithful  $\mathbb{C} \hat{A}_6$ -module not isomorphic to a basic spin module. It suffices to show that the trace  $\text{tr}_V(x t_1 t_2)$  of the element  $x t_1 t_2 \in \mathbb{C} \hat{A}_6$  on  $V$  is a non-zero integer not divisible by  $p$ . Indeed, this implies that  $x$  does not annihilate any reduction  $\bar{V}$  of  $V$  modulo  $p$ . If  $p > 5$  then  $p$  does not divide the order of  $\hat{A}_6$ , so any irreducible module looks like  $\bar{V}$ . In fact, for *faithful* modules the same is true even for  $p = 5$ , see [12, 19].

Finally, we prove that  $\text{tr}_V(x t_1 t_2)$  is a non-zero integer not divisible by  $p$ . Note that  $x = \sum_{c \in C} (\text{sgn } c) \hat{c}(t_1 t_2 + t_2 t_1) \hat{c}^{-1}$ . We will write  ${}^c y$  for  $\hat{c} y \hat{c}^{-1}$ ,  $c \in S_n$ ,  $y \in \mathbb{F} \hat{S}_n$ . By conjugating with  $t_1 t_2 t_4 t_5$ , we note that the elements  ${}^c(t_1 t_2) t_1 t_2$  are conjugate to each other for

$$c \in \{(14), (2, 5), (3, 6)\} \quad \text{and} \quad c \in \{(1, 4)(2, 5), (1, 4)(3, 6), (2, 5)(3, 6)\}.$$

The same is true for the elements of the form  ${}^c(t_2 t_1) t_1 t_2$ . So, if  $\chi$  is the character of  $V$ , then

$$\begin{aligned} \text{tr}_V(x t_1 t_2) &= \chi(t_2 t_1) + \chi(1) - 3\chi({}^{(1,4)}(t_1 t_2) t_1 t_2) - 3\chi({}^{(1,4)}(t_2 t_1) t_1 t_2) \\ &\quad + 3\chi({}^{(1,4)(2,5)}(t_1 t_2) t_1 t_2) + 3\chi({}^{(1,4)(2,5)}(t_2 t_1) t_1 t_2) \\ &\quad - \chi({}^{(1,4)(2,5)(3,6)}(t_1 t_2) t_1 t_2) - \chi({}^{(1,4)(2,5)(3,6)}(t_2 t_1) t_1 t_2). \end{aligned}$$

To identify the corresponding conjugacy classes it suffices to show, using relations, that

- (i)  $t_2 t_1$  is the lift of  $(1, 3, 2) \in S_6$  of order 3;
- (ii)  $t_2 t_1 t_1 t_2 = 1$ ;
- (iii)  ${}^{(1,4)}(t_1 t_2) t_1 t_2 = t_2 t_3 t_1 t_2$  is a lift of  $(1, 3)(2, 4) \in S_6$  (all such lifts are conjugate);
- (iv)  ${}^{(1,4)}(t_2 t_1) t_1 t_2 = t_3 t_1 t_2 t_1$  is a lift of  $(1, 4, 3) \in S_6$  of order 6;
- (v)  ${}^{(1,4)(2,5)}(t_1 t_2) t_1 t_2 = t_1 t_3 t_4 t_2$  is a lift of  $(1, 2, 4, 5, 3) \in S_6$  of order 10;
- (vi)  ${}^{(1,4)(2,5)}(t_2 t_1) t_1 t_2 = t_1 t_4 t_3 t_2$  is a lift of  $(1, 2, 5, 4, 3) \in S_6$  of order 10;

- (vii)  $(1,4)(2,5)(3,6)(t_1t_2)t_1t_2 = t_1t_2t_4t_5$  is a lift of  $(1, 2, 3)(4, 5, 6) \in S_6$  of order 3;
- (viii)  $(1,4)(2,5)(3,6)(t_2t_1)t_1t_2 = t_1t_2t_5t_4$  is a lift of  $(1, 2, 3)(4, 6, 5) \in S_6$  of order 3.

Now, using the known character values of  $\hat{S}_6$  [12], we find that  $\text{tr}_V(xt_1t_2)$  equals 18, 3, 3, 12, and 12 for irreducible non-basic spin representations  $V$  of  $\hat{S}_n$  of dimensions 4, 8, 8, 10, and 10, respectively.  $\square$

Let  $H = \hat{S}_n$  or  $\hat{A}_n$ . Recall the  $\mathbb{F}H$ -modules  $M^\lambda$  from §2. For an  $\mathbb{F}H$ -module  $D$ , we set

$$(7.1) \quad \mathcal{H}_j(D) := \text{Hom}_{\mathbb{F}H}(M^{(j, n-j)}, \text{End}_{\mathbb{F}}(D)) \quad \text{and} \quad d_j(D) := \dim \mathcal{H}_j(D).$$

The following is a crucial technical result.

**Theorem 7.2.** *Let  $p > 3$ ,  $n \geq 6$ ,  $H = \hat{S}_n$  or  $\hat{A}_n$ , and  $D$  be a non-basic irreducible faithful  $\mathbb{F}H$ -module. Then  $d_3(D) > d_2(D)$ .*

*Proof.* Note that  $M^{(j, n-j)}$  is the permutation module on the set of unordered  $j$ -tuples  $\{i_1, i_2, \dots, i_j\}$  of different numbers  $i_1, i_2, \dots, i_j \in \{1, 2, \dots, n\}$ . Set

$$f : M^{(3, n-3)} \rightarrow M^{(2, n-2)}, \quad \{i_1, i_2, i_3\} \mapsto \{i_1, i_2\} + \{i_1, i_3\} + \{i_2, i_3\}.$$

By [7, 3.1],  $f$  is surjective. Hence  $f^* : \mathcal{H}_2(D) \rightarrow \mathcal{H}_3(D)$  defined by  $f^*(\psi) = \psi \circ f$  is injective. So it suffices to prove that  $\text{Im} f^* \neq \mathcal{H}_3(D)$ . To show this, we construct an element  $\varphi \in \mathcal{H}_3 \setminus \text{Im} f^*$ . Set  $\varphi(\{1, 2, 3\})$  to be the element of  $\text{End}_F(D)$  which maps  $v \in D$  to  $(1+t_1t_2+t_2+t_1)v$ . Moreover, if  $\sigma \in H$  is the element with  $\sigma(1) = i_1, \sigma(2) = i_2, \sigma(3) = i_3$  then we set  $\varphi(\{i_1, i_2, i_3\})$  to be the element of  $\text{End}_F(D)$  which maps  $v \in D$  to  $\sigma(1+t_1t_2+t_2t_1)\sigma^{-1}v$ . As any  $\sigma \in G := H \cap \hat{S}_{(3, n-3)}$  commutes with  $(1+t_1t_2+t_2t_1)$ ,  $\varphi$  is a well defined  $\mathbb{F}H$ -homomorphism from  $M^{(3, n-3)} = \text{ind}_G^H \mathbf{1}_G$  to  $\text{End}_F(D)$ .

Now assume for a contradiction that  $\varphi \in \text{Im} f^*$ . Then  $\varphi = f^*(\psi) = \psi \circ f$  for some  $\psi \in \mathcal{H}_2$ . Let  $C$  be as in Lemma 7.1. Set  $e = \sum_{c \in C} (\text{sgn } c) \hat{c}\{1, 2, 3\} \in M^{(3, n-3)}$ . In view of [17, 4.10], we have  $f(e) = 0$ , so  $\varphi(e) = \psi(f(e)) = 0$ . By definition of  $\varphi$ , this means that  $xv = 0$  for  $x$  as in Lemma 7.1. But this contradicts Lemma 7.1.  $\square$

For  $1 \leq k \leq n$  and a subgroup  $G \leq S_n$ , we set

$$r_k(G) = \#\{G\text{-orbits on unordered } k\text{-element subsets of } \{1, 2, \dots, n\}\}.$$

If  $r_k(G) = 1$ , the group  $G$  is called  $k$ -homogeneous. If  $k \leq n/2$ , it is well known [28] that  $r_k(G) \geq r_{k-1}(G)$  for any  $G$ . The following result shows that often  $r_3(G) > r_2(G)$ .

**Lemma 7.3.** [11, §5, Corollary] *Let  $n > 5$ ,  $G < S_n$  be transitive, and  $r_2(G) = r_3(G)$ . Then one of the following holds:*

- (i)  $n$  is even and  $G$  has 2 blocks of imprimitivity of size  $n/2$ .
- (ii)  $n$  is even and  $G$  has  $n/2$  blocks of imprimitivity of size 2.
- (iii)  $G$  is 3-homogeneous.

**Theorem 7.4.** *Let  $p > 3$ ,  $n \geq 6$ ,  $H = \hat{S}_n$  or  $\hat{A}_n$ , and  $G < H$  be a subgroup. Assume that the restriction  $D \downarrow_G$  is irreducible for some non-basic faithful  $\mathbb{F}H$ -module  $D$ . Then one of the following holds:*

- (i)  $n$  is even,  $\pi(G)$  is transitive and has 2 blocks of imprimitivity of size  $n/2$ .
- (ii)  $n$  is even,  $\pi(G)$  is transitive and has  $n/2$  blocks of imprimitivity of size 2.
- (iii)  $\pi(G)$  is 3-homogeneous.



- (iv)  $\pi(G)$  has an orbit of length  $n - 1$  or  $n - 2$  on  $\{1, 2, \dots, n\}$ , and  $\pi(G)$  is 3-homogeneous on this orbit.

*Proof.* Suppose  $\pi(G)$  does not satisfy the conditions (i)–(iv) above. There are three cases:

- (1)  $\pi(G)$  is transitive. Then  $r_3(\pi(G)) > r_2(\pi(G))$  by Lemma 7.3.
- (2)  $\pi(G)$  is contained in a Young subgroup  $S_\lambda$  or  $A_\lambda$ , call it  $H_\lambda$ , with  $\lambda \neq (n - 1, 1), (n - 2, 2), (n - 2, 1, 1)$ . One easily checks that we have  $r_3(H_\lambda) > r_2(H_\lambda)$ .
- (3)  $\pi(G)$  is contained in a Young subgroup  $H_\lambda$ , with  $\lambda \in \{(n - 1, 1), (n - 2, 2), (n - 2, 1, 1)\}$ , but  $\pi(G)$  is not 3-homogeneous on the ‘long’ orbit. Here again we have  $r_3(\pi(G)) > r_2(\pi(G))$ .

Thus, if we can show that the restriction to a subgroup  $G$  with  $r_3(\pi(G)) > r_2(\pi(G))$  is reducible, the theorem will be proved (in the case (2) we will have that already the restriction to  $H_\lambda$  is reducible).

As  $r_k(\pi(G))$  equals the dimension of the  $G$ -invariant space  $(M^{(n-k,k)})^G$  and  $d_3(D) > d_2(D)$  by Theorem 7.2, the proof of Proposition 3.4 in [7] shows that  $\dim \text{End}_{\mathbb{F}G}(D \downarrow_G) > 1$ , and so the result follows from Schur’s Lemma.  $\square$

**Remark.** (i) In the case (iii) of Theorem 7.4,  $\pi(G)$  is primitive, and so this case is covered by Theorem B.

(ii) Proposition 6.2 proved in [29] shows that irreducible restrictions of a non-basic spin module from  $H$  to  $H_\lambda$  are possible only if

$$\lambda \in \{(n - 1, 1), (n - 2, 2), (n - 2, 1, 1)\}.$$

Theorem 7.4 provides an easier proof of this result, providing  $p > 3$ .

To complete the proof of Theorem D, it remains to refine the case (iv) of Theorem 7.4. We need the following proposition:

**Proposition 7.5.** *Let  $n \geq 5$ , and  $X < S_n$  be a 3-homogeneous subgroup with  $\text{soc}(X)$  not simple. Then one of the following holds:*

- (i)  $X = \text{AGL}_m(2)$  with  $n = 2^m$ ;
- (ii)  $X = 2^4.A_7$  with  $n = 16$ ;
- (iii)  $X = \text{AGL}_1(8)$  or  $\text{A}\Gamma\text{L}_1(8)$  with  $n = 8$ ;
- (iv)  $X = \text{A}\Gamma\text{L}_1(32)$  with  $n = 32$ .

*Proof.* The 3-homogeneous groups which are not 3-transitive were described in [20]. This yields the cases (iii) and (iv) above. Now we may assume that  $X$  is 3-transitive. Set  $\Omega := \{1, 2, \dots, n\}$ . Since  $\text{soc}(X)$  is not simple, it follows from the O’Nan-Scott Theorem that  $\text{soc}(X) = Z_q^m$ , for some prime  $q$ , and it acts regularly on  $\Omega$ . Then  $X = VX_0$ , where  $V = Z_q^m$  and  $X$  acts on  $\Omega = V$  as affine transformations.

Now,  $X_0$  acts 2-transitively on  $V \setminus \{0\}$ . On the other hand, if  $q > 2$  and  $m > 1$ , or if  $q > 3$  and  $m = 1$ , then  $X_0 \leq \text{GL}(V)$  acts imprimitively on  $V \setminus \{0\}$ . In our case  $q^m \geq 5$ . Hence  $q = 2$  and  $X_0 \leq \text{GL}_m(2)$ . Now consider  $Y := \text{soc}(X_0)$ . We have two cases.

1)  $Y$  is non-abelian simple. Since  $N := |V \setminus \{0\}| = 2^m - 1 \geq 7$ , in view of the classification of 2-transitive groups, (see [10, 21]) one of the following holds:

- a)  $Y = A_N$ . This is impossible as  $A_N$  with  $N = 2^m - 1$  cannot be embedded in  $\text{GL}_m(2)$ .
- b)  $Y = A_7$  and  $N = 15$ . This leads to the case (ii).

- c)  $Y = {}^2B_2(q)$  and  $N = q^2 + 1$ . Impossible as  $N = 2^m - 1$ .
- d)  $Y = {}^2A_2(q)$  and  $N = q^3 + 1$ . Impossible as  $N = 2^m - 1$ .
- e)  $Y = PSL_d(q)$  and  $N = (q^d - 1)/(q - 1)$ . Assume  $q$  is even. Then, since  $N = 2^m - 1 \geq 7$ , we see that  $q = 2$ ,  $d = m$ ,  $X_0 = GL_m(2)$ , and  $G = ASL_m(2)$ , giving the case (i) of the proposition. Finally, let  $q$  be odd. Since  $N = 2^m - 1 \geq 7$ , we see that  $d \geq 3$  and  $d$  is odd. Now,  $Y = PSL_d(q)$  embeds into  $GL_m(2)$ , giving a cross-characteristic representation of  $PSL_d(q)$ . By [14, Corollary 6.2], we have  $m \geq (q^d - 1)/(q - 1) - 2$ . This contradicts the equality  $2^m - 1 = (q^d - 1)/(q - 1)$ .
- 2)  $Y$  is elementary abelian, i.e.  $Y = Z_r^k$  for a prime  $r$ . Then  $7 \leq 2^m - 1 = r^k$ . An elementary number theory argument shows that  $k = 1$  and  $r = 2^m - 1$ . In this case  $X_0 \leq N_{GL_m(2)}(Z_r) = Z_r \cdot Z_m$ . In particular,  $|X_0| \leq rm$ . On the other hand,  $X_0$  is 2-transitive on  $V \setminus \{0\}$ , so  $|X_0| \geq r(r - 1)$ , a contradiction.  $\square$

Now, let  $G$  be as in (iv), and let  $\Omega_1$  be the ‘long’ orbit. Assume that  $D \downarrow_G$  is irreducible. Set  $s := |\Omega_1|$ , and let  $K$  be defined as in (6.2). We have  $s = n - 1$  or  $n - 2$ ,  $K \cong \hat{S}_s$  if  $H = \hat{S}_n$ , and  $K \cong \hat{A}_s$  if  $H = \hat{A}_n$ . Let  $X < S_s$  be the image of  $\pi(G)$  with respect to its action on  $\Omega_1$ . We apply Proposition 7.5 to the action of  $X$  on  $\Omega_1$ . We have that  $X$  is as in the cases (i)-(iv) in Proposition 7.5. In particular,  $X$  has no subgroups of index 2.

We claim that  $\pi(G) = X$  and  $\pi(G) \leq \pi(K)$ . This is obvious if  $s = n - 1$ , or if  $s = n - 2$  and  $\pi(G)$  is of type  $(n - 2, 1, 1)$ . Assume  $s = n - 2$  and  $\pi(G)$  is of type  $(n - 2, 2)$ . Then  $\pi(G) \cap \pi(K)$  has index 2 in  $\pi(G)$ , and it embeds in  $X$ . But  $X$  has no subgroups of index 2, so  $\pi(G) \cap \pi(K) = X$ . We may assume  $\pi(G) = \langle X, x \rangle$ , where  $x$  interchanges the two points of  $\Omega \setminus \Omega_1$ . Since  $\pi(G) \cap \pi(K) = X$ , there is  $y \in \pi(G)$  that fixes  $\Omega \setminus \Omega_1$  pointwise and induces the same action as  $x$  on  $\Omega_1$ . Replacing  $x$  by  $xy^{-1}$ , we may assume that  $x$  is a transposition. Since  $X$  has no subgroups of index 2,  $\hat{X}$  is contained in  $\hat{A}_s$  and so it is centralized by  $\hat{x}$ . Thus  $G$  is centralized by  $\hat{x}$ , a non-central involution in  $\hat{S}_n$ , so  $D \downarrow_G$  is reducible, a contradiction.

Now we know that  $G \leq K$ , and the action of  $\pi(G)$  on  $\Omega_1$  is certainly primitive, not containing  $A_s$ . Moreover, as  $D$  is non-basic, we have by Lemma 2.4 that the restriction  $D \downarrow_K$  has a non-basic composition factor  $D'$ . So we may apply Theorem B to  $(K, G, D')$ , and we also know that  $\pi(G)$  is as in (i)-(iv) of Proposition 7.5. We then get  $s = 8$ ,  $m = 3$ ,  $\pi(G) = AGL_3(2)$ ,  $\dim(D) = 24$ . Here  $n \geq 9$ , so we get a contradiction as  $H \in \hat{A}_{9,10}, \hat{S}_{9,10}$  has no non-basic representation of degree 24. The proof of Theorem D is complete.

## 8. TENSOR PRODUCTS (THEOREM F)

Recall the notation  $d_j(V)$  introduced in (7.1).

**Theorem 8.1.** *Let  $p > 3$ ,  $H = \hat{S}_n$  or  $\hat{A}_n$  with  $n \geq 7$ . Suppose  $A, B$  are irreducible  $\mathbb{F}H$ -modules such that  $d_3(A) > d_2(A)$  and  $d_3(B) > d_2(B)$ . Then the  $H$ -module  $A \otimes B$  is reducible.*

*Proof.* Consider the irreducible  $\mathbb{F}S_n$ -module  $D_j$  labelled by  $(n - j, j)$  with  $j = 0, 1, 2$ , or 3. In particular,  $D_0 = \mathbf{1}$ . It is well known that  $D_j$  is self-dual, and the modules  $D_j \downarrow_{A_n}$  are also irreducible and distinct. Throughout this proof we write  $G = \pi(H)$  for short. The condition  $d_3(A) > d_2(A)$  means that

$$\dim \operatorname{Hom}_{\mathbb{F}H}(M^{(n-3,3)}, A \otimes A^*) > \dim \operatorname{Hom}_{\mathbb{F}H}(M^{(n-2,2)}, A \otimes A^*).$$

The proof of [7, 3.4] applied to  $\text{End}_{\mathbb{F}}(A) \cong A \otimes A^*$  shows that one of the following holds.

(I) There is no  $r$  such that  $1 \leq r \leq 3$  and  $n \equiv 5 - r \pmod{p}$ . Then there is an injective  $G$ -homomorphism

$$\mathbf{1} \oplus D_3 \xrightarrow{\alpha} A \otimes A^*.$$

(II) There is (a unique)  $r$  such that  $1 \leq r \leq 3$  and  $n \equiv 5 - r \pmod{p}$ . Let  $s := 3 - r$ . Then, in the notation of [7], one of the following holds:

(A1) There is an injective  $G$ -homomorphism

$$\mathbf{1} \oplus (S^{(n-3,3)})^* = \mathbf{1} \oplus D_3 | D_s \xrightarrow{\alpha} A \otimes A^*.$$

(A2)  $r \neq 3$  and there is an injective  $G$ -homomorphism

$$\mathbf{1} \oplus S^{(n-3,3)} = \mathbf{1} \oplus D_s | D_3 \xrightarrow{\alpha} A \otimes A^*.$$

(A3)  $r = 3$  and there is an injective  $G$ -homomorphism

$$Y^{(n-3,3)} = \mathbf{1} | D_3 | \mathbf{1} \xrightarrow{\alpha} A \otimes A^*.$$

We label the corresponding cases for  $B$  by (B1), (B2), (B3), respectively, and the corresponding  $G$ -homomorphism by  $\beta$ .

We aim to show that  $\dim \text{Hom}_{\mathbb{F}G}(B \otimes B^*, A \otimes A^*) > 1$ . As

$$\text{Hom}_{\mathbb{F}H}(B \otimes B^*, A \otimes A^*) \cong \text{Hom}_{\mathbb{F}H}(A \otimes B, A \otimes B),$$

this implies that  $A \otimes B$  is not irreducible by Schur's Lemma. There is an obvious nonzero homomorphism  $B \otimes B^* \rightarrow \mathbf{1} \hookrightarrow A \otimes A^*$ . Here and below  $\rightarrow$  means a surjective homomorphism. We show that there is another homomorphism  $\gamma$  which is linearly independent from this one.

Suppose (I) holds. Then take  $\gamma$  to be the map

$$B \otimes B^* \xrightarrow{\beta^*} (\mathbf{1} \oplus D_3)^* \cong \mathbf{1} \oplus D_3 \xrightarrow{\alpha} A \otimes A^*.$$

Now suppose we are in the case (II).

If (A1) holds for  $A$  and (B1) holds for  $B$ , take  $\gamma$  to be the map

$$B \otimes B^* \xrightarrow{\beta^*} (\mathbf{1} \oplus D_3 | D_s)^* \cong \mathbf{1} \oplus D_s | D_3 \rightarrow D_3 \xrightarrow{\alpha} A \otimes A^*.$$

Now assume that  $r = 3$ . Then  $D_s \cong \mathbf{1}$ . If (A3) holds for  $A$  and (B3) holds for  $B$ , then take  $\gamma$  to be the map

$$B \otimes B^* \xrightarrow{\beta^*} (\mathbf{1} | D_3 | \mathbf{1})^* \cong \mathbf{1} | D_3 | \mathbf{1} \xrightarrow{\alpha} A \otimes A^*.$$

If (A1) holds for  $A$  and (B3) holds for  $B$ , then take  $\gamma$  to be the map

$$B \otimes B^* \xrightarrow{\beta^*} (\mathbf{1} | D_3 | \mathbf{1})^* \cong \mathbf{1} | D_3 | \mathbf{1} \rightarrow D_3 | \mathbf{1} \xrightarrow{\alpha} A \otimes A^*.$$

In the case of (A3) and (B1) we can interchange  $A$  and  $B$ .

Finally, assume that  $r \neq 3$ . In this case  $D_s \not\cong \mathbf{1}$ . If (A2) holds for  $A$  and (B2) holds for  $B$ , then take  $\gamma$  to be the map

$$B \otimes B^* \xrightarrow{\beta^*} (\mathbf{1} \oplus D_s | D_3)^* \cong \mathbf{1} \oplus D_3 | D_s \rightarrow D_s \xrightarrow{\alpha} A \otimes A^*.$$

If (A1) holds for  $A$  and (B2) holds for  $B$ , then take  $\gamma$  to be the map

$$B \otimes B^* \xrightarrow{\beta^*} (\mathbf{1} \oplus D_s | D_3)^* \cong \mathbf{1} \oplus D_3 | D_s \xrightarrow{\alpha} A \otimes A^*.$$

In the case of (A2) and (B1) we can interchange  $A$  and  $B$ . □

Now Theorem F(i) follows from Theorems 7.2 and 8.1, and Theorem F(ii) follows from Theorems 7.2, 8.1, and [7, 3.9].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OR 97403  
*E-mail address:* `klesh@math.uoregon.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FL 32611  
*E-mail address:* `tiep@math.ufl.edu`