Lectures on Infinite Dimensional Lie Algebras

Alexander Kleshchev

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Part one

Kac-Moody Algebras

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Main Definitions

1.1 Some Examples

1.1.1 Special Linear Lie Algebras

Let $\mathfrak{g} = \mathfrak{sl}_n = \mathfrak{sl}_n(\mathbb{C})$. Choose the subalgebra $\mathfrak{h}$ consisting of all diagonal matrices in $\mathfrak{g}$. Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1}$,

$$\alpha_1^\vee, \dots, \alpha_{n-1}^\vee$$

is a basis of $\mathfrak{h}$. Next define $\varepsilon_1 \dots, \varepsilon_n \in \mathfrak{h}^*$ by

$$\varepsilon_i : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i.$$

Then, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$,

$$\alpha_1, \dots, \alpha_{n-1}$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the $(n-1) \times (n-1)$ matrix $A := (a_{ij})$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

This matrix is called the *Cartan matrix*. Define

$$X_{\varepsilon_i - \varepsilon_j} := e_{ij}, \ X_{-\varepsilon_i + \varepsilon_j} := e_{ji} \qquad (1 \le i < j \le n)$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

and

$$\{\alpha_1^\vee, \dots, \alpha_{n-1}^\vee\} \cup \{X_{\varepsilon_i - \varepsilon_j} \mid 1 \le i \ne j \le n\}$$

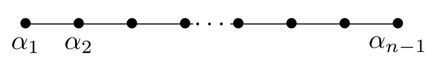
is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i < n$. It is easy to check that

$$e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}, \alpha_1^\vee, \dots \alpha_{n-1}^\vee \tag{1.1}$$

generate $\mathfrak{g}$ and the following relations hold:

$$[e_i, f_j] = \delta_{ij} \alpha_i^\vee, \tag{1.2}$$
$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \tag{1.3}$$
$$[\alpha_i^\vee, e_j] = a_{ij} e_j, \tag{1.4}$$
$$[\alpha_i^\vee, f_j] = -a_{ij} f_j, \tag{1.5}$$
$$(\mathrm{ad}\, e_i)^{1-a_{ij}}(e_j) = 0 \qquad (i \ne j), \tag{1.6}$$
$$(\mathrm{ad}\, f_i)^{1-a_{ij}}(f_j) = 0 \qquad (i \ne j). \tag{1.7}$$

A (special case of a) theorem of Serre claims that $\mathfrak{g}$ is actually generated by the elements of (1.1) subject *only* to these relations. What is important for us is the fact that the Cartan matrix contains all the information needed to write down the Serre's presentation of A. Since the Cartan matrix is all the data we need, it makes sense to find a nicer geometric way to picture the same data. Such picture is called the Dynkin diagram, and in our case it is:



Here vertices i and $i+1$ are connected because $a_{i,i+1} = a_{i+1,i} = -1$, others are not connected because $a_{ij} = 0$ for $|i - j| > 1$, and we don't have to record a_{ii} since it is always going to be 2.

1.1.2 Symplectic Lie Algebras

Let V be a $2n$-dimensional vector space and $\varphi : V \times V \to \mathbb{C}$ be a non-degenerate symplectic bilinear form on V. Let

$$\mathfrak{g} = \mathfrak{s}p(V, \varphi) = \{X \in \mathfrak{g}l(V) \mid \varphi(Xv, w) + \varphi(v, Xw) = 0 \text{ for all } v, w \in V\}.$$

An easy check shows that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. It is known from linear algebra that over $\mathbb{C}$ all non-degenerate symplectic forms are equivalent, i.e. if φ' is another such form then $\varphi'(v,w) = \varphi(gv, gw)$ for some fixed $g \in GL(V)$. It follows that

$$\mathfrak{sp}(V,\varphi') = g^{-1}(\mathfrak{sp}(V,\varphi))g \cong \mathfrak{sp}(V,\varphi),$$

thus we can speak of just $\mathfrak{sp}(V)$. To think of $\mathfrak{sp}(V)$ as a Lie algebra of matrices, choose a symplectic basis $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$, that is

$$\varphi(e_i, e_{-i}) = -\varphi(e_{-i}, e_i) = 1,$$

and all other $\varphi(e_i, e_j) = 0$. Then the Gram matrix is

$$G = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix},$$

where

$$s = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \vdots & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{1.8}$$

It follows that the matrices of $\mathfrak{sp}(V)$ in the basis of e_i's are precisely the matrices from the Lie algebra

$$\mathfrak{sp}_{2n} = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = sB^t s, C = sC^t s, D = -sA^t s\},$$

so $\mathfrak{sp}(V) \cong \mathfrak{sp}_{2n}$. Note that $sX^t s$ is the transpose of X with respect to the second main diagonal.

Choose the subalgebra $\mathfrak{h}$ consisting of all diagonal matrices in $\mathfrak{g}$. Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1} - e_{-i,-i} + e_{-i-1,-i-1}$, for $1 \le i < n$ and $\alpha_n^\vee = e_{nn} - e_{-n,-n}$,

$$\alpha_1^\vee, \dots, \alpha_{n-1}^\vee, \alpha_n^\vee$$

is a basis of $\mathfrak{h}$. Next, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i < n$, and $\alpha_n := 2\varepsilon_n$,

$$\alpha_1, \dots, \alpha_{n-1}, \alpha_n$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the Cartan matrix A is the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

Define

$$\begin{aligned} X_{2\varepsilon_i} &= e_{i,-i}, \quad (1 \le i \le n) \\ X_{-2\varepsilon_i} &= e_{-i,i}, \quad (1 \le i \le n) \\ X_{\varepsilon_i-\varepsilon_j} &= e_{ij} - e_{-j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i+\varepsilon_j} &= e_{ji} - e_{-i,-j} \quad (1 \le i < j \le n) \\ X_{\varepsilon_i+\varepsilon_j} &= e_{i,-j} + e_{j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i-\varepsilon_j} &= e_{-j,i} + e_{-i,j} \quad (1 \le i < j \le n). \end{aligned}$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

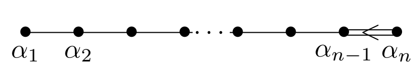
and

$$\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \cup \{X_\alpha\}$$

is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i \le n$. It is easy to check that

$$e_1, \dots, e_n, f_1, \dots, f_n, \alpha_1^\vee, \dots \alpha_n^\vee \tag{1.9}$$

generate $\mathfrak{g}$ and the relations (1.2-1.7) hold. Again, Serre's theorem claims that $\mathfrak{g}$ is actually generated by the elements of (1.11) subject *only* to these relations. The Dynkin diagram in this case is:



The vertices $n-1$ and n are connected the way they are because $a_{n-1,n} = -2$ and $a_{n,n-1} = -1$, and in other places we follow the same rules as in the case $\mathfrak{sl}$.

1.1.3 Orthogonal Lie Algebras

Let V be an N-dimensional vector space and $\varphi : V \times V \to \mathbb{C}$ be a non-degenerate symmetric bilinear form on V. Let

$$\mathfrak{g} = \mathfrak{so}(V, \varphi) = \{X \in \mathfrak{gl}(V) \mid \varphi(Xv, w) + \varphi(v, Xw) = 0 \text{ for all } v, w \in V\}.$$

An easy check shows that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. It is known from linear algebra that over $\mathbb{C}$ all non-degenerate symmetric bilinear forms are equivalent, i.e. if φ' is another such form then $\varphi'(v, w) = \varphi(gv, gw)$ for some fixed $g \in GL(V)$. It follows that

$$\mathfrak{so}(V, \varphi') = g^{-1}(\mathfrak{so}(V, \varphi))g \cong \mathfrak{so}(V, \varphi),$$

thus we can speak of just $\mathfrak{so}(V)$. To think of $\mathfrak{so}(V)$ as a Lie algebra of matrices, choose a basis $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$ if $N = 2n$ and $e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}$ if $N = 2n + 1$, such that the Gram matrix of φ in this basis is

$$\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & s \\ 0 & 2 & 0 \\ s & 0 & 0 \end{pmatrix},$$

respectively, where s is the $n \times n$ matrix as in (1.8). It follows that the matrices of $\mathfrak{so}(V)$ in the basis of e_i's are precisely the matrices from the Lie algebra

$$\mathfrak{so}_{2n} = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = -sB^t s, C = -sC^t s, D = -sA^t s\},$$

if $N = 2n$, and

$$\mathfrak{so}_{2n+1} = \{\begin{pmatrix} A & 2sx^t & B \\ y & 0 & x \\ C & 2sy^t & D \end{pmatrix} \mid B = -sB^t s, C = -sC^t s, D = -sA^t s\},$$

if $N = 2n + 1$ (here x, y are arbitrary $1 \times n$ matrices). We have in all cases that $\mathfrak{so}(V) \cong \mathfrak{so}_N$.

Choose the subalgebra $\mathfrak{h}$ consisting of all diagonal matrices in $\mathfrak{g}$. We now consider even and odd cases separately. First, let $N = 2n$.

Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1} - e_{-i,-i} + e_{-i-1,-i-1}$, for $1 \leq i < n$ and $\alpha_n^\vee = e_{n-1,n-1} + e_{nn} - e_{-n+1,-n+1} - e_{-n,-n}$,

$$\alpha_1^\vee, \dots, \alpha_{n-1}^\vee, \alpha_n^\vee$$

is a basis of $\mathfrak{h}$. Next, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i < n$, and $\alpha_n :=$

$\varepsilon_{n-1} + \varepsilon_n$,

$$\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the Cartan matrix A is the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 0 & 2 \end{pmatrix}.$$

Define

$$\begin{aligned} X_{\varepsilon_i - \varepsilon_j} &= e_{ij} - e_{-j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i + \varepsilon_j} &= e_{ji} - e_{-i,-j} \quad (1 \le i < j \le n) \\ X_{\varepsilon_i + \varepsilon_j} &= e_{i,-j} - e_{j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i - \varepsilon_j} &= e_{-j,i} - e_{-i,j} \quad (1 \le i < j \le n). \end{aligned}$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

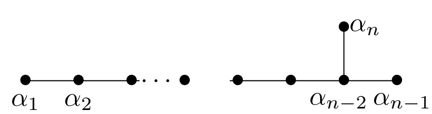
and

$$\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \cup \{X_\alpha\}$$

is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i \le n$. It is easy to check that

$$e_1, \ldots, e_n, f_1, \ldots, f_n, \alpha_1^\vee, \ldots \alpha_n^\vee \tag{1.10}$$

generate $\mathfrak{g}$ and the relations (1.2-1.7) hold. Again, Serre's theorem claims that $\mathfrak{g}$ is generated by the elements of (1.11) subject *only* to these relations. The Dynkin diagram in this case is:



Let $N = 2n+1$. Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1} - e_{-i,-i} + e_{-i-1,-i-1}$, for $1 \le i < n$ and $\alpha_n^\vee = 2e_{nn} - 2e_{-n,-n}$,

$$\alpha_1^\vee, \dots, \alpha_{n-1}^\vee, \alpha_n^\vee$$

is a basis of $\mathfrak{h}$. Next, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i < n$, and $\alpha_n := \varepsilon_n$,

$$\alpha_1, \dots, \alpha_{n-1}, \alpha_n$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the Cartan matrix A is the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -2 & 2 \end{pmatrix}.$$

(It is transpose to the one in the symplectic case). Define

$$\begin{aligned} X_{\varepsilon_i} &= 2e_{i,0} + e_{0,-i}, \quad (1 \le i \le n) \\ X_{-\varepsilon_i} &= 2e_{-i,0} + e_{0,i}, \quad (1 \le i \le n) \\ X_{\varepsilon_i - \varepsilon_j} &= e_{ij} - e_{-j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i + \varepsilon_j} &= e_{ji} - e_{-i,-j} \quad (1 \le i < j \le n) \\ X_{\varepsilon_i + \varepsilon_j} &= e_{i,-j} - e_{j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i - \varepsilon_j} &= e_{-j,i} - e_{-i,j} \quad (1 \le i < j \le n). \end{aligned}$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

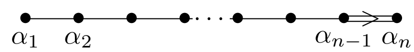
and

$$\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \cup \{X_\alpha\}$$

is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i \le n$. It is easy to check that

$$e_1, \dots, e_n, f_1, \dots, f_n, \alpha_1^\vee, \dots \alpha_n^\vee \tag{1.11}$$

generate $\mathfrak{g}$ and the relations (1.2-1.7) hold. Again, Serre's theorem claims that $\mathfrak{g}$ is actually generated by the elements of (1.11) subject *only* to these relations. The Dynkin diagram in this case is:



1.2 Generalized Cartan Matrices

Definition 1.2.1 A matrix $A \in M_n(\mathbb{Z})$ is a *generalized Cartan matrix* (GCM) if

(C1) $a_{ii} = 2$ for all i;
(C2) $a_{ij} \le 0$ for all $i \ne j$;
(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Two GCMs A and A' are *equivalent* if they have the same degree n and there is $\sigma \in S_n$ such that $a'_{ij} = a_{\sigma(i),\sigma(j)}$. A GCM is called *indecomposable* if it is not equivalent to a diagonal sum of smaller GCMs.

Throughout we are going to assume that $A = (a_{ij})_{1\le i,j\le n}$ is a generalized Cartan matrix of rank ℓ.

Definition 1.2.2 A *realization* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ such that

(i) both Π and $\Pi^\vee$ are linearly independent;
(ii) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for all i, j;
(iii) $\dim \mathfrak{h} = 2n - \ell$.

Two realizations $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}', \Pi', (\Pi')^\vee)$ are *isomorphic* if there exists an isomorphism $\varphi : \mathfrak{h} \to \mathfrak{h}'$ of vector spaces such that $\varphi(\alpha_i^\vee) = ((\alpha_i')^\vee)$ and $\varphi^*(\alpha_i') = (\alpha_i)$ for $i = 1, 2, \dots, n$.

Example 1.2.3 (i) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. We have $n = \ell = 3$. Let $e_1, \dots, e_4$ be the standard basis of $\mathbb{C}^4$, $\varepsilon_1, \dots, \varepsilon_4$ be the dual basis, and $\mathfrak{h} = \{(a_1, \dots, a_4) \mid a_1 + \dots + a_4 = 0\}$. Finally, take $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4\}$ and $\Pi^\vee = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$.

Another realization comes as follows. Let $\mathfrak{h} = \mathbb{C}^3$, and α_i denote the ith coordinate function. Now take $\alpha_i^\vee$ to be the ith row of A. It is clear that the two realizations are isomorphic.

(ii) Let $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. We have $n = 3$, $\ell = 2$. Take $\mathfrak{h} = \mathbb{C}^4$ and let α_i denote the ith coordinate function (we only need the first three). Now take $\alpha_i^\vee$ to be the ith row of the matrix $\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 1 \end{pmatrix}$.

Proposition 1.2.4 *For each A there is a unique up to isomorphism realization. Realizations of matrices A and B are isomorphic if and only if $A = B$.*

Proof Assume for simplicity that A is of the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{11} is a non-singular $\ell \times \ell$ matrix. Let

$$C = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}.$$

Note $\det C = \pm \det A_{11}$, so C is non-singular. Let $\mathfrak{h} = \mathbb{C}^{2n-\ell}$. Define $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ to be the first n coordinate functions, and $\alpha_1^\vee, \dots, \alpha_n^\vee$ to be the first n row vectors of C.

Now, let $(\mathfrak{h}', \Pi', (\Pi')^\vee$ be another realization of A. We complete $(\alpha')_1^\vee, \dots, (\alpha')_n^\vee$ to a basis $(\alpha')_1^\vee, \dots, (\alpha')_{2n-\ell}^\vee$ of $\mathfrak{h}'$. Then the matrix $(\langle(\alpha_i')^\vee, \alpha_j'\rangle)$ has form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ B_1 & B_2 \end{pmatrix}.$$

By linear independence, this matrix has rank n. Thus it has n linearly independent rows. Since the rows rows $\ell + 1, \dots, n$ are linear combinations of rows $1, \dots, \ell$, the matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ B_1 & B_2 \end{pmatrix}$$

is non-singular. We now complete $\alpha_1', \dots, \alpha_n'$ to $\alpha_1', \dots, \alpha_{2n-\ell}'$, so that the matrix $(\langle(\alpha_i')^\vee, \alpha_j'\rangle)$ is

$$\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ B_1 & B_2 & 0 \end{pmatrix}.$$

This matrix is non-singular, so $\alpha'_1, \dots, \alpha'_{2n-\ell}$ is a basis of $(\mathfrak{h}')^*$. Since A_{11} is non-singular, by adding suitable linear combinations of the first ℓ rows to the last $n-\ell$ rows, we may achieve $B_1 = 0$. Thus it is possible to choose $(\alpha')^\vee_{n+1}, \dots, (\alpha')^\vee_{2n-\ell}$, so that $(\alpha')^\vee_1, \dots, (\alpha')^\vee_{2n-\ell}$ are a basis of $\mathfrak{h}'$ and

$$(\langle(\alpha'_i)^\vee, \alpha'_j\rangle) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ 0 & B'_2 & 0 \end{pmatrix}.$$

The matrix B'_2 must be non-singular since the whole matrix is non-singular. We now make a further change to $(\alpha')^\vee_{n+1}, \dots, (\alpha')^\vee_{2n-\ell}$ equivalent to multiplying the above matrix by

$$\begin{pmatrix} I_\ell & 0 & 0 \\ 0 & I_{n-\ell} & 0 \\ 0 & 0 & (B'_2)^{-1} \end{pmatrix}.$$

Then we obtain

$$(\langle(\alpha'_i)^\vee, \alpha'_j\rangle) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}.$$

This is equal to the matrix C above. Thus the map $\alpha_i^\vee \mapsto (\alpha'_i)^\vee$ gives an isomorphism $\mathfrak{h} \to \mathfrak{h}'$ whose dual is given by $\alpha'_i \mapsto \alpha_i$. This shows that the realizations $(\mathfrak{h}, \Pi, \Pi^\vee$ and $(\mathfrak{h}', \Pi', (\Pi')^\vee$ are isomorphic.

Finally, assume that $\varphi : (\mathfrak{h}, \Pi, \Pi^\vee) \to (\mathfrak{h}', \Pi', (\Pi')^\vee)$ is an isomorphism of realizations of A and B respectively. Then

$$b_{ij} = \langle(\alpha'_i)^\vee, \alpha'_j\rangle = \langle\varphi(\alpha_i^\vee), \alpha'_j\rangle = \langle\alpha_i^\vee, \varphi^*(\alpha'_j)\rangle = \langle\alpha_i^\vee, \alpha_j\rangle = a_{ij}.$$

□

Throughout we assume that $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of A.

We refer to the elements of Π as *simple roots* and the elements of $\Pi^\vee$ as *simple coroots*, to Π and $\Pi^\vee$ as *root basis* and coroot basis, respectively. Also set

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i, \quad Q_+ = \oplus_{i=1}^n \mathbb{Z}_+\alpha_i.$$

We call Q *root lattice*. *Dominance ordering* is a partial order $\geq$ on $\mathfrak{h}^*$ defined as follows: $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. For $\alpha = \sum_{i=1}^n k_i\alpha_i \in Q$, the number

$$\text{ht}\ \alpha := \sum_{i=1}^n k_i$$

is called the *height* of α.

1.3 The Lie algebra $\tilde{\mathfrak{g}}(A)$

Definition 1.3.1 *The Lie algebra $\tilde{\mathfrak{g}}(A)$ is defined as the algebra with generators e_i, f_i $(i = 1, \dots, n)$ and $\mathfrak{h}$ and relations*

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee, \tag{1.12}$$
$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h}), \tag{1.13}$$
$$[h, e_i] = \langle \alpha_i, h\rangle e_i \qquad (h \in \mathfrak{h}), \tag{1.14}$$
$$[h, f_i] = -\langle \alpha_i, h\rangle f_i \qquad (h \in \mathfrak{h}). \tag{1.15}$$

It follows from the uniqueness of realizations that $\tilde{\mathfrak{g}}(A)$ depends only on A (this boils down to the following calculation: $\langle \alpha_i', \varphi(h)\rangle = \langle \varphi^*(\alpha_i'), h\rangle = \langle \alpha_i, h\rangle$).

Denote by $\tilde{\mathfrak{n}}_+$ (resp. $\tilde{\mathfrak{n}}_-$) the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by all e_i (resp. f_i).

Lemma 1.3.2 (Weight Lemma) *Let V be an $\mathfrak{h}$-module such that $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ where the* weight space *V_λ is defined as $\{v \in V \mid hv = \langle \lambda, h\rangle v$ for all $h \in \mathfrak{h}\}$. Let U be a submodule of V. Then $U = \bigoplus_{\lambda \in \mathfrak{h}^*}(U \cap V_\lambda)$.*

Proof Any elementr $v \in V$ can be written in the form $v = v_1 + \dots + v_m$ where $v_j \in V_{\lambda_j}$, and theere is $h \in \mathfrak{h}$ such that $\lambda_j(h)$ are all distinct. For $v \in U$, we have

$$h^k(v) = \sum_{j=1}^m \lambda_j(h)^k v_j \in U \qquad (k = 0, 1, \dots, m-1).$$

We got a system of linear equations with non-singular matrix. It follows that all $v_j \in U$. □

Theorem 1.3.3 *Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$. Then*

(i) $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.
(ii) $\tilde{\mathfrak{n}}_+$ *(resp. $\tilde{\mathfrak{n}}_-$) is freely generated by the e_i's (resp. f_i's).*
(iii) *The map $e_i \mapsto f_i$, $f_i \mapsto e_i$, $h \mapsto -h$ $(h \in \mathfrak{h})$ extends uniquely to an involution $\tilde{\omega}$ of $\tilde{\mathfrak{g}}$.*

(iv) *One has the root space decomposition with respect to* $\mathfrak{h}$*:*

$$\tilde{\mathfrak{g}} = \Big(\bigoplus_{\alpha \in Q_+, \alpha \neq 0} \tilde{\mathfrak{g}}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha \in Q_+, \alpha \neq 0} \tilde{\mathfrak{g}}_{\alpha}\Big),$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}} \mid [h,x] = \alpha(h)x$ *for all* $h \in \mathfrak{h}\}$*. Moreover, each* $\tilde{\mathfrak{g}}_\alpha$ *is finite dimensional, and* $\tilde{\mathfrak{g}}_{\pm\alpha} \subset \tilde{\mathfrak{n}}_\pm$ *for* $\pm\alpha \in Q_+$, $\alpha \neq 0$.

(v) *Among the ideals of* $\tilde{\mathfrak{g}}$ *which have trivial intersection with* $\mathfrak{h}$*, there is unique maximal ideal* $\mathfrak{r}$*. Moreover,*

$$\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+) \qquad (\textit{direct sum of ideals}).$$

Proof Let V be a complex vector space with basis $v_1, \dots, v_n$ and let $\lambda \in \mathfrak{h}^*$. Define the action of the generators on the tensor algebra $T(V)$ as follows:

(a) $f_i(a) = v_i \otimes a$ for $a \in T(V)$.

(b) $h(1) = \langle \lambda, h \rangle$ and then inductively on s,

$$h(v_j \otimes a) = -\langle \alpha_j, h \rangle v_j \otimes a + v_j \otimes h(a)$$

for $a \in T^{s-1}(V)$.

(c) $e_i(1) = 0$ and then inductively on s,

$$e_i(v_j \otimes a) = \delta_{ij} \alpha_i^\vee(a) + v_j \otimes e_i(a)$$

for $a \in T^{s-1}(V)$.

To see that these formulas define a representation of $\tilde{\mathfrak{g}}$, let us check the relations. For the first relation:

$$\begin{aligned}(e_i f_j - f_j e_i)(a) &= e_i(v_j \otimes a) - v_j \otimes e_i(a) \\ &= \delta_{ij}\alpha_i^\vee(a) + v_j \otimes e_i(a) - v_j \otimes e_i(a) \\ &= \delta_{ij}\alpha_i^\vee(a).\end{aligned}$$

The second relation is obvious since $\mathfrak{h}$ acts diagonally. For the third relation, apply induction on s, the relation being obvious for $s = 0$. For $s > 0$, take $a = v_k \otimes a_1$ where $a_1 \in T^{s-1}(V)$. Then using induction we have

$$\begin{aligned}(he_j - e_jh)(v_k \otimes a_1) &= h(\delta_{jk}\alpha_j^\vee(a_1)) + h(v_k \otimes e_j(a_1)) \\ &\quad -e_j(-\langle \alpha_k, h\rangle v_k \otimes a_1) - e_j(v_k \otimes h(a_1)) \\ &= \delta_{jk}\alpha_j^\vee(h(a_1)) - \langle \alpha_k, h\rangle v_k \otimes e_j(a_1) \\ &\quad +v_k \otimes h(e_j(a_1)) + \langle \alpha_k, h\rangle \delta_{jk}\alpha_j^\vee(a_1) \\ &\quad +\langle \alpha_k, h\rangle v_k \otimes e_j(a_1) - \delta_{jk}\alpha_j^\vee(h(a_1)) \\ &\quad -v_k \otimes e_jh(a_1) \\ &= v_k \otimes [h, e_j](a_1) + \langle \alpha_j, h\rangle \delta_{jk}\alpha_j^\vee(a_1) \\ &= v_k \otimes \langle \alpha_j, h\rangle e_j(a_1) + \langle \alpha_j, h\rangle \delta_{jk}\alpha_j^\vee(a_1) \\ &= \langle \alpha_j, h\rangle (v_k \otimes e_j(a_1) + \delta_{jk}\alpha_j^\vee(a_1)) \\ &= \langle \alpha_j, h\rangle e_j(v_k \otimes a_1).\end{aligned}$$

Finally, for the fourth relation:

$$\begin{aligned}(hf_j - f_jh)(a) &= h(v_j \otimes a) - v_j \otimes h(a) \\ &= -\langle \alpha_j, h\rangle v_j \otimes a + v_j \otimes a - v_j \otimes h(a) \\ &= -\langle \alpha_j, h\rangle v_j \otimes a.\end{aligned}$$

Now we prove (i)-(v).

(iii) is easy to check using the defining relations.

(ii) Consider the map $\varphi : \tilde{\mathfrak{n}}_- \to T(V),\ u \mapsto u(1)$. We have $\varphi(f_i) = v_i$, and for any Lie word $w(f_1, \dots, f_n)$ we have

$$\varphi(w(f_1, \dots, f_n)) = w(v_1, \dots, v_n).$$

Now, for two words w and w', we have

$$\begin{aligned}\varphi([w(f_1, \dots f_n), w'(f_1, \dots f_n)]) &= [w(v_1, \dots v_n), w'(v_1, \dots v_n)] \\ &= [\varphi(w(f_1, \dots f_n)), \varphi(w'(f_1, \dots f_n))],\end{aligned}$$

so φ is a Lie algebra homomorphism. Now $T(V) = F(v_1, \dots, v_n)$, the free associative algebra on $v_1, \dots, v_n$. Moreover, the free Lie algebra $FL(v_1, \dots, v_n)$ lies in $T(V)$ and is spanned by all Lie words in $v_1, \dots, v_n$. Thus $FL(v_1, \dots, v_n)$ is the image of φ. But there is a Lie algebra homomorphism $\varphi' : FL(v_1, \dots, v_n) \to \tilde{\mathfrak{n}}_-,\ v_i \mapsto f_i$, which is inverse to φ, so φ is an isomorphism. It follows that the f_i generate $\tilde{\mathfrak{n}}_-$ freely. The similar result for $\tilde{\mathfrak{n}}_+$ follows by applying the automorphism $\tilde{\omega}$.

(i) It is clear from relations that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$. Let $u = n_- + h + n_+ = 0$. Then in T(V) we have $0 = u(1) = n_-(1) + \langle \lambda, h\rangle$. It follows that

$\langle \lambda, h\rangle = 0$ for all λ, whence $h = 0$. Now $0 = n_-(1) = \varphi(n_-)$, whence $n_- = 0$.

(iv) It follows from the last two defining relations that

$$\tilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha\in Q_+,\ \alpha\neq 0} \tilde{\mathfrak{g}}_{\pm\alpha}.$$

Moreover,

$$\dim \tilde{\mathfrak{g}}_\alpha \leq n^{|\,\mathrm{ht}\ \alpha|}. \tag{1.16}$$

(v) By Lemma 1.3.2, for any ideal $\mathfrak{i}$ of $\tilde{\mathfrak{g}}$, we have $\mathfrak{i} = \bigoplus_{\alpha\in\mathfrak{h}^*}(\tilde{\mathfrak{g}}_\alpha \cap \mathfrak{i})$. Since $\mathfrak{h} = \tilde{\mathfrak{g}}_0$, the sum of the ideals which have trivial intersection with $\mathfrak{h}$ is the unique maximal ideal with this property. It is also clear that the sum in (v) is direct. Finally, $[f_i, \mathfrak{r}\cap\tilde{\mathfrak{n}}_+] \subset \tilde{\mathfrak{n}}_+$. Hence $[\tilde{\mathfrak{g}}, \mathfrak{r}\cap\tilde{\mathfrak{n}}_+] \subset \mathfrak{r}\cap\tilde{\mathfrak{n}}_+$. Similarly for $\mathfrak{r}\cap\tilde{\mathfrak{n}}_-$. □

Remark 1.3.4 Note that the formula (b) in the proof of the theorem implies that the natural homomorphism $\mathfrak{h} \to \tilde{\mathfrak{g}}$ is an injection. This justifies out notation.

1.4 The Lie algebra $\mathfrak{g}(A)$

Definition 1.4.1 We define the *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ to be the quotient $\tilde{\mathfrak{g}}(A)/\mathfrak{r}$ where $\mathfrak{r}$ is the ideal from Theorem 1.3.3(v).

We refer to A as the *Cartan matrix* of $\mathfrak{g}$, and to n as the *rank* of $\mathfrak{g}$.

In view of Remark 1.3.4, we have a natural embedding $\mathfrak{h} \to \mathfrak{g}(A)$. The image of this embedding is also denoted $\mathfrak{h}$ and is called a *Cartan subalgebra* of $\mathfrak{g}$.

We keep the same notation for the images of the elements $e_i, f_i, \mathfrak{h}$ in $\mathfrak{g}$. The elements e_i and f_i are called *Chevalley generators*.

We have the following root decomposition with respect to $\mathfrak{h}$:

$$\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}_\alpha,$$

with $\mathfrak{g}_0 = \mathfrak{h}$. The number

$$\mathrm{mult}\,\alpha := \dim \mathfrak{g}_\alpha$$

is called the *multiplicity* of α. The element $\alpha \in Q$ is called a *root* if $\alpha \neq 0$ and $\mathrm{mult}\,\alpha \neq 0$. A root $\alpha > 0$ is called *positive*, a root $\alpha < 0$ is called *negative*. Every root is either positive or negative. We denote by $\Delta, \Delta_+, \Delta_-$ the sets of the roots, positive roots, and negative roots, respectively.

The subalgebra of $\mathfrak{g}$ generated by the e_i's (resp. f_i's) is denoted by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$). From Theorem 1.3.3, we have

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

It follows that $\mathfrak{g}_\alpha \subset \mathfrak{n}_+$ if $\alpha > 0$ and $\mathfrak{g}_\alpha \subset \mathfrak{n}_-$ if $\alpha < 0$. So for $\alpha > 0$, $\mathfrak{g}_\alpha$ is a span of the elements of the form $[\dots[[e_{i_1}, e_{i_2}], e_{i_3}]\dots e_{i_s}]$ such that $\alpha_{i_1} + \dots + \alpha_{i_s} = \alpha$. Similarly for $\alpha < 0$. It follows that

$$\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i, \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i, \quad \mathfrak{g}_{s\alpha_i} = 0 \qquad (s \neq \pm 1).$$

Since every root is either positive or negative, we deduce

Lemma 1.4.2 *If $\beta \in \Delta_+ \setminus \{\alpha_i\}$, then $(\beta + \mathbb{Z}\alpha_i) \cap \Delta \subset \Delta_+$.*

From Theorem 1.3.3(v), $\mathfrak{r}$ is $\tilde{\omega}$-invariant, so we get the *Chevalley involution*

$$\omega : \mathfrak{g} \to \mathfrak{g},\ e_i \mapsto -f_i,\ f_i \mapsto -e_i,\ h \mapsto -h \qquad (h \in \mathfrak{h}). \tag{1.17}$$

It is clear that $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, so $\operatorname{mult}\alpha = \operatorname{mult}(-\alpha)$ and $\Delta_- = -\Delta_+$.

Proposition 1.4.3 *Let A_1 be an $n \times n$ GCM, A_2 be an $m \times m$ GCM, and $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ be the direct sum matrix. Let $(\mathfrak{h}_i, \Pi_i, \Pi_i^\vee)$ be a realization of A_i. Then $(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \Pi_1 \sqcup \Pi_2, \Pi_1^\vee \sqcup \Pi_2^\vee)$ is a realization of A, and $\mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2) \cong \mathfrak{g}(A)$, the isomorphism sending $(h_1, h_2) \mapsto (h_1, h_2)$, $(e_i, 0) \mapsto e_i$, $(0, e_j) \mapsto e_{n+j}$, $(f_i, 0) \mapsto f_i$, $(0, f_j) \mapsto f_{n+j}$.*

Proof The first statement is obvious. For the second one, observe that generators $(h_1, h_2), (e_i, 0), (0, e_j), (f_i, 0), (0, f_j)$ of $\mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$ satisfy the defining relations of $\tilde{\mathfrak{g}}(A)$. So there exists a surjective homomorphism

$$\tilde{\pi} : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$$

which acts on the generators as the inverse of the isomorphism promised in the proposition. Moreover, since $\mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$ has no ideals with intersect $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ trivially, it follows that $\tilde{\pi}$ factors through the surjective homomorphism

$$\pi : \mathfrak{g}(A) \to \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2).$$

It suffices to show that π is injective. If not, its kernel must be an ideal whose intersection with $\mathfrak{h}$ is non-trivial. But then $\dim \pi(\mathfrak{h}) < \dim \mathfrak{h}$

giving a contradiction with the fact that $\pi(h) = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ has dimension $\dim \mathfrak{h}_1 + \dim \mathfrak{h}_2 = \dim \mathfrak{h}$. □

Denote by

$$\mathfrak{g}' = \mathfrak{g}'(A)$$

the subalgebra of $\mathfrak{g}(A)$ generated by all Chevalley generators e_i and f_j.

Proposition 1.4.4 *Let $\mathfrak{h}' \subset \mathfrak{h}$ be the span of $\alpha_1^\vee, \dots, \alpha_n^\vee$.*

(i) $\mathfrak{g}' = \mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+$.
(ii) $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

Proof (i) It is clear that $\mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+ \subset \mathfrak{g}'$. Conversely, if a Lie word in the Chevalley generators is not equal to zero and belongs to $\mathfrak{h}$, it follows from the relations that it belongs to $\mathfrak{h}'$.

(ii) It is clear that $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, and it follows from (i) that $\mathfrak{g}/\mathfrak{g}' \cong \mathfrak{h}/\mathfrak{h}'$ is abelian, so $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}'$. Conversely, $\alpha_i^\vee = [e_i, f_i]$, $e_i = [\frac{1}{2}\alpha_i^\vee, e_i]$, and $f_i = [f_i, \frac{1}{2}\alpha_i^\vee]$, so $\mathfrak{g}' \subset [\mathfrak{g}, \mathfrak{g}]$. □

Let $s = (s_1, \dots, s_n) \in \mathbb{Z}^n$. The *s-grading*

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j(s)$$

of $\mathfrak{g}$ is obtained by setting

$$\mathfrak{g}_j(s) = \bigoplus \mathfrak{g}_\alpha$$

where the sum is over all $\alpha = \sum_i k_i \alpha_i \in Q$ such that $\sum_i s_i k_i = j$. Note that

$$\deg e_i = -\deg f_i = -s_i, \quad \deg \mathfrak{h} = 0.$$

The case $s = (1, \dots, 1)$ gives the *principal grading* of $\mathfrak{g}$.

Lemma 1.4.5 *If an element a of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) commutes with all f_i (resp. all e_i), then $a = 0$.*

Proof Note that in the principal grading $\mathfrak{g}_{-1} = \mathrm{span}(f_1, \dots, f_n)$ and $\mathfrak{g}_1 = \mathrm{span}(e_1, \dots, e_n)$. So $[a, \mathfrak{g}_{-1}] = 0$. Then

$$\sum_{i,j \geq 0} (\mathrm{ad}\, \mathfrak{g}_1)^i (\mathrm{ad}\, \mathfrak{h})^j a$$

is an ideal of $\mathfrak{g}$ contained in $\mathfrak{n}_+$. This ideal must be zero, whence $a = 0$. □

Proposition 1.4.6 *The center of $\mathfrak{g}$ and $\mathfrak{g}'$ is*

$$\mathfrak{c} = \{h \in \mathfrak{h} \mid \langle \alpha_i, h \rangle = 0 \text{ for all } i = 1, \dots, n\}. \tag{1.18}$$

Moreover, $\dim \mathfrak{c} = n - \ell$.

Proof Let $c \in \mathfrak{g}$ be central and $c = \sum_i c_i$ be decomposition with respect to the principal grading. Then $[c, \mathfrak{g}_{-1}] = 0$ implies $[c_i, \mathfrak{g}_{-1}] = 0$, whence $c_i = 0$ for $i > 0$ and similarly $c_i = 0$ for $i < 0$. So $c \in \mathfrak{h}$, and then $0 = [c, e_i] = \langle \alpha_i, c \rangle e_i$ implies $c \in \mathfrak{c}$. Converse is clear. Finally, $\mathfrak{c} \subset \mathfrak{h}'$, since otherwise $\dim \mathfrak{c} > n - \ell$. □

Lemma 1.4.7 *Let I_1, I_2 be disjoint subsets of $\{1, \dots, n\}$ such that $a_{ij} = 0 = a_{ji}$ for all $i \in I_1, j \in I_2$. Let $\beta_s = \sum_{i \in I_s} k_i^{(s)} \alpha_i$ $(s = 1, 2)$. If $\alpha = \beta_1 + \beta_2$ is a root of $\mathfrak{g}$, then either β_1 or β_2 is zero.*

Proof Let $i \in I_1, j \in I_2$. Then $[\alpha_i^\vee, e_j] = 0, [\alpha_j^\vee, e_i] = 0, [e_i, f_j] = 0, [e_j, f_i] = 0$. Using Leibnitz formula and Lemma 1.4.5, we conclude that $[e_i, e_j] = [f_i, f_j] = 0$. Denote by $\mathfrak{g}^{(s)}$ be the subalgebra generated by all e_i, f_i for $i \in I_s$. We have shown that $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ commute. Now, since $\mathfrak{g}_\alpha$ is contained in the subalgebra generated by $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ it follows that it is contained in one of them. □

Proposition 1.4.8

(i) *$\mathfrak{g}$ is a simple Lie algebra if and only if* $\det A \neq 0$ *and for each pair of indices i, j the following condition holds:*

$$\text{there are indices } i_1, \dots, i_s \text{ such that } a_{ii_1} a_{i_1 i_2} \dots a_{i_s j} \neq 0. \tag{1.19}$$

(ii) *If the condition (1.19) holds then every ideal of $\mathfrak{g}$ either contains $\mathfrak{g}'$ or is contained in the center.*

Proof (i) If $\det A = 0$, then the center of $\mathfrak{g}$ is non-trivial by Proposition 1.4.6. If (1.19) is violated, then we can split $\{1, \dots, n\}$ into two non-trivial sunsets I_1 and I_2 such that $a_{ij} = a_{ji} = 0$ whenever $i \in I_1, j \in I_2$. Then $\mathfrak{g}$ is a direct sum of two ideals by Proposition 1.4.3. Conversely, let $\det A \neq 0$ and (1.19) hold. If $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{i}$ contains a non-zero element $h \in \mathfrak{h}$. By Proposition 1.4.6, $\mathfrak{c} = 0$, and hence $[h, e_j] = ae_j \neq 0$ for some j. Hence $e_j \in \mathfrak{i}$, and $\alpha_j^\vee = [e_j, f_j] \in \mathfrak{i}$. Now from (1.19) it follows that $e_j, f_j, \alpha_j^\vee \in \mathfrak{i}$ for all $\mathfrak{i}$. Since $\det A \neq 0$, $\mathfrak{h}$ is a span of the $\alpha_j^\vee$'s, and $\mathfrak{i} = \mathfrak{g}$.

(ii) is proved similarly—exercise. □

We finish with some terminology concerning duality. Note that A^t is also GCM, and $(\mathfrak{h}^*, \Pi^\vee, \Pi)$ is its realization. The algebras $\mathfrak{g}(A)$ and $\mathfrak{g}(A^t)$ are called *dual* to each other. Then the *dual root lattice*

$$Q^\vee := \sum_{i=1}^n \mathbb{Z}\alpha_i^\vee$$

corresponding to $\mathfrak{g}(A)$ is the root lattice corresponding to $\mathfrak{g}(A^t)$. Also, denote by

$$\Delta^\vee \subset Q^\vee$$

the root system $\Delta(A^t)$ and refer to it as the *dual root system of* $\mathfrak{g}$.

1.5 Examples

The following clumsy but easy result will be useful for dealing with examples:

Proposition 1.5.1 *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ be a finite dimensional abelian subalgebra of $\mathfrak{g}$ with $\dim \mathfrak{h} = 2n - \ell$. Suppose $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a linearly independent system of $\mathfrak{h}^*$ and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ a linearly independent system of $\mathfrak{h}$ satisfying $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. Suppose also that $e_1, \dots, e_n, f_1, \dots, f_n$ are elements of $\mathfrak{g}$ satisfying relations (1.12)-(1.15). Suppose $e_1, \dots, e_n, f_1, \dots, f_n$ and $\mathfrak{h}$ generate $\mathfrak{g}$ and that $\mathfrak{g}$ has no non-zero ideals $\mathfrak{i}$ with $\mathfrak{i} \cap \mathfrak{h} = 0$. Then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}(A)$.*

Proof There is surjective homomorphism $\theta : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}$. The restriction of θ to $\mathfrak{h} \subset \tilde{\mathfrak{g}}(A)$ is an isomorphism onto $\mathfrak{h} \subset \mathfrak{g}$, cf. Remark 1.3.4. So $\ker \theta \cap \mathfrak{h} = 0$. It follows that $\ker \theta \subset \mathfrak{r}$. In fact, $\ker \theta = \mathfrak{r}$, since $\mathfrak{g}$ has no nonzero ideal $\mathfrak{i}$ with $\mathfrak{i} \cap \mathfrak{h} = 0$. □

Example 1.5.2 Let

$$A = A_n := \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

We claim that $\mathfrak{g}(A) \cong \mathfrak{sl}_{n+1}$. We take $\mathfrak{h} \subset \mathfrak{sl}_{n+1}$ to be diagonal matrices of trace 0. Let $\varepsilon_i \in \mathfrak{h}^*$ be the ith coordinate function, i.e.

$$\varepsilon_i(\mathrm{diag}(a_1, \dots, a_n)) = a_i \qquad (1 \le i \le n).$$

Now take

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad \alpha_i^\vee = e_{ii} - e_{i+1,i+1} \qquad (1 \le i \le n),$$

and

$$e_i = e_{i,i+1}, \quad f_i = e_{i+1,i} \qquad\qquad (1 \le i \le n).$$

It is easy to see that all assumptions of Proposition 1.5.1 are satisfied. For example, to see that $\mathfrak{sl}_{n+1}$ does not contain nonzero ideals $\mathfrak{i}$ with $\mathfrak{i} \cap \mathfrak{h} = 0$, note that any such ideal would have to be a direct sum of the root subspaces, and it is easy to see that no such is an ideal. In fact, an argument along these lines shows that $\mathfrak{sl}_{n+1}$ is a simple Lie algebra, i.e. it has no non-trivial ideals. Note that the roots of $\mathfrak{sl}_{n+1}$ are precisely

$$\varepsilon_i - \varepsilon_j \qquad (1 \le i \ne j \le n+1),$$

with the corresponding root spaces $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}e_{ij}$.

Moreover, a similar argument shows that if $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra with Cartan matrix A, then $\mathfrak{g} \cong \mathfrak{g}(A)$.

Before doing the next example we explain several general constructions.

Let $\mathfrak{g}$ be an arbitrary Lie algebra. A 2-*cocycle* on $\mathfrak{g}$ is a bilinear map

$$\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

satisfying

$$\psi(y, x) = -\psi(x, y) \qquad (x, y \in \mathfrak{g}), \qquad (1.20)$$

$$\psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0 \qquad (x, y, z \in \mathfrak{g}). \qquad (1.21)$$

If ψ is a 2-cocyle and

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}c$$

for some formal element c, then $\tilde{\mathfrak{g}}$ is a Lie algebra with respect to

$$[x + \lambda c, y + \mu c] = [x, y] + \psi(x, y)c.$$

We refer to $\tilde{\mathfrak{g}}$ as the *central extension* of $\mathfrak{g}$ with respect to the cocycle ψ.

Let $D : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra derivation, i.e. D is a linear map and

$$D([x, y]) = [D(x), y] + [x, D(y)] \qquad (x, y \in \mathfrak{g}).$$

Let

$$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}d$$

for some formal element d. Then $\hat{\mathfrak{g}}$ is a Lie algebra with respect to

$$[x + \lambda d, y + \mu d] = [x, y] + \lambda d(y) - \mu d(x).$$

We refer to $\hat{g}$ as the Lie algebra obtained from $\mathfrak{g}$ by *adjoining* the derivation D. Sometimes we use the same letter d for both d and D.

A typical example of derivation comes as follows. Let $\mathfrak{g} = \oplus_j \in \mathbb{Z}\mathfrak{g}_j$ be a Lie algebra grading on $\mathfrak{g}$. Then the map $\mathfrak{g}$ sending x to jx for any $x \in \mathfrak{g}$ is a derivation.

Let

$$\mathcal{L} = \mathbb{C}[t, t^{-1}],$$

and for any Lie algebra $\mathfrak{g}$ define the corresponding *loop algebra*

$$\mathcal{L}(\mathfrak{g}) := \mathcal{L} \otimes \mathfrak{g}.$$

This is an infinite dimensional Lie algebra with bracket

$$[P \otimes x, Q \otimes y] = PQ \otimes [x, y] \qquad (P, Q \in \mathcal{L},\ x, y \in \mathfrak{g}).$$

If $(\cdot|\cdot)$ is a bilinear form on $\mathfrak{g}$, it can be extended to a $\mathcal{L}$-valued bilinear form

$$(\cdot|\cdot)_t : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathcal{L}$$

by setting

$$(P \otimes x | Q \otimes y)_t = PQ(x|y).$$

We define the *residue* function

$$\text{Res }: \mathcal{L} \to \mathbb{C},\ \sum_{i \in \mathbb{Z}} c_i t^i \mapsto c_{-1}.$$

Lemma 1.5.3 *Let $(\cdot|\cdot)$ be a symmetric invariant bilinear form on $\mathfrak{g}$. The function $\psi : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathbb{C}$ defined by*

$$\psi(a, b) = \text{Res}\,(\frac{da}{dt}|b)_t$$

is a 2-cocycle on $\mathcal{L}(\mathfrak{g})$. Moreover, $\psi(t^i \otimes x, t^j \otimes y) = i\delta_{i,-j}(x|y)$.

Proof Note that

$$\begin{aligned}\psi(t^i \otimes x, t^j \otimes y) &= \operatorname{Res}\,(it^{i-1} \otimes x | t^j \otimes y)_t \\ &= \operatorname{Res}\, it^{i+j-1}(x|y) \\ &= \begin{cases} i(x|y) & \text{if } i+j=0 \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

from which (1.20) follows. Moreover, we have

$$\begin{aligned}\psi([t^i \otimes x, t^j \otimes y], t^k \otimes z) &= \psi(t^{i+j}[x,y], t^k \otimes z) \\ &= \begin{cases} (i+j)([x,y]|z) & \text{if } i+j+k=0 \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Now, if $i+j+k \neq 0$, (1.21) is clear. If $i+j+k=0$, the required sum is

$$\begin{aligned}& -k([x,y]|z) - i([y,z]|x) - j([z,x]|y) \\ = \;& -k([x,y]|z) - i([x,y]|z) - j([x,y]|z) = 0\end{aligned}$$

since the form is symmetric and invariant. □

If $\mathfrak{g}$ is a simple finite dimensional Lie algebra it possesses unique up to a scalar non-degenerate symmetric invariant form $(\cdot|\cdot)$, so Lemma 1.5.3 allows us to define a 2-cocycle ψ on $\mathcal{L}(\mathfrak{g})$, and the previous discussion then allows us to consider the corresponding central extension

$$\bar{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c.$$

Moreover, $\bar{\mathcal{L}}(\mathfrak{g})$ is graded with $\deg t^j \otimes x = j,\ \deg c = 0$. We then have the corresponding derivation

$$d : \bar{\mathcal{L}}(\mathfrak{g}) \to \bar{\mathcal{L}}(\mathfrak{g}),\ t^j \otimes x \mapsto jt^j \otimes x,\ c \mapsto 0.$$

Finally, by adjoining d to $\bar{\mathcal{L}}(\mathfrak{g})$ we get the Lie algebra

$$\hat{\mathcal{L}}(\mathfrak{g}) := \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with operation

$$\begin{aligned}&[t^m \otimes x + \lambda c + \mu d, t^n \otimes y + \lambda_1 c + \mu_2 d] \\ &= (t^{m+n} \otimes [x,y] + \mu n t^n \otimes y - \mu_1 m t^m \otimes x) + m\delta_{m,-n}(x|y)c.\end{aligned}$$

Example 1.5.4 Let $A = A_1^{(1)} := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We claim that

$$\mathfrak{g}(A) \cong \hat{\mathcal{L}}(\mathfrak{s}l_2),$$

sometimes also denoted $\widehat{\mathfrak{sl}_2}$. First of all recall that the non-degenerate symmetric invariant form on $\mathfrak{sl}_2$ is just the trace form

$$(x|y) = \operatorname{tr}(xy) \qquad (x, y \in \mathfrak{sl}_2).$$

Then

$$(e, f) = 1,\ (h, h) = 2,\ (e, e) = (e, h) = (f, h) = (f, f) = 0.$$

Now set

$$\mathfrak{h} = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$$

and note that $\dim \mathfrak{h} = 2n - \ell$. Next define

$$\alpha_0^\vee = c - 1 \otimes h, \quad \alpha_1^\vee = 1 \otimes h$$

and $\alpha_0, \alpha_1 \in \mathfrak{h}^*$ via

$$\langle \alpha_i, \alpha_i^\vee \rangle = 2, \quad \langle \alpha_i, \alpha_j^\vee \rangle = -2 \qquad (0 \le i \ne j \le 1)$$

and

$$\langle \alpha_0, c \rangle = 0,\ \langle \alpha_0, d \rangle = 1,\ \langle \alpha_1, c \rangle = 0,\ \langle \alpha_1, d \rangle = 0.$$

It is clear that we have defined a realization of A. Next set

$$e_0 = t \otimes f,\ e_1 = 1 \otimes e,\ f_0 = t^{-1} \otimes e,\ f_1 = 1 \otimes f.$$

It is now easy to check the remaining conditions of Proposition 1.5.1. Indeed,

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee,\ [h, e_i] = \alpha_i(h)e_i,\ [h, f_i] = -\alpha_i(h)f_i \qquad (h \in \mathfrak{h})$$

follow from definitions. Next, $\widehat{\mathfrak{sl}_2}$ is generated by $\mathfrak{h}, e_0, e_1, f_0, f_1$: if $\mathfrak{m}$ is the subalgebra generated by them, then clearly $1 \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$. Set $\mathfrak{i} := \{x \in \mathfrak{sl}_2 \mid t \otimes x \in \mathfrak{m}$. We have $f \in \mathfrak{i}$, so $\mathfrak{i} \ne 0$. Also, if $x \in \mathfrak{i}, y \in \mathfrak{sl}_2$, then $[x, y] \in \mathfrak{i}$, thus $\mathfrak{i}$ is an ideal of $\mathfrak{sl}_2$, whence $\mathfrak{i} = \mathfrak{sl}_2$, and $t \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$. We may now use the relation

$$[t \otimes x, t^{k-1} \otimes y] = t^k \otimes [x, y] \qquad (k > 0)$$

to deduce by induction on k that $t^k \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$ for all $k > 0$. Analogously $t^k \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$ for all $k < 0$.

It remains to show that $\widehat{\mathfrak{sl}_2}$ has no non-zero ideals $\mathfrak{i}$ having trivial intersection with $\mathfrak{h}$. For this we study root space decomposition of $\widehat{\mathfrak{sl}_2}$. Let $\delta \in \mathfrak{h}^*$ be defined from

$$\delta(\alpha_1^\vee) = \delta(\alpha_2^\vee) = 0, \quad \delta(d) = 1.$$

We claim that the roots are precisely

$$\{\pm\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\}.$$

Indeed,

$$\mathfrak{g}_{\alpha_1+k\delta} = \mathbb{C}(t^k \otimes e),\ \mathfrak{g}_{-\alpha_1+k\delta} = \mathbb{C}(t^k \otimes f), \qquad (k \in \mathbb{Z})$$

and

$$\mathfrak{g}_{k\delta} = \mathbb{C}(t^k \otimes h) \qquad (k \in \mathbb{Z} \setminus \{0\}).$$

Since $\delta = \alpha_1 + \alpha_2$, positive roots are of the form $\{(k+1)\alpha_1 + k\alpha_1, k\alpha_1 + (k+1)\alpha_2, (k+1)\alpha_1 + (k+1)\alpha_2\}$ for $k \in \mathbb{Z}_{\geq 0}$.

Let $\mathfrak{i}$ be a non-zero ideal of $\widehat{\mathfrak{s}l_2}$ which has trivial intersection with $\mathfrak{h}$. It follows from Lemma 1.3.2 that some $t^i \otimes x \in \mathfrak{i}$ where $x = e, f$ or h. Take y to be f, e or h, respectively. Then $(x|y) \neq 0$, and

$$[t^i \otimes x, t^{-i} \otimes y] = [x, y] + i(x|y)c \in \mathfrak{i} \cap \mathfrak{h},$$

and hence

$$[x, y] + i(x|y)c = 0.$$

since $[x, y]$ is a multiple of $1 \otimes h$, we must have $i = 0$, whence $[x, y] = 0$. But since $i = 0$ we cannot have $x = h$, and then $[x, y] = 0$ is a contradiction.

In conclusion we introduce the element $\Lambda_0 \in \mathfrak{h}^*$ which is defined from

$$\Lambda_0 : \alpha_0^\vee \mapsto 1,\ \alpha_1^\vee \mapsto 0,\ d \mapsto 0.$$

Then $\{\alpha_0, \alpha_1, \Lambda_0\}$ and $\{\alpha_1, \delta, \Lambda_0\}$ are bases of $\mathfrak{h}^*$.

2
Invariant bilinear form and generalized Casimir operator

2.1 Symmetrizable GCMs

A GCM $A = (a_{ij})$ is called *symmetrizable* if there exists a non-singular diagonal matrix $D = \mathrm{diag}(\varepsilon_1, \dots, \varepsilon_n)$ and a symmetric matrix B such that

$$A = DB. \tag{2.1}$$

If A is symmetrizable, we also call $\mathfrak{g} = \mathfrak{g}(A)$ symmetrizable.

Lemma 2.1.1 *Let A be a GCM. Then A is symmetrizable if and only if*

$$a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} \dots a_{i_1 i_k}$$

for all $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$.

Proof If A is symmetrisable then $a_{ij} = \varepsilon_i b_{ij}$, hence

$$\begin{aligned} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} &= d_{i_1} \dots d_{i_k} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1}, \\ a_{i_2 i_1} a_{i_3 i_2} \dots a_{i_1 i_k} &= d_{i_1} \dots d_{i_k} b_{i_2 i_1} b_{i_3 i_2} \dots b_{i_1 i_k}, \end{aligned}$$

and these are equal since B is symmetric.

For the converse, we may assume that A is indecomposable. Thus for each $i \in \{1, \dots, n\}$ there exists a sequence $1 = j_1, \dots, j_t = i$ with

$$a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{t-1} j_t} \neq 0.$$

We choose a number $\varepsilon_1 \neq 0$ in $\mathbb{R}$ and define

$$\varepsilon_i = \frac{a_{j_t j_{t-1}} \dots a_{j_2 j_1}}{a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{t-1} j_t}} \varepsilon_1. \tag{2.2}$$

To see that this definition depends only on i, not on the sequence chosen from 1 to i, let $1 = k_1, \ldots, k_u = i$ be another such sequence. Then

$$\frac{a_{j_t j_{t-1}} \cdots a_{j_2 j_1}}{a_{j_1 j_2} a_{j_2 j_3} \cdots a_{j_{t-1} j_t}} = \frac{a_{k_u k_{u-1}} \cdots a_{k_2 k_1}}{a_{k_1 k_2} a_{k_2 k_3} \cdots a_{k_{u-1} k_u}},$$

since it is equivalent to

$$a_{1k_2} a_{k_2 k_3} \cdots a_{k_{t-1} i} a_{i j_{t-1}} \cdots a_{j_2 1} = a_{k_2 1} a_{k_3 k_2} \cdots a_{i k_{u-1}} a_{j_{t-1} j_t} \cdots a_{1 j_2},$$

which is one of the given conditions on the matrix A. Thus $\varepsilon_i \in \mathbb{R}$ is well defined and $\varepsilon_i \neq 0$.

Let $b_{ij} = a_{ij}/\varepsilon_i$. It remains to show that $b_{ij} = b_{ji}$ or $a_{ij}/\varepsilon_i = a_{ji}/\varepsilon_j$. If $a_{ij} = 0$ this is clear since then $a_{ji} = 0$. If $a_{ij} \neq 0$, let $1 = j_1, \ldots, j_t = i$ be a sequence from 1 to i of the type described above. Then $1 = j_1, \ldots, j_t, j$ is another such sequence from 1 to j. These sequences may be used to obtain ε_i and ε_j respectively, and we have

$$\varepsilon_j = \frac{a_{ji}}{a_{ij}} \varepsilon_i,$$

as required. □

Lemma 2.1.2 *Let A be a symmetrizable indecomposable GCM. Then A can be expressed in the form $A = DB$ where $D = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, B is symmetric, with $\varepsilon_1, \ldots, \varepsilon_n$ positive integers and $b_{ij} \in \mathbb{Q}$. Also D is determined by these conditions up to a scalar multiple.*

Proof We choose ε_1 to be any positive rational number. Then (2.2) shows that we can choose all ε_i to be positive rational numbers. Multiplying by a positive scalar we can make all ε_i positive integers. Also $b_{ij} = a_{ij}/\varepsilon_i \in \mathbb{Q}$. The proof of Lemma 2.1.1 also shows that D is unique up to a scalar multiple. □

Remark 2.1.3 If A is symmetrizable, in view of the above lemma, we may and **always will assume** that $\varepsilon_1, \ldots, \varepsilon_n$ are positive integers and B is a rational matrix.

2.2 Invariant bilinear form on $\mathfrak{g}$

Let A be a symmetrizable GCM as above. Fix a linear complement $\mathfrak{h}''$ to $\mathfrak{h}'$ in $\mathfrak{h}$:

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''.$$

Define a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ by the following two conditions:

$$(\alpha_i^\vee|h) = \langle\alpha_i, h\rangle\varepsilon_i \qquad (h \in \mathfrak{h}); \tag{2.3}$$

$$(h'|h'') = 0 \qquad (h', h'' \in \mathfrak{h}''). \tag{2.4}$$

Note that

$$(\alpha_i^\vee|\alpha_j^\vee) = b_{ij}\varepsilon_i\varepsilon_j. \tag{2.5}$$

Lemma 2.2.1

(i) *The kernel of the restriction* $(\cdot|\cdot)|_{\mathfrak{h}'}$ *is* $\mathfrak{c}$.

(ii) $(\cdot|\cdot)$ *is non-degenerate on* $\mathfrak{h}$.

Proof (i) is clear from (1.18).

(ii) It follows from (i) and Proposition 1.4.6 that the kernel of $(\cdot|\cdot)$ is contained in $\mathfrak{h}'$. Now if for all $h \in \mathfrak{h}$ we have

$$0 = (\sum_{i=1}^m c_i\alpha_i^\vee|h) = \langle\sum_{i=1}^m c_i\varepsilon_i\alpha_i, h\rangle,$$

whence $\sum_{i=1}^m c_i\varepsilon_i\alpha_i = 0$, and so all $c_i = 0$. □

Since $(\cdot|\cdot)$ is non-degenerate we have an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ such that

$$\langle\nu(h_1), h_2\rangle = (h_1|h_2) \qquad (h_1, h_2 \in \mathfrak{h}),$$

and the induced bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$. Note from (2.3) that

$$\nu(\alpha_i^\vee) = \varepsilon_i\alpha_i. \tag{2.6}$$

So, by (2.5),

$$(\alpha_i|\alpha_j) = b_{ij} = a_{ij}\varepsilon_i^{-1}. \tag{2.7}$$

Since all $\varepsilon_i > 0$ (Remark 2.1.3), it follows that

$$(\alpha_i|\alpha_i) > 0 \qquad (1 \le i \le n). \tag{2.8}$$

$$(\alpha_i|\alpha_j) \le 0 \qquad (i \ne j). \tag{2.9}$$

$$\alpha_i^\vee = \frac{2}{(\alpha_i|\alpha_i)}\nu^{-1}(\alpha_i). \tag{2.10}$$

So we get the usual expression for Cartan matrix:

$$a_{ij} = \frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}.$$

Example 2.2.2 (i) If A is as in Example 1.5.2, then the Gram matrix of $(\cdot|\cdot)$ in the basis $\alpha_1^\vee, \dots, \alpha_n^\vee$ is A itself. In fact, we may take $(\cdot|\cdot)$ to be the trace form restricted to $\mathfrak{h}$.

(ii) If A is as in Example 1.5.4, choose $\mathfrak{h}'' := \mathbb{C}d$. Then the Gram matrix of $(\cdot|\cdot)$ in the basis $\alpha_0^\vee, \alpha_1^\vee, d$ and the transported form in the basis $\alpha_0, \alpha_1, \Lambda_0$ is

$$\begin{pmatrix} 2 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

while the Gram matrix of the same forms in the bases $\alpha_1^\vee, c, d$ and $\alpha_1, \delta, \Lambda_0$ is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Theorem 2.2.3 *Let* $\mathfrak{g}$ *be symmetrizable. Fix decomposition (2.1) for* A*. Then there exists a non-degenerate symmetric bilinear form* $(\cdot|\cdot)$ *on* $\mathfrak{g}$ *such that*

(i) $(\cdot|\cdot)$ *is invariant, i.e. for all* $x, y, z \in \mathfrak{g}$ *we have*

$$([x, y]|z) = (x|[y, z]). \tag{2.11}$$

(ii) $(\cdot|\cdot)|_{\mathfrak{h}}$ *is as above.*
(iii) $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ *of* $\alpha + \beta \neq 0$.
(iv) $(\cdot|\cdot)|_{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}}$ *is non-degenerate for* $\alpha \neq 0$.
(v) $[x, y] = (x|y)\nu^{-1}(\alpha)$ *for* $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}, \alpha \in \Delta$.

Proof Set $\mathfrak{g}(N) := \oplus_{j=-N}^{N} \mathfrak{g}_j$, $N = 0, 1, \dots$, where $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is the principal grading. Start with the form $(\cdot|\cdot)$ on $\mathfrak{g}(0) = \mathfrak{h}$ defined above and extend it to $\mathfrak{g}(1)$ as follows:

$$(f_j|e_i) = (e_i|f_j) = \delta_{ij}\varepsilon_i, \ (\mathfrak{g}_0|\mathfrak{g}_{\pm 1}) = 0, \ (\mathfrak{g}_{\pm 1}|\mathfrak{g}_{\pm 1}) = 0.$$

An explicit check shows that the form $(\cdot|\cdot)$ on $\mathfrak{g}(1)$ satisfies (2.11) if both $[x, y]$ and $[y, z]$ belong to $\mathfrak{g}(1)$. Now we proceed by induction to extend the form to an arbitrary $\mathfrak{g}(N)$, $N \geq 2$. By induction we assume that the form has been extended to $\mathfrak{g}(N-1)$ so that it satisfies $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ for $|i|, |j| \leq N-1$ with $i + j \neq 0$, and (2.11) for all $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j, z \in \mathfrak{g}_k$ with $|i+j|, |j+k| \leq N-1$. We show that the form can be extended to $\mathfrak{g}(N)$ with analogous properties. First we require that $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ for all $|i|, |j| \leq N$ with $i + j \neq 0$. It remains to define $(x|y) = (y|x)$ for

$x \in \mathfrak{g}_N, y \in \mathfrak{g}_{-N}$. Note that y is a linear combination of Lie monomials in $f_1, \ldots, f_n$ of degree N. Since $N \geq 2$, each Lie monomial is a bracket of Lie monomials of degrees s and t with $s + t = N$. It follows that y can be written in the form

$$y = \sum_i [u_i, v_i] \qquad (u_i \in \mathfrak{g}_{-a_i},\ v_i \in \mathfrak{g}_{-b_i}) \tag{2.12}$$

where $a_i, b_i > 0$ and $a_i + b_i = N$. The expression of y in this form need not be unique. Now define

$$(x|y) := \sum_i ([x, u_i]|v_i). \tag{2.13}$$

The RHS is known since $[x, u_i]$ and v_i lie in $\mathfrak{g}(N-1)$. We must therefore show that RHS remains the same if a different expression (2.12) for y is chosen. In a similar way we can write x in the form

$$x = \sum_j [w_j, z_j] \qquad (w_j \in \mathfrak{g}_{c_j},\ z_j \in \mathfrak{g}_{d_j})$$

where $c_j, d_j > 0$ and $c_j + d_j = N$. We will show that

$$\sum_j (w_j|[z_j, y]) = \sum_i ([x, u_i]|v_i).$$

This will imply that the RHS of (2.13) is independent of the given expression for y. In fact it is sufficient to show that

$$(w_j|[z_j, [u_i, v_i]]) = ([[w_j, z_j], u_i]|v_i).$$

Now

$$\begin{aligned}
([[w_j, z_j], u_i]|v_i) &= ([[w_j, u_i], z_j]|v_i) + ([w_j, [z_j, u_i]]|v_i) \\
&= ([w_j, u_i]|[z_j, v_i]) - ([z_j, u_i]|[w_j, v_i]) \\
&= ([w_j, u_i]|[z_j, v_i]) - ([w_j, v_i]|[z_j, u_i]) \\
&= (w_j|[u_i, [z_j, v_i]]) - (w_j|[v_i, [z_j, u_i]]) \\
&= (w_j|[z_j, [u_i, v_i]]).
\end{aligned}$$

We must now check (2.11) for all $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j, z \in \mathfrak{g}_k$ with $|i + j|, |j + k| \leq N$. We may assume that $i + j + k = 0$ and at least one of $|i|, |j|, |k|$ is N. We suppose first that just one of $|i|, |j|, |k|$ is N. Then the other two are non-zero. If $|i| = N$ then (2.11) holds by definition of the form on $\mathfrak{g}(N)$. Similarly for $|k| = N$. So suppose $|j| = N$. We may assume that y has form $y = [u, v]$ where $u \in \mathfrak{g}_a, v \in \mathfrak{g}_b$, $a + b = |N|$, and $0 < |a| < |j|, 0 < |b| < |j|$. Then

$$\begin{aligned}
([x, y]|z) &= ([x, [u, v]]|z) \\
&= ([[v, x], u]|z) + ([[x, u], v]|z) \\
&= ([v, x]|[u, z]) + ([x, u]|[v, z]) \\
&= ([x, v]|[z, u]) + ([x, u]|[v, z]) \\
&= (x|[v, [z, u]]) + (x|[u, [v, z]]) \\
&= (x|[[u, v], z]) \\
&= (x|[y, z]).
\end{aligned}$$

Now suppose two of $|i|, |j|, |k|$ are equal to N. Then i, j, k are $N, -N, 0$ in some order. Thus one of x, y, z lies in $\mathfrak{h}$. Suppose $x \in \mathfrak{h}$. We may again assume that $y = [u, v]$. Then

$$\begin{aligned}
([x, y]|z) &= ([x, [u, v]]|z) \\
&= ([[x, u], v]|z) - ([[x, v], u]|z) \\
&= ([x, u]|[v, z]) - ([x, v]|[u, z]) \quad \text{(by definition of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N)) \\
&= (x|[u, [v, z]]) - (x|[v, [u, z]]) \quad \text{(by invariance of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N-1)) \\
&= (x|[[u, v], z]) \\
&= (x|[y, z]).
\end{aligned}$$

If $z \in \mathfrak{h}$ the result follows by symmetry. Finally, let $y \in \mathfrak{h}$. Then we may assume that $z = [u, v]$ where $u \in \mathfrak{g}_a, v \in \mathfrak{g}_b$, $a + b = k$, and $0 < |a| < |k|, 0 < |b| < |k|$. Then

$$\begin{aligned}
(x|[y, z]) &= (x|[y, [u, v]]) \\
&= (x|[u, [y, v]]) + (x|[[y, u], v]) \\
&= ([x, u]|[y, v]) + ([x, [y, u]]|v) \quad \text{(by definition of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N)) \\
&= ([[x, u], y]|v) + ([x, [y, u]]|v) \quad \text{(by invariance of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N-1)) \\
&= ([[x, y], u]|v) \\
&= ([x, y]|[u, v]) \quad \text{(by definition of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N)) \\
&= ([x, y]|z).
\end{aligned}$$

By induction, we have defined a symmetric bilinear form on $\mathfrak{g}$ which satisfies (i) and (ii). Let $\mathfrak{i}$ be the radical of $(\cdot|\cdot)$. Then $\mathfrak{i}$ is an ideal in $\mathfrak{g}$. If $\mathfrak{i} \neq 0$ then $\mathfrak{i} \cap \mathfrak{h} \neq 0$, which contradicts Lemma 2.2.1(ii). Thus $(\cdot|\cdot)$ is non-degenerate.

The form also satisfies (iii), since for all $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$, using invariance, we have

$$0 = ([h,x]|y) + (x|[h,y]) = (\langle \alpha, h\rangle + \langle \beta, h\rangle)(x|y).$$

Now (iv) also follows from the non-degeneracy of the form.

Finally, let $\alpha \in \Delta$, $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta, h \in \mathfrak{h}$. Then

$$([x,y] - (x|y)\nu^{-1}(\alpha)|h) = (x|[y,h]) - (x|y)\langle \alpha, h\rangle = 0,$$

which implies (v). □

The form $(\cdot|\cdot)$ constructed in the theorem above is called the *standard invariant form* on $\mathfrak{g}$. It is uniquely determined by the conditions (i) and (ii) of the theorem (indeed, if $(\cdot|\cdot)_1$ is another such form then $(\cdot|\cdot) - (\cdot|\cdot)_1$ is too, but its radical is non-trivial ideal containing $\mathfrak{h}$, which is a contradiction).

Throughout $(\cdot|\cdot)$ denotes the standard invariant form on symmetrizable $\mathfrak{g}$.

Example 2.2.4 (i) The standard invariant form is just the trace form on $\mathfrak{sl}_{n+1}$ is the trace form.

(ii) The standard invariant form on $\widehat{\mathfrak{sl}}_2$ is given by

$$\begin{aligned} (t^m \otimes x|t^n \otimes y) &= \delta_{m,-n}\mathrm{tr}\,(xy), \\ (\mathbb{C}c + \mathbb{C}d|\mathcal{L}(\mathfrak{sl}_2)) &= 0, \\ (c|c) = (d|d) &= 0, \\ (c|d) &= 1. \end{aligned}$$

2.3 Generalized Casimir operator

Let $\mathfrak{g}$ be symmetrizable. By Theorem 2.2.3(iii),(iv), we can choose dual bases $\{e_\alpha^{(i)}\}$ and $\{e_{-\alpha}^{(i)}\}$ in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Then

$$(x|y) = \sum_s (x|e_{-\alpha}^{(s)})(y|e_\alpha^{(s)}) \qquad (x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha}). \tag{2.14}$$

Lemma 2.3.1 *If $\alpha, \beta \in \Delta$ and $z \in \mathfrak{g}_{\beta-\alpha}$, then in $\mathfrak{g} \otimes \mathfrak{g}$ we have*

$$\sum_s e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}] = \sum_t [e_{-\beta}^{(t)}, z] \otimes e_\beta^{(t)}. \tag{2.15}$$

Proof Define a bilinear form $(\cdot|\cdot)$ on $\mathfrak{g} \otimes \mathfrak{g}$ via

$$(x \otimes y | x_1 \otimes y_1) := (x|x_1)(y|y_1).$$

Taje $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\beta}$. It suffices to prove that pairing of both sides of (2.15) with $e \otimes f$ gives the same result. We have, using (2.14),

$$\begin{aligned}\sum_s (e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}] | e \otimes f) &= \sum_s (e_{-\alpha}^{(s)}|e)([z, e_\alpha^{(s)}]|f) \\ &= \sum_s (e_{-\alpha}^{(s)}|e)(e_\alpha^{(s)}|[f,z]) \\ &= (e|[f,z]).\end{aligned}$$

Similarly,

$$\sum_t ([e_{-\beta}^{(t)}, z] \otimes e_\beta^{(t)} | e \otimes f) = ([z,e]|f),$$

as required. □

Corollary 2.3.2 *In the notation of Lemma 2.3.1, we have*

$$\sum_s [e_{-\alpha}^{(s)}, [z, e_\alpha^{(s)}]] = -\sum_t [[z, e_{-\beta}^{(t)}], e_\beta^{(t)}] \qquad (\text{in } \mathfrak{g}), \qquad (2.16)$$

$$\sum_s e_{-\alpha}^{(s)} [z, e_\alpha^{(s)}] = -\sum_t [z, e_{-\beta}^{(t)}] e_\beta^{(t)} \qquad (\text{in } U(\mathfrak{g})). \qquad (2.17)$$

Definition 2.3.3 A $\mathfrak{g}$-module V is called *restricted* if for every $v \in V$ we have $\mathfrak{g}_\alpha v = 0$ for all but finitely many positive roots α.

Let $\rho \in \mathfrak{h}^*$ be any functional satisfying

$$\langle \rho, \alpha_i^\vee \rangle = 1 \qquad (1 \le i \le n).$$

Then, by (2.10),

$$(\rho|\alpha_i) = (\alpha_i|\alpha_i)/2 \qquad (1 \le i \le n). \qquad (2.18)$$

For a restricted $\mathfrak{g}$-module V we define a linear operator Ω_0 on V as follows:

$$\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}.$$

One can check that this definition is independent on choice of dual bases. Let $u_1, u_2, \dots$ and $u^1, u^2, \dots$ be dual bases of $\mathfrak{h}$. Note that

$$(\lambda|\mu) = \sum_i \langle \lambda, u^i \rangle \langle \mu, u_i \rangle \qquad (\lambda, \mu \in \mathfrak{h}^*). \qquad (2.19)$$

Indeed,

$$\begin{aligned}(\lambda|\mu) &= (\nu^{-1}(\lambda)|\nu^{-1}(\mu)) \\ &= \sum_i (\nu^{-1}(\lambda)|u^i)(\nu^{-1}(\mu)|u_i) \\ &= \sum_i \langle\lambda, u^i\rangle\langle\mu, u_i\rangle.\end{aligned}$$

Also,

$$[\sum_i u^i u_i, x] = x((\alpha|\alpha) + 2\nu^{-1}(\alpha)) \qquad (x \in \mathfrak{g}_\alpha). \tag{2.20}$$

Indeed,

$$\begin{aligned}[\sum_i u^i u_i, x] &= \sum_i \langle\alpha, u^i\rangle x u_i + \sum_i u^i \langle\alpha, u_i\rangle x \\ &= \sum_i \langle\alpha, u^i\rangle\langle\alpha, u_i\rangle x + x\left(\sum_i u^i\langle\alpha, u_i\rangle + u_i\langle\alpha, u^i\rangle\right).\end{aligned}$$

Define the generalized *Casimir operator* to be the following linear operator Ω on V:

$$\Omega := 2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0.$$

Example 2.3.4 (i) Let $\mathfrak{g} = \mathfrak{sl}_2$. Then we have

$$\Omega = h + h(1/2)h + 2fe = ef + fe + h(1/2)h,$$

i.e. $\Omega = \sum v^i v_i$ for a pair $\{v^i\}$ and $\{v_i\}$ of dual bases of $\mathfrak{sl}_2$. This is a general fact for a finite dimensional simple Lie algebra.

(ii) Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. We can take a pair of dual bases u^i and u_i of $\mathfrak{h}$ as follows

$$\{\alpha_1^\vee, c, d\} \quad \text{and} \quad \{(1/2)\alpha_1^\vee, d, c\},$$

and

$$2\nu^{-1}(\rho) = \alpha_1^\vee + 4d.$$

Finally,

$$\Omega_0 = \sum_{k=1}^{+\infty}(t^{-k}h)(t^k h) + 2\sum_{k=0}^{+\infty}(t^{-k}f)(t^k e) + 2\sum_{k=1}^{+\infty}(t^{-k}e)(t^k f).$$

For the purposes of he following theorem consider root space decomposition of $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \bigoplus_{\beta \in Q} U_\beta,$$

where

$$U_\beta = \{u \in U(\mathfrak{g}) | [h, u] = \langle \beta, h \rangle u \text{ for all } h \in \mathfrak{h}\}.$$

Set

$$U'_\beta = U_\beta \cap U(\mathfrak{g}'),$$

so that $U(\mathfrak{g}') = \bigoplus_{\beta \in Q} U'_\beta$.

Theorem 2.3.5 *Let* $\mathfrak{g}$ *be symmetrizable.*

(i) *If* V *be a restricted* $\mathfrak{g}'$*-module and* $u \in U'_\alpha$ *then*

$$[\Omega_0, u] = -u\big(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)\big). \tag{2.21}$$

(ii) *If* V *is a restricted* $\mathfrak{g}$*-module then* Ω *commutes with the action of* $\mathfrak{g}$ *on* V.

Proof Note that elements of $\mathfrak{h}$ commute with Ω since Ω is of weight 0. Now (ii) follows from (i) and (2.20). Next, note that if (i) holds for $u \in U'_\alpha$ and $u_1 \in U'_\beta$, then it also holds for $uu_1 \in U'_{\alpha+\beta}$:

$$\begin{aligned}
[\Omega_0, uu_1] &= [\Omega_0, u]u_1 + u[\Omega_0, u_1] \\
&= -u\big(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)\big)u_1 \\
&\quad -uu_1\big(2(\rho|\beta) + (\beta|\beta) + 2\nu^{-1}(\beta)\big) \\
&= -uu_1\big(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha) \\
&\quad +2(\alpha|\beta) + 2(\rho|\beta) + (\beta|\beta) + 2\nu^{-1}(\beta)\big) \\
&= -uu_1\big(2(\rho|\alpha+\beta) + (\alpha+\beta|\alpha+\beta) + 2\nu^{-1}(\alpha+\beta)\big).
\end{aligned}$$

Since the e_{α_i}'s and $e_{-\alpha_i}$'s generate $\mathfrak{g}'$, it suffices to check (2.21) for $u = e_{\alpha_i}$ and $e_{-\alpha_i}$. We explain the calculation for e_{α_i}, the case of $e_{-\alpha_i}$ being similar. We have

$$\begin{aligned}
[\Omega_0, e_{\alpha_i}] &= 2\sum_{\alpha \in \Delta_+} \sum_s ([e_{-\alpha}^{(s)}, e_{\alpha_i}]e_\alpha^{(s)} + e_{-\alpha}^{(s)}[e_\alpha^{(s)}, e_{\alpha_i}]) \\
&= 2[e_{-\alpha_i}, e_{\alpha_i}]e_{\alpha_i} + 2\sum_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} \sum_s ([e_{-\alpha}^{(s)}, e_{\alpha_i}]e_\alpha^{(s)} + e_{-\alpha}^{(s)}[e_\alpha^{(s)}, e_{\alpha_i}]).
\end{aligned}$$

Note using Theorem 2.2.3(v) that

$$2[e_{-\alpha_i}, e_{\alpha_i}]e_{\alpha_i} = -2\nu^{-1}(\alpha_i)e_{\alpha_i} = -2(\alpha_i|\alpha_i)e_{\alpha_i} - 2e_{\alpha_i}\nu^{-1}(\alpha_i),$$

which is the RHS of (2.21) for $u = e_{\alpha_i}$. So it remains to prove that

$$\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}} \sum_s ([e_{-\alpha}^{(s)}, e_{\alpha_i}]e_\alpha^{(s)} + e_{-\alpha}^{(s)}[e_\alpha^{(s)}, e_{\alpha_i}]) = 0. \tag{2.22}$$

Applying (2.17) to $z = e_{\alpha_i}$, we get

$$\begin{aligned} &\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}} \sum_s ([e_{-\alpha}^{(s)}, e_{\alpha_i}]e_\alpha^{(s)} + e_{-\alpha}^{(s)}[e_\alpha^{(s)}, e_{\alpha_i}]) \\ = &\sum_{\alpha\in\Delta_+\setminus\{\alpha_i\}} \sum_s ([e_{-\alpha}^{(s)}, e_{\alpha_i}]e_\alpha^{(s)} - \sum_{\alpha\in\Delta\setminus\{\alpha_i\}} \sum_t [e_{-\alpha-\alpha_i}^{(t)}, e_{\alpha_i}]e_{\alpha+\alpha_i}^{(t)}. \end{aligned}$$

If $\alpha + \alpha_i \notin \Delta$, the last term is interpreted as zero. If $\alpha - \alpha_i \notin \Delta$, then $[e_{-\alpha}^{(s)}, e_{\alpha_i}] = 0$. Thus we may assume $\alpha = \beta + \alpha_i$ in the first term with $\beta \in \Delta_+$ in view of Lemma 1.4.2, which makes that term equal to $\sum_{\beta\in\Delta\setminus\{\alpha_i\}} \sum_t [e_{-\beta-\alpha_i}^{(t)}, e_{\alpha_i}]e_{\beta+\alpha_i}^{(t)}$, which completes the proof of (2.22). □

Corollary 2.3.6 *If in the assumptions of Theorem 2.3.5, $v \in V$ is a high weight vector of weight Λ then*

$$\Omega(v) = (\Lambda + 2\rho|\Lambda)v.$$

If, additionally, v generates V, then

$$\Omega = (\Lambda + 2\rho|\Lambda)I_V.$$

Proof The second statement follows from the first and the theorem. The first statement is a consequence of the definition of Ω and (2.19). □

3

Integrable representations of $\mathfrak{g}$ and the Weyl group

3.1 Integrable modules

Let

$$\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}\alpha_i^\vee + \mathbb{C}f_i.$$

It is clear that $\mathfrak{g}_{(i)}$ is isomorphic to $\mathfrak{sl}_2$ with standard basis.

Lemma 3.1.1 (Serre Relations) *If $i \neq j$ then*

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0. \tag{3.1}$$

Proof We prove the second equality, the first then follows by application of ω. Let $v = f_j$, $\theta_{ij} = (\operatorname{ad} f_i)^{1-a_{ij}} f_j$. We consider $\mathfrak{g}$ as a $\mathfrak{g}_{(i)}$-module via adjoint action. We have $e_i v = 0$ and $\alpha_i^\vee v = -a_{ij} v$. So, by representation theory of $\mathfrak{sl}_2$,

$$e_i \theta_{ij} = (1 - a_{ij})(-a_{ij} - (1 - a_{ij}) + 1)(\operatorname{ad} f_i)^{-a_{ij}} f_j = 0 \qquad (i \neq j).$$

It is also clear from relations that $e_k \theta_{ij} = 0$ if $k \neq i, j$ or if $k = j$ and $a_{ij} \neq 0$. Finally, if $k = j$ and $a_{ij} = 0$, then

$$e_j \theta_{ij} = [e_j, [f_i, f_j]] = [f_i, \alpha_j^\vee] = a_{ji} f_i = 0.$$

It remains to apply Lemma 1.4.5. □

Let V be a $\mathfrak{g}$-module and $x \in \mathfrak{g}$. Then x is *locally nilpotent* on V if for every $v \in V$ there is N such that $x^N v = 0$.

Lemma 3.1.2 *Let $\mathfrak{g}$ be a Lie algebra, V be a $\mathfrak{g}$-module, and $x \in \mathfrak{g}$.*

(i) *If $y_1, y_2, \ldots$ generate $\mathfrak{g}$ and $(\operatorname{ad} x)^{N_i} y_i = 0$, $i = 1, 2, \ldots,$ then $\operatorname{ad} x$ is locally nilpotent on $\mathfrak{g}$.*

(ii) *If* $v_1, v_2, \ldots$ *generate* V *as* $\mathfrak{g}$*-module,* $\operatorname{ad} x$ *is locally nilpotent on* $\mathfrak{g}$*, and* $x^{N_i} v_i = 0,\ i = 1, 2, \ldots,$ *then* x *is locally nilpotent on* V*.*

Proof Since $\operatorname{ad} x$ is a derivation, we have

$$(\operatorname{ad} x)^k [y, z] = \sum_{i=0}^{k} \binom{k}{i} [(\operatorname{ad} x)^i y, (\operatorname{ad} x)^{k-i} z].$$

This yields (i) by induction on the length of commutators in the y_i's.

(ii) follows from the formula

$$x^k a = \sum_{i=0}^{k} \binom{k}{i} ((\operatorname{ad} x)^i a) x^{k-i}, \tag{3.2}$$

which holds in any associative algebra. □

Lemma 3.1.3 *Operators* $\operatorname{ad} e_i$ *and* $\operatorname{ad} f_i$ *are locally nilpotent on* $\mathfrak{g}$*.*

Proof Follows from the defining relations, Serre relations, and Lemma 3.1.2(i). □

A $\mathfrak{g}$-module V is called $\mathfrak{h}$-*diagonalizable* if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where the *weight space* V_λ is defined to be

$$V_\lambda = \{v \in V | hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$ we call λ a *weight* of V, and $\dim V_\lambda$ the *multiplicity of the weight* λ denoted $\operatorname{mult}_V \lambda$. $\mathfrak{h}'$-diagonalizable $\mathfrak{g}'$-modules are defined similarly.

A $\mathfrak{g}$ (resp. $\mathfrak{g}'$)-module V is called *integrable* if it is $\mathfrak{h}$ (resp. $\mathfrak{h}'$)-diagonalizable and all e_i, f_i act locally nilpotently on V. For example the adjoint $\mathfrak{g}$-module is integrable.

Proposition 3.1.4 *Let* V *be an integrable* $\mathfrak{g}$*-module. As a* $\mathfrak{g}_{(i)}$*-module,* V *decomposes into a direct sum of finite dimensional irreducible* $\mathfrak{h}$*-invariant submodules.*

Proof For $v \in V_\lambda$ we have

$$e_i f_i^k v = k(1 - k + \langle \lambda, \alpha_i^\vee \rangle) f_i^{k-1} v + f_i^k e_i v.$$

It follows that the subspace

$$U := \sum_{k,m \geq 0} \mathbb{C} f_i^k e_i^m v$$

is $(\mathfrak{g}_{(i)} + \mathfrak{h})$-invariant. Since e_i and f_i are locally nilpotent on V, $\dim U < \infty$. By Weyl's Complete Reducibility Theorem, U is a direct sum of irreducible $\mathfrak{h}$-invariant $\mathfrak{g}_{(i)}$-submodules (for $\mathfrak{h}$-invariance use the fact that $f_i^k e_i^m v$ and $f_i^{k'} e_i^{m'} v$ are of the same $\alpha_i^\vee$-weight if and only if they are of the same $\mathfrak{h}$-weight). It follows that each $v \in V$ lies in a direct sum of finite dimensional $\mathfrak{h}$-invariant irreducible $\mathfrak{g}_{(i)}$-modules, which implies the proposition. □

Proposition 3.1.5 *Let V be an integrable $\mathfrak{g}$-module, $\lambda \in \mathfrak{h}^*$ be a weight of V, and α_i a simple root of $\mathfrak{g}$. Denote by M the set of all $t \in \mathbb{Z}$ such that $\lambda + t\alpha_i$ is a weight of V, and let $m_t := \mathrm{mult}_V(\lambda + t\alpha_i)$. Then:*

- (i) *M is a closed interval $[-p, q]$ of integers, where both p and q are either non-negative integers or ∞; $p - q = \langle \lambda, \alpha_i^\vee \rangle$ when both p and q are finite; if $\mathrm{mult}_V \lambda < \infty$ then p and q are finite.*
- (ii) *The map $e_i : V_{\lambda + t\alpha_i} \to V_{\lambda+(t+1)\alpha_i}$ is an embedding for $t \in [-p, -\langle \lambda, \alpha_i^\vee \rangle/2)$; in particular, the function $t \mapsto m_t$ is increasing on this interval.*
- (iii) *The function $t \mapsto m_t$ is symmetric with respect to $t = -\langle \lambda, \alpha_i^\vee \rangle/2$.*
- (iv) *If λ and $\lambda + \alpha_i$ are weights then $e_i(V_\lambda) \neq 0$.*
- (v) *If $\lambda + \alpha_i$ (resp. $\lambda - \alpha_i$) is not a weight, then $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ (resp. $\langle \lambda, \alpha_i^\vee \rangle \leq 0$).*
- (vi) *$\lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ is also a weight of V and*

$$\mathrm{mult}_V(\lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i) = \mathrm{mult}_V \lambda.$$

Proof Set $U := \sum_{k \in \mathbb{Z}} V_{\lambda + k\alpha_i}$. This is a $(\mathfrak{g}_{(i)} + \mathfrak{h})$-module, which in view of Proposition 3.1.4 is a direct sum of finite dimensional $\mathfrak{h}$-invariant irreducible $\mathfrak{g}_{(i)}$-modules. Let $p := -\inf M$ and $q := \sup M$. Then $p, q \in \mathbb{Z}_+$ since $0 \in M$. Now everything follows from representation theory of $\mathfrak{sl}_2$ using the fact $\langle \lambda + t\alpha_i, \alpha_i^\vee \rangle = 0$ for $t = -\langle \lambda, \alpha_i^\vee \rangle/2$. □

3.2 Weyl group

For each $i = 1, \dots, n$ define the *fundamental reflection* r_i of $\mathfrak{h}^*$ by the formula

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \qquad (\lambda \in \mathfrak{h}^*).$$

It is clear that r_i is a reflection with respect to the hyperplane

$$T_i = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}.$$

The subgroup $W = W(A)$ of $GL(\mathfrak{h}^*)$ generated by all fundamental reflections is called the *Weyl group* of $\mathfrak{g}$. The action r_i on $\mathfrak{h}^*$ induces the dual fundamental reflection $r_i^\vee$ on $\mathfrak{h}$. Hence the Weyl groups of dual Kac-Moody algebras are contragredient linear groups which allows us to identify them. We will always do this and write r_i for $r_i^\vee$. A simple check shows that the dual fundamental reflection $r_i^\vee$ is given by

$$r_i^\vee(h) = h - \langle h, \alpha_i \rangle \alpha_i^\vee.$$

Proposition 3.2.1

(i) *Let V be an integrable $\mathfrak{g}$-module. Then* $\operatorname{mult}_V \lambda = \operatorname{mult}_V w(\lambda)$ *for any $\lambda \in \mathfrak{h}^*$ and $w \in W$. In particular, the set of weights of V is W-invariant.*

(ii) *The root system Δ is W-invariant and* $\operatorname{mult} \alpha = \operatorname{mult} w(\alpha)$ *for all $\alpha \in \Delta, w \in W$.*

Proof Follows from Proposition 3.1.5. □

Lemma 3.2.2 *If $\alpha \in \Delta_+$ and $r_i(\alpha) < 0$ then $\alpha = \alpha_i$. In particular, $\Delta_+ \setminus \{\alpha_i\}$ is invariant with respect to r_i.*

Proof Follows from Lemma 1.4.2. □

If a is a locally nilpotent operator on a vector space V, and b is another operator on V such that $(\operatorname{ad} a)^n b = 0$ for some N, then

$$(\exp a) b (\exp -a) = (\exp(\operatorname{ad} a))(b). \tag{3.3}$$

Indeed, using induction and (3.2), we get

$$(\operatorname{ad} a)^k(b) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} a^{k-j} b a^j,$$

and so

$$\begin{aligned}(\sum_{i\geq 0} \frac{a^i}{i!}) b (\sum_{j \geq 0} (-1)^j \frac{a^j}{j!}) &= \sum_{k\geq 0} \frac{1}{k!} \sum_{i+j=k} \frac{k!}{i!j!} (-1)^j (a^i b a^j) \\ &= \sum_{k \geq 0} \frac{1}{k!} (\operatorname{ad} a)^k (b).\end{aligned}$$

Lemma 3.2.3 *Let π be an integrable representation of $\mathfrak{g}$ in V. For $i = 1, \dots, n$ set*

$$r_i^\pi := (\exp \pi(f_i))(\exp \pi(-e_i))(\exp \pi(f_i)).$$

Then

(i) $r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$;
(ii) $r_i^{\mathrm{ad}} \in \mathrm{Aut}\,\mathfrak{g}$;
(iii) $r_i^{\mathrm{ad}}|_{\mathfrak{h}} = r_i$.

Proof Let $v \in V_\lambda$. Then

$$h(r_i^\pi(v)) = r_i^\pi(h(v)) = \langle \lambda, h\rangle r_i^\pi(v) \quad \text{if} \quad \langle \alpha_i, h\rangle = 0. \tag{3.4}$$

Next we prove that

$$\alpha_i^\vee(r_i^\pi(v)) = -\langle \lambda, \alpha_i^\vee\rangle r_i^\pi(v). \tag{3.5}$$

This follows from

$$(r_i^\pi)^{-1}\pi(\alpha_i^\vee) r_i^\pi = \pi(-\alpha_i^\vee), \tag{3.6}$$

and, in view of (3.3), it is enough to check (3.6) holds for the adjoint representation of $\mathfrak{sl}_2$. Applying (3.3) one more time, we see that it is enough to check (3.6) for the natural 2-dimensional representation of $\mathfrak{sl}_2$. But in that representation we have

$$\exp f_i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \exp(-e_i) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad r_i^\pi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which implies (3.6) easily.

Now, any $h \in \mathfrak{h}$ can be written in the form $h = h' + c\alpha_i^\vee$, where c is a constant and $\langle \alpha_i, h'\rangle = 0$. Then using (3.4) and (3.5), we have

$$h(r_i^\pi(v)) = (\langle \lambda, h'\rangle - \langle \lambda, c\alpha_i^\vee\rangle) r_i^\pi(v) = \langle \lambda, r_i(h)\rangle r_i^\pi(v) = \langle r_i(\lambda), h\rangle r_i^\pi(v),$$

which proves (i).

For (iii), take $h \in \mathfrak{h}$ and write it again in the form $h = h' + c\alpha_i^\vee$ as above. Then it is clear that $r_i^{\mathrm{ad}} h' = h'$, and we just have to prove that $r_i^{\mathrm{ad}}(\alpha_i^\vee) = -\alpha_i^\vee$. This can be done as above calculating with 2×2

matrices, or, if you prefer, here is another argument.

$$\begin{aligned}(\exp \operatorname{ad} f_i)(\alpha_i^\vee) &= \alpha_i^\vee + 2f_i;\\ (\exp \operatorname{ad}(-e_i))(\alpha_i^\vee + 2f_i) &= \alpha_i^\vee + 2e_i + 2f_i - 2\alpha_i^\vee - 2e_i\\ &= -\alpha_i^\vee + 2f_i;\\ (\exp \operatorname{ad} f_i)(-\alpha_i^\vee + 2f_i) &= -\alpha_i^\vee - 2f_i + 2f_i\\ &= -\alpha_i^\vee.\end{aligned}$$

(ii) follows from (3.3) applied to the adjoint representation:

$$\begin{aligned} r_i^{\mathrm{ad}}[x,y] &= (\exp \operatorname{ad} f_i)(\exp \operatorname{ad}(-e_i))(\exp \operatorname{ad}(f_i))(\operatorname{ad} x)(y)\\ &= (\exp \operatorname{ad} f_i)(\exp \operatorname{ad}(-e_i))(\exp \operatorname{ad}(f_i))(\operatorname{ad} x)\\ &\quad \times(\exp \operatorname{ad}(-f_i))(\exp \operatorname{ad} e_i)(\exp \operatorname{ad}(-f_i))\\ &\quad \times(\exp \operatorname{ad} f_i)(\exp \operatorname{ad}(-e_i))(\exp \operatorname{ad}(f_i))(y)\\ &= r_i^{\mathrm{ad}}(x)(r_i^{\mathrm{ad}}(y))\\ &= [r_i^{\mathrm{ad}}(x), r_i^{\mathrm{ad}}(y)]. \end{aligned}$$

□

Proposition 3.2.4 *The bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ is W-invariant.*

Proof Note that $|r_i(\alpha_i)|^2 = |-\alpha_i|^2 = |\alpha_i|^2$. Now let $\Lambda, \Phi \in \mathfrak{h}^*$ and write $\Lambda = c\alpha_i + \lambda, \Phi = d\alpha_i + \varphi$ where $(\lambda|\alpha_i) = (\varphi|\alpha_i) = 0$, and c, d are constants. Then $r_i(\Lambda) = \lambda - c\alpha_i,\ r_i(\Phi) = \varphi - d\alpha_i$, so

$$(r_i(\Lambda)|r_i(\Phi) = (\lambda - c\alpha_i|\varphi - d\alpha_i) = (\lambda, \varphi) + (c\alpha_i|d\alpha_i) = (\Lambda|\Phi).$$

□

3.3 Weyl group as a Coxeter group

Lemma 3.3.1 *If α_i is a simple root and $r_{i_1} \dots r_{i_t}(\alpha_i) < 0$ then there exists s such that $1 \le s \le t$ and*

$$r_{i_1} \dots r_{i_t} r_i = r_{i_1} \dots \widehat{r_{i_s}} \dots r_{i_t}.$$

Proof Set $\beta_k = r_{i_{k+1}} \dots r_{i_t}(\alpha_i)$ for $k < t$ and $\beta_t = \alpha_i$. Then $\beta_t > 0$ and $\beta_0 < 0$. Hence for some s we have $\beta_{s-1} < 0$ and $\beta_s > 0$. But $\beta_{s-1} = r_{i_s}\beta_s$, so by Lemma 3.2.2, $\beta_s = \alpha_{i_s}$, and we get

$$\alpha_{i_s} = w(\alpha_i), \quad \text{where} \quad w = r_{i_{s+1}} \dots r_{i_t}. \tag{3.7}$$

By Lemma 3.2.3, $w = \tilde{w}|_{\mathfrak{h}}$ for some $\tilde{w}$ from the subgroup of $\operatorname{Aut}\mathfrak{g}$ generated by the r_i^{ad}. Applying $\tilde{w}$ to both sides of the equation $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] = \mathbb{C}\alpha_i^\vee$, we see that $\mathbb{C}w(\alpha_i^\vee) = \mathbb{C}\alpha_{i_s}^\vee$. Since $\langle w(\alpha_i), w(\alpha_i^\vee)\rangle = \langle \alpha_i, \alpha_i^\vee\rangle = 2$, we now conclude that

$$w(\alpha_i^\vee) = \alpha_{i_s}^\vee. \tag{3.8}$$

It now follows that $r_{i_s} = wr_iw^{-1}$:

$$wr_iw^{-1}(\lambda) = w(w^{-1}(\lambda) - \langle w^{-1}(\lambda), \alpha_i^\vee\rangle\alpha_i) = \lambda - \langle \lambda, \alpha_{i_s}^\vee\rangle\alpha_{i_s} = r_{i_s}(\lambda).$$

It remains to multiply both sides of $r_{i_s} = wr_iw^{-1}$ by $r_{i_1} \dots r_{i_{s-1}}$ on the left and by $r_{i_{s+1}} \dots r_{i_t} r_i$ on the right. □

Decomposition $w = r_{i_1} \dots r_{i_s}$ is called *reduced* if s is minimal among all presentations of w as a product of simple reflections r_i. Then s is called the *length* of w and is denoted $\ell(w)$. Note that $\det r_i = -1$, so

$$\det w = (-1)^{\ell(w)} \qquad (w \in W). \tag{3.9}$$

Lemma 3.3.2 *Let $w = r_{i_1} \dots r_{i_t} \in W$ be a reduced decomposition and α_i be a simple root. Then*

(i) *$\ell(wr_i) < \ell(w)$ if and only if $w(\alpha_i) < 0$;*
(ii) (Exchange Condition) *If $\ell(wr_i) < \ell(w)$ then there exists s such that $1 \le s \le t$ and*

$$r_{i_s} r_{i_{s+1}} \dots r_{i_t} = r_{i_{s+1}} \dots r_{i_t} r_i$$

Proof By Lemma 3.3.1, $w(\alpha_i) < 0$ implies $\ell(wr_i) < \ell(w)$. Now, if $w(\alpha_i) > 0$, then $wr_i(\alpha_i) < 0$ and it follows that $\ell(w) = \ell(wr_ir_i) < \ell(wr_i)$, completing the proof of (i).

(ii) If $\ell(wr_i) < \ell(w)$ then (i) implies $w(\alpha_i) < 0$, and we deduce the required Exchange Condition from Lemma 3.3.1 by multiplying it with $r_{i_{s-1}} \dots r_{i_1}$ on the left and r_i on the right. □

Lemma 3.3.3 *$\ell(w)$ equals the number of roots $\alpha > 0$ such that $w(\alpha) < 0$.*

Proof Denote

$$n(w) := |\{\alpha \in \Delta_+ \mid w(\alpha) < 0\}.$$

It follows from Lemma 3.2.2 that $n(wr_i) = n(w) \pm 1$, whence $n(w) \le \ell(w)$.

We now apply induction on $\ell(w)$ to prove that $\ell(w) = n(w)$. If $\ell(w) = 0$ then $w = 1$ (by convention), and clearly $n(w) = 0$. Assume that

$\ell(w) = t > 0$, and $w = r_{i_1} \dots r_{i_{t-1}} r_{i_t}$. Denote $w' = r_{i_1} \dots r_{i_{t-1}}$. By induction, $n(w') = t - 1$. Let $\beta_1, \dots, \beta_{t-1}$ be the positive roots which are sent to negative roots by w'. By Lemma 3.3.2(i), $w'(\alpha_{i_t}) > 0$, whence $w(\alpha_{i_t}) < 0$. It follows from Lemma 3.2.2 that $r_{i_t}(\beta_1), \dots, r_{i_t}(\beta_{t-1}), \alpha_{i_t}$ are distinct positive roots which are mapped to negative roots by w, so $n(w) \geq \ell(w)$. □

Lemma 3.3.4 (Deletion Condition) *Let $w = r_{i_1} \dots r_{i_s}$. Suppose $\ell(w) < s$. Then there exist $1 \leq j < k \leq s$ such that*

$$w = r_{i_1} \dots \widehat{r_{i_j}} \dots \widehat{r_{i_k}} \dots r_{i_s}.$$

Proof Since $\ell(w) < s$ there exists $2 \leq k \leq s$ such that

$$\ell(r_{i_1} \dots r_{i_k}) < \ell(r_{i_1} \dots r_{i_{k-1}}) = k - 1$$

. Then by Lemmas 3.3.2(i) and 3.3.1,

$$r_{i_1} \dots r_{i_k} = r_{i_1} \dots \widehat{r_{i_j}} \dots r_{i_{k-1}}$$

for some $1 \leq j < k$. □

Now for $1 \leq i \neq j \leq n$ define

$$m_{ij} := \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0, \\ 3 & \text{if } a_{ij}a_{ji} = 1, \\ 4 & \text{if } a_{ij}a_{ji} = 2, \\ 5 & \text{if } a_{ij}a_{ji} = 3, \\ \infty & \text{if } a_{ij}a_{ji} \geq 4. \end{cases}$$

Lemma 3.3.5 *Let $1 \leq i \neq j \leq n$. Then the order of $(r_i r_j)$ is m_{ij}.*

Proof The subspace $\mathbb{R}\alpha_i + \mathbb{R}\alpha_j$ is invariant with respect to r_i and r_j, and we can make all calculations in this 2-dimensional space. The matrices of r_i and r_j in the basis α_i, α_j are $\begin{pmatrix} -1 & -a_{ij} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -a_{ji} & -1 \end{pmatrix}$, respectively. So the matrix of $r_i r_j$ is $\begin{pmatrix} -1 + a_{ij}a_{ji} & a_{ij} \\ -a_{ji} & -1 \end{pmatrix}$. The characteristic polynomial of this matrix is $\lambda^2 + (2 - a_{ij}a_{ji})\lambda + 1$, and now the result is an elementary calculation. □

Proposition 3.3.6 *W is generated by $r_1, \ldots, r_n$ subject only to the Coxeter relations*

$$r_i^2 = 1 \qquad (1 \le i \le n), \tag{3.10}$$
$$(r_i r_j)^{m_{ij}} = 1 \qquad (1 \le i \ne j \le n), \tag{3.11}$$

where w^∞ is interpreted as 1. So W is a Coxeter group.

Proof This is a general fact. All we need is Deletion Condition. We need to show that every relation

$$r_1 \ldots r_{i_s} = 1$$

in W is a consequence of (3.10) and (3.11). We have $\det r_i = -1$ for all i, so $s = 2q$. We apply induction on q. If $q = 1$ the relation looks like $s_{i_1} s_{i_2} = 1$. Hence $s_{i_2} = s_{i_1}^{-1} = s_{i_1}$. So our relation is $s_{i_1}^2 = 1$, which is one of (3.10).

For inductive step, rewrite the given relation as follows:

$$r_{i_1} \ldots r_{i_q} r_{i_{q+1}} = r_{i_{2q}} \ldots r_{i_{q+2}}. \tag{3.12}$$

Then $\ell(r_{i_1} \ldots r_{i_q} r_{i_{q+1}}) < q + 1$, so by the Deletion Condition,

$$r_{i_1} \ldots r_{i_q} r_{i_{q+1}} = r_{i_1} \ldots \widehat{r_{i_j}} \ldots \widehat{r_{i_k}} \ldots r_{i_{q+1}} \tag{3.13}$$

for some $1 \le j < k \le q + 1$. Now, unless $j = 1$ and $k = q + 1$, this is a consequence of a relation with fewer than $2q$ terms—for example, if $j > 1$, (3.13) is equivalent to

$$r_{i_2} \ldots r_{i_q} r_{i_{q+1}} = r_{i_2} \ldots \widehat{r_{i_j}} \ldots \widehat{r_{i_k}} \ldots r_{i_{q+1}}.$$

So, by induction, (3.13) can be deduced from the defining relations. The relation

$$r_{i_1} \ldots \widehat{r_{i_j}} \ldots \widehat{r_{i_k}} \ldots r_{i_{q+1}} = r_{i_{2q}} \ldots r_{i_{q+2}}$$

has $2q - 2$ terms, so is also a consequence of the defining relations. Therefore (3.12) is a consequence of the defining relations, unless $j = 1$ and $k = q + 1$.

In the exceptional case (3.13) is

$$r_{i_1} \ldots r_{i_q} r_{i_{q+1}} = r_{i_2} \ldots r_{i_q},$$

or

$$r_{i_1} \ldots r_{i_q} = r_{i_2} \ldots r_{i_{q+1}}. \tag{3.14}$$

Now we write (3.12) in the alternative form

$$r_{i_2} \dots r_{i_{2q}} r_{i_1} = 1. \tag{3.15}$$

In exactly the same way this relation will be a consequence of the defining relations unless

$$r_{i_2} \dots r_{i_{q+1}} = r_{i_3} \dots r_{i_{q+2}}. \tag{3.16}$$

If this relation is a consequence of the defining relations then (3.12) is also a consequence of the defining relations by the above argument, and we are done. Now, (3.12) is equivalent to

$$r_{i_3} r_{i_2} r_{i_3} \dots r_{i_q} r_{i_{q+1}} r_{i_{q+2}} r_{i_{q+1}} \dots r_{i_4} = 1, \tag{3.17}$$

and this will be a consequence of the defining relations unless

$$r_{i_3} r_{i_2} r_{i_3} \dots r_{i_q} = r_{i_2} r_{i_3} \dots r_{i_q} r_{i_{q+1}},$$

We may therefore assume that this is true. But we must also have (3.17). So $r_{i_1} = r_{i_3}$. Hence the given relation will be a consequence of the defining relations unless $r_{i_1} = r_{i_3}$. However, an equivalent forms of the given relation are also $r_{i_2} \dots r_{i_{2q}} r_{i_1} = 1$, $r_{i_3} \dots r_{i_{2q}} r_{i_1} r_{i_2} = 1$, etc. Thus this relation will be a consequence of the defining relations unless $r_{i_1} = r_{i_3} = \dots = r_{i_{2q-1}}$ and $r_{i_2} = r_{i_4} = \dots = r_{i_{2q}}$. Thus we may assume that the given relation has form $(r_{i_1} r_{i_2})^q = 1$. Then $m_{i_1 i_2}$ divides q, and the relation is a consequence of the Coxeter relation (3.11). □

3.4 Geometric properties of Weyl groups

Let $(\mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^\vee)$ be a realization of A over $\mathbb{R}$, so that

$$(\mathfrak{h}, \Pi, \Pi^\vee) = (\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{h}_{\mathbb{R}}, \Pi, \Pi^\vee).$$

Note that $\mathfrak{h}_{\mathbb{R}}$ is W-invariant since $\mathbb{Q}^\vee \subset \mathfrak{h}_{\mathbb{R}}$. The set

$$C = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha_i, h \rangle \geq 0 \text{ for } i = 1, \dots, n\}$$

is called the *fundamental chamber*, the sets of the form $w(C)$ are called *chambers*, and their union

$$X := \bigcup_{w \in W} w(C)$$

is called the *Tits cone*. There are corresponding dual objects $C^\vee, X^\vee$, etc. in $\mathfrak{h}^*_{\mathbb{R}}$.

Proposition 3.4.1

(i) *For $h \in C$, the group $W_h := \{w \in W \mid w(h) = h\}$ is generated by the fundamental reflections contained in it.*
(ii) *The fundamental chamber is the fundamental domain for the action of W on X, i.e. every W-orbit intersects C in exactly one point. In particular, W acts regularly on the set of chambers.*
(iii) $X = \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha, h\rangle < 0$ *for a finite number of* $\alpha \in \Delta_+\}$. *In particular X is a convex cone.*
(iv) $C = \{h \in \mathfrak{h}_{\mathbb{R}} \mid h - w(h) = \sum_i c_i \alpha_i^\vee,$ *where* $c_i \geq 0$, *for any* $w \in W\}$.
(v) *The following conditions are equivalent:*
   (a) $|W| < \infty$;
   (b) $X = \mathfrak{h}_{\mathbb{R}}$;
   (c) $|\Delta| < \infty$;
   (d) $|\Delta^\vee| < \infty$.
(vi) *If $h \in X$ then $|W_h| < \infty$ if and only if h is an interior point of X.*

Proof Take $w \in W$ and let $w = r_{i_1} \dots r_{i_s}$ be a reduced decomposition. Take $h \in C$ and assume that $h' = w(h) \in C$. We have $\langle \alpha_{i_s}, h\rangle \geq 0$, hence $\langle w(\alpha_{i_s}), w(h)\rangle = \langle w(\alpha_{i_s}), h'\rangle \geq 0$. It follows from Lemma 3.3.2(i) that $w(\alpha_{i_s}) < 0$, hence $\langle w(\alpha_{i_s}), h'\rangle \leq 0$, and $\langle w(\alpha_{i_s}), h'\rangle = 0$, whence $\langle \alpha_{i_s}, h\rangle = 0$. Hence $r_{i_s}(h) = h$. Now for the proof of (i) and (ii) it suffices to apply induction on $\ell(w)$.

(iii) Set $X' := \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha, h\rangle < 0$ for a finite number of $\alpha \in \Delta_+\}$. Let $h \in X'$ and $w \in W$. Then $\langle \alpha, w(h)\rangle = \langle w^{-1}\alpha, h\rangle$. Only finitely many positive α's are sent to negatives by w^{-1}, see Lemma 3.3.3. So X' is W-invariant, and clearly $C \subset X'$. Therefore $X \subset X'$. To prove the converese embedding, take $h \in X'$ and set $M_h := \{\alpha \in \Delta_+ \mid \langle \alpha, h\rangle < 0\}$. By definition M_h is finite. If $M_h \neq \varnothing$, then some simple root $\alpha_i \in M_h$. But then it follows from Lemma 3.2.2 that $|M_{r_i(h)}| < |M_h|$. Now induction on $|M_h|$ completes the proof of (iii).

(iv) $\supset$ is clear. The converse embedding is proved by induction on $s = \ell(w)$. For $s = 0$ the result is clear and for $s = 1$ it is equivalent to the definition of C. Let $s > 1$ and $w = r_{i_1} \dots r_{i_s}$. We have

$$h - w(h) = (h - r_{i_1} \dots r_{i_{s-1}}(h)) + r_{i_1} \dots r_{i_{s-1}}(h - r_{i_s}(h)).$$

It follows from (the dual version of) Lemma 3.3.2(i) that $r_{i_1} \dots r_{i_{s-1}}(\alpha_{i_s}^\vee) \in Q_+^\vee$, which implies that the second summand is in $Q_+^\vee$. The first summand is there too by inductive assumption.

(v) (a) $\Rightarrow$ (b). Let $h \in \mathfrak{h}_{\mathbb{R}}$, and choose an element h' from the (finite) orbit $W \cdot h$ for which $\mathrm{ht}\,(h'-h)$ is maximal. Then $h' \in C$, whence $h \in X$.

(b) $\Rightarrow$ (c) Take h in the interior of C. Then $\langle \alpha, -h\rangle < 0$ for all $\alpha \in \Delta_+$, and it remains to apply (iii).

(c) $\Rightarrow$ (a) It suffices to prove that the action of W on the roots is faithful. Assume that $w(\alpha) = \alpha$ for all $\alpha \in \Delta$, and $w = r_{i_1} \dots r_{i_s}$ be a reduced decomposition. But then $w(\alpha_{i_s}) < 0$ by Lemma 3.3.2(i).

(d) $\Leftrightarrow$ (a) is similar to (c) $\Leftrightarrow$ (a), but using dual root system.

(vi) In view of (ii) we may assume that $h \in C$. Then by (i), W_h is generated by the fundamental reflections with respect to the roots orthogonal to h. The action of W_h on $\mathfrak{h}$ induces the action of W_h on $\mathfrak{h}' := \mathfrak{h}_{\mathbb{R}}/\mathbb{R}h$. Moreover, this induced action allows us to identify W_h with a Weyl group W' acting naturally on $\mathfrak{h}'$. By (v), this group is finite if and only if its $X' = \mathfrak{h}'$ □

Example 3.4.2 (i) Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then r_i acts on $\varepsilon_1, \dots, \varepsilon_{n+1}$ by swapping ε_i and ε_{i+1}, from which it follows that $W \cong S_{n+1}$. Introduce $\Lambda_1, \dots, \Lambda_n \in \mathfrak{h}^*$ as the dual basis to $\alpha_1^\vee, \dots, \alpha_n^\vee$:

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij} \qquad (1 \le i, j \le n).$$

Then

$$C^\vee = \mathbb{R}_{\ge 0}\Lambda_1 \oplus \dots \oplus \mathbb{R}_{\ge 0}\Lambda_n.$$

and $X^\vee = \mathfrak{h}_{\mathbb{R}}^*$.

(ii) Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. Then $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\alpha_1 \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$ and the sum $(\mathbb{R}\alpha_1) \oplus (\mathbb{R}\delta \oplus \mathbb{R}\Lambda_0)$ is orthogonal. Moreover,

$$\begin{aligned} r_0 : \alpha_1 &\mapsto -\alpha_1 + 2\delta, \quad \delta \mapsto \delta, \quad \Lambda_0 \mapsto \alpha_1 - \delta + \Lambda_0; \\ r_1 : \alpha_1 &\mapsto -\alpha_1, \qquad\quad \delta \mapsto \delta, \qquad \Lambda_0 \mapsto \Lambda_0, \end{aligned}$$

whence

$$r_0 r_1(\lambda\alpha_1 + \mu\delta + \nu\Lambda_0) = (\lambda + \nu)\alpha_1 + (\mu - 2\lambda - \nu)\delta + \nu\Lambda_0. \qquad (3.18)$$

Consider the affine subspace

$$\mathfrak{h}_1^* = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, c\rangle = 1\} \subset \mathfrak{h}_{\mathbb{R}}^*,$$

invariant with respect to the action of W. So W acts on $\mathfrak{h}_1^*$ with affine transformations. Elements of $\mathfrak{h}_1^*$ are of the form

$$\lambda\alpha_1 + \mu\delta + \Lambda_0 \qquad (\lambda, \mu \in \mathbb{R}).$$

Moreover, it is clear that r_0 and r_1 act trivially on δ. So the action of

W on $\mathfrak{h}_1^*$ factors through to give an action of W on $\mathfrak{h}_1^*/\mathbb{R}\delta$ which can be identified with $\mathbb{R}\alpha_1$. We will denote the induced affine action of $w \in W$ on $\mathbb{R}\alpha_1$ via $\bar{w}$. An easy calculation gives:

$$\bar{r}_1 : \lambda\alpha_1 \mapsto -\lambda\alpha_1, \quad \bar{r}_0 : \lambda\alpha_1 \mapsto -\lambda\alpha_1 + \alpha_1,$$

whence

$$\bar{r}_0\bar{r}_1(\lambda\alpha_1) = \lambda\alpha_1 + \alpha_1$$

is a 'shift' by α_1. It follows that the image $\bar{W}$ of W is a semidirect product

$$\bar{W} = \mathbb{Z} \rtimes S_2.$$

In fact the map $w \mapsto \bar{w}$ is injective. This follows from the fact that every element of W can be written uniquely in the form $r_1^{\varepsilon}(r_0r_1)^k$ where $k \in \mathbb{Z}$ and $\varepsilon = 0$ or 1. Thus

$$W = \mathbb{Z} \rtimes S_2.$$

Next,

$$C = \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid 0 \le \lambda \le \frac{1}{2}\nu\}.$$

It follows from (3.18) that

$$\begin{aligned} (r_0r_1)^k C &= \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid \nu \ge 0,\ k\nu \le \lambda \le (k+\frac{1}{2})\nu\} \\ r_1(r_0r_1)^k C &= \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid \nu \ge 0,\ -(k+\frac{1}{2})\nu \le \lambda \le -k\nu\}, \end{aligned}$$

whence

$$X = \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid \nu \ge 0\}.$$

In terms of the affine action, C gets identified with the *fundamental alcove*

$$C_{\text{af}} = \{\lambda\alpha_1 \mid 0 \le \lambda \le \frac{1}{2}\},$$

which is the fundamental domain for the affine action of W on $\mathbb{R}\alpha_1$.

4

The Classification of Generalized Cartan Matrices

4.1 A trichotomy for indecomposable GCMs

Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. We write

$$v \geq 0 \qquad \text{if all } v_i \geq 0$$

and

$$v > 0 \qquad \text{if all } v_i > 0.$$

We consider $v \in \mathbb{R}^n$ as row or column as convenient.

Definition 4.1.1 A GCM A has *finite type* if the following three conditions hold:

(i) $\det A \neq 0$;
(ii) there exists $u > 0$ with $Au > 0$;
(iii) $Au \geq 0$ implies $u > 0$ or $u = 0$.

A GCM A has *affine type* if the following three conditions hold:

(i) corank $A = 1$ (i.e. rank $A = n - 1$);
(ii) there exists $u > 0$ with $Au = 0$;
(iii) $Au \geq 0$ implies $Au = 0$.

A GCM A has *indefinite type* if the following two conditions hold:

(i) there exists $u > 0$ with $Au < 0$;
(ii) $Au \geq 0$ and $u \geq 0$ imply $u = 0$.

Remark 4.1.2 What we really have in mind in this. Let $\gamma = u_1\alpha_1 + \cdots + u_n\alpha_n$, and $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$ be the corresponding column vector. Then Au is the column vector $(\langle \gamma, \alpha_1^\vee \rangle, \ldots, \langle \gamma, \alpha_n^\vee \rangle)$.

Example 4.1.3 Let a, b be positive integers, and $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$. Then A is of finite (resp. affine, resp. indefinite) type if and only if $ab \leq 3$ (resp. $ab = 4$, resp. $ab > 4$).

We will prove that an indecomposable GCM has exactly one of the three types above.

Lemma 4.1.4 *Let $v^i = (v_{i1}, \ldots, v_{in}) \in \mathbb{R}^n$ for $i = 1, \ldots, m$. Then there exist $x_1, \ldots, x_n \in \mathbb{R}$ with*

$$\sum_{j=1}^{n} v_{ij} x_j > 0 \qquad (i = 1, \ldots, m)$$

if and only if

$$\lambda_1 v^1 + \cdots + \lambda_m v^m = 0, \quad \lambda_1, \ldots, \lambda_m \geq 0$$

implies $\lambda_1 = \cdots = \lambda_m = 0$.

Proof Consider the usual scalar product $(x, y) = x_1 y_1 + \ldots x_n y_n$ for two vectors $x, y \in \mathbb{R}^n$. Suppose there exists a column vector $x = (x_1, \ldots, x_n)$ such that $(v^i, x) > 0$ for all i. Suppose $\lambda_1 v^1 + \cdots + \lambda_m v^m = 0$ with all $\lambda_i \geq 0$. Then

$$\lambda_1 (v^1, x) + \cdots + \lambda_m (v^m, x) = 0.$$

This implies $\lambda_i = 0$ for all i.

Conversely, suppose $\lambda_1 v^1 + \cdots + \lambda_m v^m = 0$, $\lambda_i \geq 0$ implies $\lambda_i = 0$ for all i. Let

$$S := \left\{ \sum_{i=1}^{m} \lambda_i v^i \mid \lambda_i \geq 0, \ \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Define $f : S \to \mathbb{R}$ by $f(y) = ||y|| := \sqrt{y_1^2 + \cdots + y_n^2}$. Then S is a compact subset of $\mathbb{R}^n$ and f is a continuous function. Thus $f(S)$ is a compact subset of $\mathbb{R}$. Hence there exists $x \in S$ with $||x|| \leq ||x'||$ for all $x' \in S$. Clearly $x \neq 0$ since $0 \notin S$ by assumption. We will show $(v_i, x) > 0$ for all i as required. In fact we will show more, namely, that $(y, x) > 0$ for all $y \in S$.

Now S is a convex subset of $\mathbb{R}^n$. So for $y \neq x$ we have $ty + (1-t)x \in S$ for all $0 \leq t \leq 1$. By the choice of x,

$$(ty + (1-t)x, ty + (1-t)x) \geq (x, x)$$

or

$$t(y-x, y-x) + 2(y-x, x) \geq 0.$$

As t can be made arbitrarily small, this implies $(y-x, x) \geq 0$ or $(y, x) \geq (x, x) > 0$. □

Proposition 4.1.5 *Let C be an $m \times n$ matrix over $\mathbb{R}$. Suppose $u \geq 0$ and $C^t u \geq 0$ imply $u = 0$. Then there exists $v > 0$ with $Cv < 0$.*

Proof Let $C = (c_{ij})$ and consider the following system of inequalities:

$$\begin{aligned} -\sum_{j=1}^{n} c_{ij} x_j &> 0 \qquad (i = 1, \ldots, m), \\ x_j &> 0 \qquad (j = 1, \ldots, n). \end{aligned}$$

We want to use Lemma 4.1.4 to show that this system has a solution. Thus we consider an equation of the form

$$\sum_{i=1}^{m} \lambda_i(-c_{i1}, \ldots, -c_{in}) + \sum_{j=1}^{n} \mu_j \varepsilon_j = 0,$$

where $\lambda_i, \mu_j \geq 0$ and ε_j is the jth coordinate vector in $\mathbb{R}^n$. Then

$$\sum_{i=1}^{m} \lambda_i c_{ij} = \mu_j \qquad (j = 1, \ldots, n).$$

Let $u = (\lambda_1, \ldots, \lambda_m)$. Then $C^t u = (\mu_1, \ldots, \mu_n)$. Thus we have $u \geq 0$ and $C^t u \geq 0$. This implies $u = 0$ and $C^t u = 0$. Thus all λ_i and μ_j are zero. Hence Lemma 4.1.4 shows that the above inequalities have a solution. Thus there exists $v > 0$ with $Cv < 0$. □

We now consider three classes of GCM A. Let

$$\begin{aligned} S_F &= \{A \mid A \text{ has finite type}\} \\ S_A &= \{A \mid A \text{ has affine type}\} \\ S_I &= \{A \mid A \text{ has indeterminate type}\} \end{aligned}$$

It is easy to see that no GCM can lie in more than one of these classes. We want to show that each indecomposable *GCM* lies in one of the three classes.

Lemma 4.1.6 *Let A be an indecomposable GCM. Then $u \geq 0$ and $Au \geq 0$ imply that $u > 0$ or $u = 0$.*

Proof Suppose $u \geq 0$, $u \neq 0$ and $u \not> 0$. Then we can reorder $1, \ldots, n$ so that $u_1 = \cdots = u_s = 0$ and $u_{s+1}, \ldots, u_n > 0$. Let $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ where P is $s \times s$ and S is $(n-s) \times (n-s)$. Now all entries of the block Q are ≤ 0 since A is GCM, and if Q has a negative entry, then Au has a negative coefficient, giving a contradiction. Thus $Q = 0$, whence $R = 0$ by definition of GCM. Now A is decomposable, a contradiction. □

Now let A be an indecomposable GCM and define

$$K_A = \{u \mid Au \geq 0\}.$$

K_A is a convex cone. We consider its intersection with the convex cone $\{u \mid u \geq 0\}$. We will distinguish between two cases:

$$\begin{aligned} \{u \mid u \geq 0,\ Au \geq 0\} &\neq \{0\}, \\ \{u \mid u \geq 0,\ Au \geq 0\} &= \{0\}. \end{aligned}$$

The first of these cases splits into two subcases, as is shown by the next lemma.

Lemma 4.1.7 *Suppose* $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$. *Then just one of the following cases occurs:*

$$\begin{aligned} K_A &\subset \{u \mid u > 0\} \cup \{0\}, \\ K_A &= \{u \mid Au = 0\} \text{ and } K_A \text{ is a 1-dimensional subspace of } \mathbb{R}^n. \end{aligned}$$

Proof We know there exists $u \neq 0$ with $u \geq 0$ and $Au \geq 0$. By Lemma 4.1.6, $u > 0$. Suppose the first case does not hold. Then there is $v \neq 0$ with $Av \geq 0$ such that some coordinate of v is ≤ 0. If $v \geq 0$ then $v > 0$ by Lemma 4.1.6, so some coordinate of v is negative.

We have $Au \geq 0$ and $Av \geq 0$, hence $A(tu + (1-t)v) \geq 0$ for all $0 \leq t \leq 1$. Since all coordinates of u are positive and some coordinate of v is negative, there exists $0 < t < 1$ with $tu + (1-t)v \geq 0$ and some coordinate of $tu + (1-t)v$ is zero. But then $tu + (1-t)v = 0$ by Lemma 4.1.6. Thus v is a scalar multiple of u. We also have

$$0 = A(tu + (1-t)v) = tAu + (1-t)Av.$$

Since $Au \geq 0$ and $Av \geq 0$ this implies $Av = Au = 0$.

Now let $w \in K_A$. Then $Aw \geq 0$. Either $w \geq 0$ or some coordinate of w is negative. If $w \geq 0$ then $w > 0$ or $w = 0$ by Lemma 4.1.6. Suppose $w > 0$. Then by the above argument with u replaced by w, v

is a scalar multiple of w, hence w is a scalar multiple of u. Now suppose some coordinate of w is negative. Then by the above argument with v replaced by w, w is a scalar multiple of u. Thus in all cases w is a scalar multiple of u. Hence $K_A = \mathbb{R}u = \{u \mid Au = 0\}$.

Finally, both cases cannot hold simultaneously since in the first case K_A cannot contain a 1-dimensional subspace. □

We can now identify the first case in the lemma above with the case of matrices of finite type.

Proposition 4.1.8 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A has finite type;*
(ii) $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$ *and* $K_A \subset \{u \mid u > 0\} \cup \{0\}$.

Proof (i) $\Rightarrow$ (ii) Suppose A is of finite type. Then there exists $u > 0$ with $Au > 0$. Hence $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$. Also, $\det A \neq 0$. Thus $\{u \mid Au = 0\}$ is not a 1-dimensional subspace. Hence (ii) holds by Lemma 4.1.7.

(ii) $\Rightarrow$ (i) There cannot exist $u \neq 0$ with $Au = 0$ for this would give a 1-dimensional subspace in K_A. Thus $\det A \neq 0$. Now there exists $u \neq 0$ with $u \geq 0$ and $Au \geq 0$. By Lemma 4.1.6, $u > 0$. If $Au > 0$, A has finite type. So suppose to the contrary that some coordinate of Au is zero. Choose the numbering of $1, \ldots, n$ so that the first s coordinates of Au are 0 and the last $n - s$ are positive. Let $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ where P is $s \times s$ and S is $(n-s) \times (n-s)$. The block $Q \neq 0$, since A is indecomposable. We choose numbering so that the first row of Q is not the zero vector. Then

$$Au = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} Pu^1 + Qu^2 \\ Ru^1 + Su^2 \end{pmatrix},$$

and $Pu^1 + Qu^2 = 0$ and $Ru^1 + Su^2 > 0$. We also have $u^1, u^2 > 0$. Thus $Qu^2 \leq 0$ since the entries of Q are non-positive, and the first coordinate of Qu^2 is negative. Hence $Pu^1 \geq 0$ and the first coordinate of Pu^1 is positive. Since $Ru^1 + Su^2 > 0$ we can chose $\varepsilon > 0$ such that $R(1+\varepsilon)u^1 + Su^2 > 0$.

We now consider instead of our original vector $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$, the vector

$\begin{pmatrix} (1+\varepsilon)u^1 \\ u^2 \end{pmatrix} > 0$. We have

$$A \begin{pmatrix} (1+\varepsilon)u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} Pu^1 + Qu^2 + \varepsilon Pu^1 \\ Ru^1 + Su^2 + \varepsilon Ru^1 \end{pmatrix} = \begin{pmatrix} \varepsilon Pu^1 \\ R(1+\varepsilon)u^1 + Su^2 \end{pmatrix}.$$

The first coordinate and the last $n - s$ coordinates of this vector are positive and the remaining coordinates are ≥ 0. Thus $A \begin{pmatrix} (1+\varepsilon)u^1 \\ u^2 \end{pmatrix} \geq 0$ and the number of non-zero coordinates in this vector is greater than that in Au. We may now iterate this process, obtaining at each stage at least one more non-zero coordinate than we had before. We eventually obtain a vector $v > 0$ such that $Av > 0$. □

We next identify the second case in Lemma 4.1.7 with that of an affine GCM.

Proposition 4.1.9 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A has affine type;*
(ii) *$\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$, $K_A = \{u \mid Au = 0\}$, and K_A is a 1-dimensional subspace of $\mathbb{R}^n$.*

Proof (i) $\Rightarrow$ (ii) Suppose A is of affine type. Then there exists $u > 0$ with $Au = 0$. It follows that $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$. Also $\lambda u \in K_A$ for all $\lambda \in \mathbb{R}$. It follows from Lemma 4.1.7 that we are in the second case of that lemma.

(ii) $\Rightarrow$ (i) Note first that corank $A = 1$. Also there exists $u \neq 0$ with $u \geq 0$ and $Au \geq 0$. By Lemma 4.1.6, $u > 0$. So there exists $u > 0$ with $Au \geq 0$. But $K_A = \{u \mid Au = 0\}$, so $Au = 0$. Finally, $Au \geq 0$ implies $Au = 0$. □

Proposition 4.1.10 *Let A be an indecomposable GCM. Then*

(i) *A has finite type if and only if A^t has finite type;*
(ii) *A has affine type if and only if A^t has affine type.*

Proof Let A be of finite type. There does not exist $v > 0$ with $Av < 0$ ($Av < 0 \Rightarrow A(-v) > 0 \Rightarrow (-v) > 0 \Rightarrow v < 0$). So by Proposition 4.1.5, there exists $u \neq 0$ with $u \geq 0$ and $A^t u \geq 0$. So

$$\{u \mid u \geq 0,\ A^t u \geq 0\} \neq \{0\}.$$

By Lemma 4.1.7, either

$$K_{A^t} \subset \{u \mid u > 0\} \cup \{0\}$$

or $K_{A^t} = \{u \mid A^t u = 0\}$ and this is a 1-dimensional subspace. Now $\det A \neq 0$, so $\det A^t \neq 0$. Thus the latter case cannot occur. The former case must therefore occur, so by Proposition 4.1.8, A^t is of finite type.

Let A be of affine type. Again, there does not exist $v > 0$ with $Av < 0$ ($Av < 0 \Rightarrow A(-v) > 0$, which is impossible in the affine case). So by Proposition 4.1.5, there exists $u \neq 0$ with $u \geq 0$ and $A^t u \geq 0$. So

$$\{u \mid u \geq 0,\ A^t u \geq 0\} \neq \{0\}.$$

By Lemma 4.1.7, either

$$K_{A^t} \subset \{u \mid u > 0\} \cup \{0\}$$

or $K_{A^t} = \{u \mid A^t u = 0\}$ and this is a 1-dimensional subspace. Now corank $A = 1$ so corank $A^t = 1$. This shows that we cannot have the first possibility. Thus the second possibility holds, and by Proposition 4.1.9, we see that A^t has affine type. □

We may now identify the case not appearing in Lemma 4.1.7.

Proposition 4.1.11 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A has indefinite type;*
(ii) $\{u \mid u \geq 0,\ Au \geq 0\} = \{0\}$.

Proof If A has indefinite type then $u \geq 0$ and $Au \geq 0$ imply $u = 0$.

Conversely, suppose $\{u \mid u \geq 0,\ Au \geq 0\} = \{0\}$. Then the same condition holds for A^t, i.e. $\{u \mid u \geq 0,\ A^t u \geq 0\} = \{0\}$. Indeed this follows from Lemma 4.1.7 and Propositions 4.1.8, 4.1.9, 4.1.10. But then Proposition 4.1.5 implies that there exists $v > 0$ with $Av < 0$. Thus A has indefinite type. □

Theorem 4.1.12 (Trichotomy Theorem) *Let A be an indecomposable GCM. Then exactly one of the following three possibilities holds: A has finite type, A has affine type, or A has indefinite type. Moreover, the type of A is the same as the type of A^t. Finally,*

(i) *A has finite type if and only if there exists $u > 0$ with $Au > 0$.*
(ii) *A has affine type if and only if there exists $u > 0$ with $Au = 0$. This u is unique up to a (positive) scalar.*

(iii) *A has indefinite type if and only if there exists $u > 0$ with $Au < 0$.*

Proof The first two statements have already been proved. We prove the third statement. Let $u > 0$.

(i) Assume that $Au > 0$. A cannot have affine type as then $Au \geq 0$ would imply $Au = 0$. A cannot have indefinite type as then $u \geq 0$ and $Au \geq 0$ would imply $u = 0$. Thus A has finite type. The converse is clear.

(ii) Assume that $Au = 0$. A cannot have finite type as then $\det A = 0$. A cannot have indefinite type as then $u \geq 0$ and $Au \geq 0$ would imply $u = 0$. Thus A has affine type. The converse is clear, and the remaining statement follows from Proposition 4.1.9.

(iii) Assume that $Au < 0$. Then $A(-u) > 0$. A cannot have finite type as this would imply $-u > 0$ or $-u = 0$. A cannot have affine type as and $A(-u) > 0$ would then imply $-u = 0$. Thus A has indefinite type. The converse is clear. □

Lemma 4.1.13 *Let A be an indecomposable GCM.*

(i) *If A is of finite type then every principal minor A_J is also of finite type.*
(ii) *If A is of affine type then every proper principal minor A_J is of finite type.*

Proof By passing to an equivalent GCM we may assume that $J = \{1, \ldots, m\}$ for some $m \leq n$. Let $K = \{m+1, \ldots, n\}$. Write

$$A = \begin{pmatrix} A_J & Q \\ R & S \end{pmatrix}.$$

(i) We have $Au > 0$ for some $u = \begin{pmatrix} u_J \\ u_K \end{pmatrix} > 0$. We have

$$Au = \begin{pmatrix} A_J u_J + Q u_K \\ R u_J + S u_K \end{pmatrix}.$$

We have $A_J u_J + Q u_K > 0$. But $Q u_K \leq 0$, so $A_J u_J > 0$.

(ii) As in (i) we get $A_J u_J + Q u_K = 0$, and $Q u_K \leq 0$ implies $A_J u_J \geq 0$. Suppose if possible $A_J u_J = 0$. Then $Q u_K = 0$, and since $u_K > 0$ this implies that $Q = 0$, which contradicts the assumption that A is indecomposable. Hence we have $u_J > 0$, $A_J u_J \geq 0$, $A_J u_J \neq 0$. This implies that A_J cannot have affine or indefinite type. □

Remark 4.1.14 In proving results of this section we have never used the full force of the assumption that A is a GCM. Namely we nowhere needed that $a_{ii} = 2$ and $a_{ij} \in \mathbb{Z}$.

4.2 Indecomposable symmetrizable GCMs

Proposition 4.2.1 *Suppose A is a symmetric indecomposable GCM. Then:*

(i) *A has finite type if and only if A is positive definite.*

(ii) *A has affine type if and only if A is positive semidefinite of corank 1.*

(iii) *A has indefinite type otherwise.*

Proof (i) Let A be of finite type. Then there exists $u > 0$ with $Au > 0$. Hence for all $\lambda > 0$ we have $(A + \lambda I)u > 0$. Thus $A + \lambda I$ has finite type by Trichotomy Theorem. (Note that $A + \lambda I$ need not be GCM, but see Remark 4.1.14.) Thus $\det(A + \lambda I) \neq 0$ when $\lambda \geq 0$, that is $\det(A - \lambda I) \neq 0$ when $\lambda \leq 0$. Now the eigenvalues of the real symmetric matrix A are all real. Thus all the eigenvalues of A must be positive.

Conversely, suppose A is positive definite. Then $\det A \neq 0$, so A has finite or indefinite type. If A has indefinite type there exists $u > 0$ with $Au < 0$. But then $u^t Au < 0$, contradicting the fact that A is positive definite. Thus A must have finite type.

(ii) Let A have affine type. Then there is $u > 0$ with $Au = 0$. The same argument as in (i) shows that all eigenvalues of A are non-negative. But A has corank 1, so 0 appears with multiplicity 1.

Conversely, suppose A is positive semidefinite of corank 1. Then $\det A = 0$ so A cannot have finite type. Suppose A has indefinite type. Then there exists $u > 0$ with $Au < 0$. Thus $u_t Au < 0$, which contradicts the fact that A is positive semidefinite.

(iii) follows from (i) and (ii). □

Lemma 4.2.2 *Let A an indecomposable GCM of finite or affine type. Suppose that $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k} a_{i_k i_1} \neq 0$ for some integers $i_1, \dots, i_k$ with $k \geq 3$ such that $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{k-1} \neq i_k, i_k \neq i_1$. Then A is* *of the form*

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}. \tag{4.1}$$

Proof Choose integers $i_1, \dots, i_k$ as in the assumption with minimal possible k. We thus have

$$a_{i_r i_s} \neq 0 \text{ is } (r,s) \in \{(1,2),(2,3),\dots,(k,1),(2,1),(3,2),\dots(1,k)\}.$$

The minimality of k implies that $a_{i_r,i_s} = 0$ if (r,s) does not lie in the above set.

Let $J = \{i_1, \dots, i_k\}$. Then the principal minor A_J of A has form

$$A_J = \begin{pmatrix} 2 & -r_1 & 0 & 0 & \dots & 0 & 0 & -s_k \\ -s_1 & 2 & -r_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & -s_2 & 2 & -r_3 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -s_{k-2} & 2 & -r_{k-1} \\ -r_k & 0 & 0 & 0 & \dots & 0 & -s_{k-1} & 2 \end{pmatrix} \tag{4.2}$$

with positive integers r_i, s_i. In particular we see that A_J is indecomposable. Now A_J must be finite or affine type by Lemma 4.1.13. Thus there exists $u = (u_1, \dots, u_k) > 0$ with $A_J u \geq 0$. We define the $k \times k$ matrix

$$M := \mathrm{diag}(u_1^{-1}, \dots, u_k^{-1}) A_J \mathrm{diag}(u_1, \dots, u_k).$$

Then $m_{ij} = u_i^{-1} a_{ij} u_j$. Thus

$$\sum_j m_{ij} = u_i^{-1} \sum_j (A_J)_{ij} u_j \geq 0.$$

In particular, $\sum_{ij} m_{ij} \geq 0$. Now we have

$$M = \begin{pmatrix} 2 & -r'_1 & 0 & 0 & \dots & 0 & 0 & -s'_k \\ -s'_1 & 2 & -r'_2 & 0 & \dots & 0 & 0 & 0 \\ 0 & -s'_2 & 2 & -r'_3 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -s'_{k-2} & 2 & -r'_{k-1} \\ -r'_k & 0 & 0 & 0 & \dots & 0 & -s'_{k-1} & 2 \end{pmatrix},$$

where $r'_i = u_i^{-1} r_i u_{i+1}$, $s'_i = u_{i+1}^{-1} s_i u_i$ and u_{k+1} is interpreted as u_1. We note that $r'_i, s'_i > 0$ and $r'_i s'_i = r_i s_i \in \mathbb{Z}$. We also have

$$\sum_{ij} m_{ij} = 2k - (r'_1 + s'_1) - \cdots - (r'_k + s'_k).$$

Now $\frac{r'_i + s'_i}{2} \geq \sqrt{r'_i s'_i} = \sqrt{r_i s_i} \geq 1$, hence $r'_i + s'_i \geq 2$. Since $\sum_{ij} m_{ij} \geq 0$, we deduce that $r'_i + s'_i = 2$ and $r'_i s'_i = 1$. Hence $r_i s_i = 1$, and since r_i, s_i are positive integers, we deduce that $r_i = s_i = 1$, i.e. A_J is of the form (4.1).

Let $v = (1, \dots, 1)$. Then $v > 0$ and $A_J v = 0$. Thus A_J is affine type by Theorem 4.1.12. Lemma 2.1.1 shows that this can only happen when $A_J = A$. □

Theorem 4.2.3 *Indecomposable GCM of finite or affine type is symmetrisable.*

Proof If there is a set of integers $i_1, \dots, i_k$ as in Lemma 4.2.2, then we know that A is of the form (4.1), in particular it is symmetric. Otherwise A is symmetrizable by Lemma 2.1.1. □

Theorem 4.2.4 *Let A be an indecomposable GCM. Then:*

(i) *A has finite type if and only if all its principal minors have positive determinant.*
(ii) *A has affine type if and only if* $\det A = 0$ *and all proper principal minors have positive determinant.*
(iii) *A has indefinite type if and only if neither of the above conditions holds.*

Proof (i) Suppose A has finite type. Then A is symmetrizable by Theorem 4.2.3, hence $A = DB$ where $D = \mathrm{diag}(d_1, \dots, d_n)$ with $d_i > 0$ and B symmetric, see Lemma 2.1.2. Theorem 4.1.12 shows that A and B

have the same type. By Lemma 4.1.13 all principal minors of B have finite type, hence by Proposition 4.2.1 they all have positive determinant. Then the same is true for A.

Conversely, let all principal minors of A have positive determinant. Suppose there is a set of integers $i_1, \dots, i_k$ with $k \geq 3$ such that $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{k-1} \neq i_k, i_k \neq i_1$ and $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k} a_{i_k i_1} \neq 0$. As in the proof of the previous theorem, A_J has form (4.2). Analyzing 2×2 and 3×3 principal subminors we conclude that A_J is of the form (4.1). But then $\det A_J = 0$, giving a contradiction. Thus there is no such sequence $i_1, \dots, i_k$ and so A is symmetrizable by Lemma 2.1.1. Hence $A = DB$ where $D = \operatorname{diag}(d_1, \dots, d_n)$ with $d_i > 0$ and B symmetric of the same type as A. Now, it follows from the assumption that all principal minors of B have positive determinant, so B is of finite type.

(ii) If A has affine type, then $\det A = 0$ and all proper principal minors have finite type so have positive determinants by (i).

Conversely, supppose $\det A = 0$ and all proper principal minors have positive determinants. As above, we have two cases:

(a) there is a principal minor of the form (4.1). Since $\det A_J = 0$ we must have $A = A_J$, which is affine type.

(b) A is symmetrizable, in which case we reduce to the symmetric case as above. □

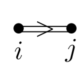
4.3 The classification of finite and affine GCMs

To every GCM A we associate the graph $S(A)$, called the *Dynkin diagram* of A, as follows. The vertices of the Dynkin diagram are labelled by $1, \dots, n$ (or the corresponding simple roots $\alpha_1, \dots, \alpha_n$). Let i, j be distinct vertices of $S(A)$. The rules are as follows:

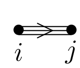
(a) If $a_{ij} a_{ji} = 0$, vertices i, j are not joined.

(b) If $a_{ij} = a_{ji} = -1$, vertices i, j are joined by a single edge.

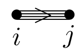
(c) If $a_{ij} = -1$, $a_{ji} = -2$, vertices i, j are joined as follows



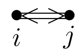
(d) If $a_{ij} = -1$, $a_{ji} = -3$, vertices i, j are joined as follows



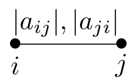
(e) If $a_{ij} = -1$, $a_{ji} = -4$, vertices i, j are joined as follows



(f) If $a_{ij} = -2$, $a_{ji} = -2$, vertices i, j are joined as follows



(g) If $a_{ij}a_{ji} =\geq 5$, vertices i, j are joined as follows



It is clear that the GCM is determined by its Dynkin diagram. Moreover, A is indecomposable if and only if $S(A)$ is connected.

Theorem 4.3.1 *Let A be an indecomposable GCM. Then:*

(i) *A is of finite type if and only if its Dynkin diagram belongs to Figure 4.1. Numbers on the right give* $\det A$.

(ii) *A is of affine type if and only if its Dynkin diagram belongs to Figures 4.2 and 4.3. All diagrams there have $\ell+1$ vertices. Numeric marks are the coordinates of the unique vector $\delta = (a_0, a_1, \ldots, a_\ell)$ such that $A\delta = 0$ and the a_i are positive mutually prime integers. Each diagram $X_\ell^{(1)}$ in Figure 4.2 is obtained from the diagram X_ℓ in Figure 4.1 by adding a vertex labeled α_0 and preserving the labeling of other vertices.*

Proof We first prove that the numeric marks in the diagrams from Figures 4.2 and 4.3 are the coordinates of the unique vector $\delta = (a_0, a_1, \ldots, a_\ell)$ such that $A\delta = 0$ and the a_i are positive mutually prime integers. Note that $A\delta = 0$ is equivalent to

$$2a_i = \sum_j m_j a_j \qquad \text{for all } i$$

where the sum is over all j which are linked with i; moreover if the number of edges between i and j is equal to $s > 1$ and the arrow points to i then then $m_j = s$, otherwise $m_j = 1$. Now check that the marks work in all cases. Now from Theorem 4.1.12 we conclude that all diagrams from Figures 4.2 and 4.3 are affine and δ is unique.

Since all diagrams from Figure 4.1 are proper subdiagrams of diagrams from Figures 4.2 and 4.3, Theorem 4.2.4 implies that they are of finite type. It remains to show that if A is of finite (resp. affine) type then

$S(A)$ appears in Figure 4.1 (resp. Figures 4.2 and 4.3). We establish this by induction on n. The case $n = 1$ is clear. Also, using the condition $\det A \geq 0$ and Theorem 4.2.4, we obtain:

$$\text{finite diagrams of rank 2 are } A_2, C_2, G_2; \quad (4.3)$$

$$\text{affine diagrams of rank 2 are } A_1^{(1)}, A_2^{(2)}; \quad (4.4)$$

$$\text{finite diagrams of rank 3 are } A_3, B_3, C_3; \quad (4.5)$$

$$\text{affine diagrams of rank 3 are } A_2^{(1)}, C_2^{(1)}, G_2^{(1)}, D_3^{(2)}, A_4^{(2)}, D_4^{(3)}. \quad (4.6)$$

Next, from Lemma 4.2.2, we have

$$\text{if } S(A) \text{ contains a cycle, then } S(A) = A_\ell^{(1)}. \quad (4.7)$$

Moreover, by induction and Lemma 4.1.13,

$$\text{Any proper subdiagram of } S(A) \text{ appears in Figure 4.1.} \quad (4.8)$$

Now let $S(A)$ be a finite diagram. Then it does not have graphs appearing in Figures 4.2 and 4.3 as subgraphs and does not have cycles. This implies that every branch vertex has type D_4 since otherwise we would get an affine subdiagram or a contradiction with (4.8). Using (4.8) again we see that there is at most one branch vertex, in which case it also follows that $S(A)$ is D_ℓ, E_6, E_7, or E_8. Similarly one checks that if $S(A)$ has multiple edges then it must be B_ℓ, C_ℓ, F_4, or G_2. Finally, a graph without branch vertices, cycles and multiple edges must be A_ℓ.

Let $S(A)$ be affine. In view of (4.7) we may assume that $S(A)$ has no cycles. In view of (4.8), $S(A)$ is obtained from a diagram in Figure 4.1 by adjoining one vertex in such a way that every subdiagram is again in Figure 4.1. It is easy to see that in this way we can only get diagrams from Figures 4.2 and 4.3. □

Proposition 4.3.2 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A is of finite type.*
(ii) *A is symmetrizable and the $(\cdot|\cdot)$ on $\mathfrak{h}_\mathbb{R}$ is positive definite.*
(iii) *$|W| < \infty$.*
(iv) *$|\Delta| < \infty$.*
(v) *$\mathfrak{g}(A)$ is a finite dimensional simple Lie algebra.*
(vi) *There exists $\alpha \in \Delta_+$ such that $\alpha + \alpha_i \notin \Delta$ for all $i = 1, \dots, n$.*

Proof (i) $\Rightarrow$ (ii) follows from Theorems 4.2.3 and 4.2.4.

(ii) $\Rightarrow$ (iii). In view of Proposition 3.2.4, W is a subgroup of the orthogonal group $G := O((\cdot|\cdot))$, which is known to be compact. If we can check that W is a discrete subgorup, it will follow from general theory that W is finite. To see that W is discrete it suffices to find an open neighborhood U of identity e in G with $U \cap W = \{e\}$. Consider the action of G on $\mathfrak{h}_{\mathbb{R}}$ and fix an element h in the interior C of the fundamental chamber. We get a continuos map $\varphi : G \to G \cdot h$. Take $U := \varphi^{-1}(C)$.

(iii) $\Rightarrow$ (iv) follows from Proposition 3.4.1(v).

(iv) $\Rightarrow$ (vi) is obvious.

(vi) $\Rightarrow$ (i). Let $\alpha \in \Delta_+$ be such that $\alpha + \alpha_i \notin \Delta$ for all i. By Proposition 3.1.5(v), $\langle \alpha, \alpha_i^\vee \rangle \geq 0$ for all i. Write $\alpha = u_1\alpha_1 + \cdots + u_n\alpha_n$ with non-negative coefficients u_i. Then $u = (u_1, \ldots, u_n) \geq 0$, $u \neq 0$, and $Au \geq 0$. By Trichotomy Theorem, A is finite or affine type, and in the latter case we have $\langle \alpha, \alpha_i^\vee \rangle = 0$ for all i. But then $\alpha \neq \alpha_i$, and so $\alpha - \alpha_i \in \Delta_+$ for some i by Lemma 1.4.5, hence $\alpha - \alpha_i + 2\alpha_i = \alpha + \alpha_i \in \Delta_+$ in view of Proposition 3.1.5(vi), giving a contradiction.

Finally, (i) $\Rightarrow$ (v) follows from Proposition 1.4.8 and (v) $\Rightarrow$ (iv) is obvious. □

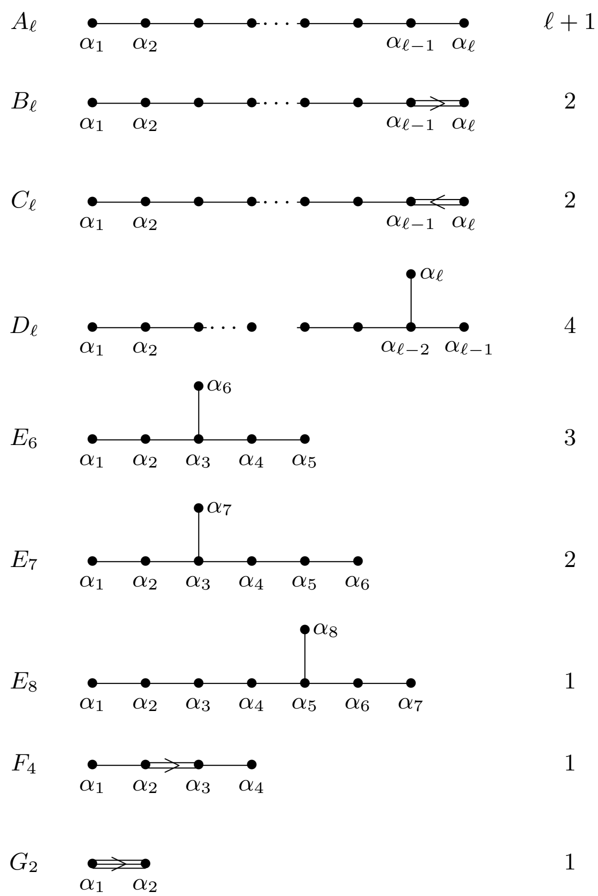


Fig. 4.1. Dynkin diagrams of finite GCMs

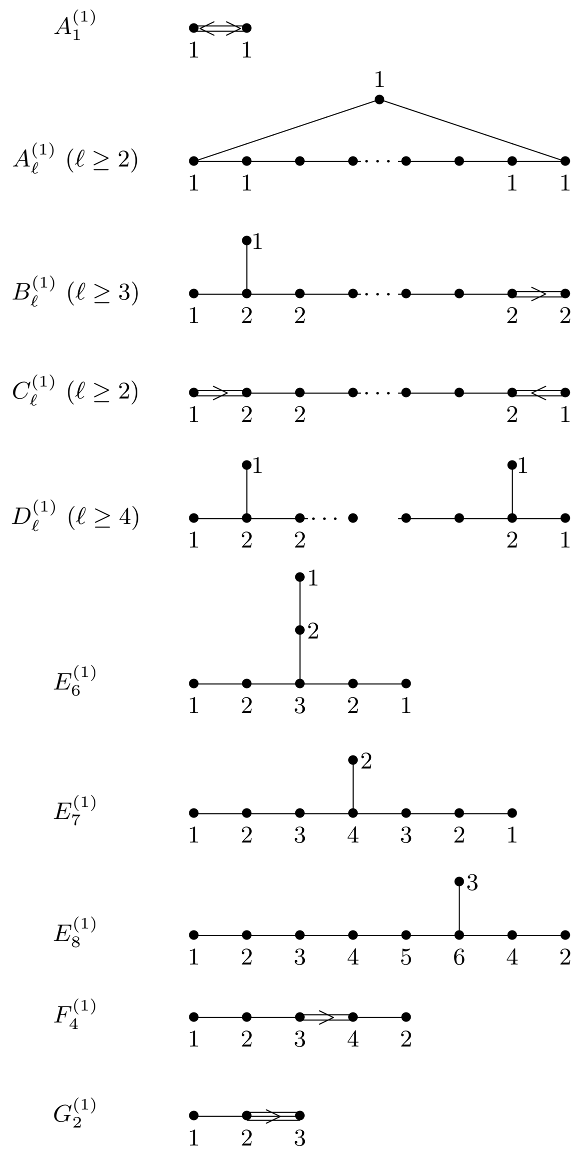


Fig. 4.2. Dynkin diagrams of untwisted affine GCMs

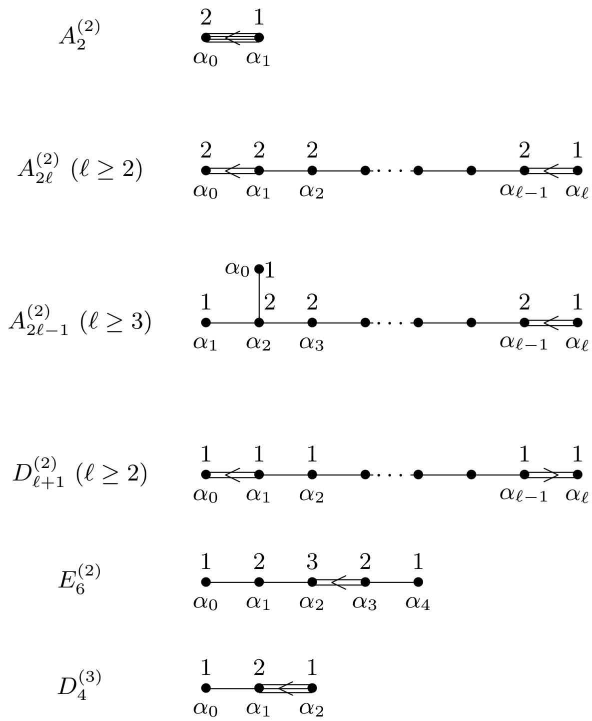


Fig. 4.3. Dynkin diagrams of twisted affine GCMs

5
Real and Imaginary Roots

5.1 Real roots

A root $\alpha \in \Delta$ is called *real* if there exists $w \in W$ such that $w(\alpha)$ is a simple root. Denote by Δ^{re} and Δ_+^{re} the sets of the real and positive real roots respectively. If A is of finite type, then induction on height shows that every root is real.

Let $\alpha \in \Delta^{\mathrm{re}}$. Then $\alpha = w(\alpha_i)$ for some w and some i. Define the *dual real root* $\alpha^\vee \in (\Delta^\vee)^{\mathrm{re}}$ by setting

$$\alpha^\vee = w(\alpha_i^\vee).$$

This definition is independent of the choice of the presentation $\alpha = w(\alpha_i)$. Indeed, we have to show that the equality $u(\alpha_i) = \alpha_j$ implies $u(\alpha_i^\vee) = \alpha_j^\vee$, but this has been proved in Lemma 3.3.1, see (3.8). Thus we have a canonical W-equivariant bijection bijection $\Delta^{\mathrm{re}} \to (\Delta^\vee)^{\mathrm{re}}$.

For $\alpha \in \Delta^{\mathrm{re}}$, define the reflection

$$r_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*,\ \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Since $\langle \alpha, \alpha^\vee \rangle = 2$, it is indeed a reflection. If $\alpha = w(\alpha_i)$, then $w r_i w^{-1} = r_\alpha$, so we have $r_\alpha \in W$.

Proposition 5.1.1 *Let $\alpha \in \Delta^{\mathrm{re}}$. Then:*

(i) $\mathrm{mult}\, \alpha = 1$*;*

(ii) *$k\alpha$ is a root if and only if $k = \pm 1$.*

(iii) *If $\beta \in \Delta$, then there exist non-negative integers p, q such that $p - q = \langle \beta, \alpha^\vee \rangle$ such that $\beta + k\alpha \in \Delta \cup \{0\}$ if and only if $-p \leq k \leq q,\ k \in \mathbb{Z}$.*

(iv) *Suppose that A is symmetrizable and let $(\cdot|\cdot)$ is the standard invariant bilinear form on $\mathfrak{g}$. Then*

(a) $(\alpha|\alpha) > 0$;
(b) $\alpha^\vee = 2\nu^{-1}(\alpha)/(\alpha|\alpha)$;
(c) *if* $\alpha = \sum_i k_i\alpha_i$, *then* $k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ *for all* i.

(v) *if* $\pm\alpha \notin \Pi$, *then there exists* i *such that*

$$|\,\mathrm{ht}\ r_i(\alpha)| < |\,\mathrm{ht}\ \alpha|.$$

(vi) *if* $\alpha > 0$ *then* $\alpha^\vee > 0$.

Proof The proposition is true if α is a simple root, see (2.8), (2.10), and Proposition 3.1.5. Now (i)-(iii) follow from Proposition 3.2.1(ii), and (iv)(a),(b) from Proposition 3.2.4.

(iv)(c) follows from the fact that $\alpha^\vee \in \sum_i \mathbb{Z}\alpha_i^\vee$ and the formula

$$\alpha^\vee = \sum_i \frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} k_i\alpha_i^\vee, \tag{5.1}$$

which in turn follows from (iv)(b).

(v) Assume the statement does not hold. We may assume that $\alpha > 0$. Then $-\alpha \in C^\vee$, and by Proposition 3.4.1(iv) applied to dual root system, $-\alpha + w(\alpha) \geq 0$ for any $w \in W$. Taking w such that $w(\alpha) \in \Pi$ we get a contradiction.

(vi) Apply induction on ht α. For ht $\alpha > 1$ we have by (v) that ht $r_i\alpha <$ ht α, for some i, and $r_i\alpha > 0$. By induction, $r_i(\alpha^\vee) = (r_i\alpha)^\vee > 0$, whence $\alpha^\vee > 0$. □

Lemma 5.1.2 *Assume that* A *is symmetrizable. Then the set of all* $\alpha = \sum_i k_i\alpha_i \in Q$ *such that*

$$k_i(\alpha_i|\alpha_i) \in (\alpha|\alpha)\mathbb{Z} \quad \textit{for all } i \tag{5.2}$$

is W*-invariant.*

Proof It suffices to check that $r_i\alpha$ again satisfies (5.2), i.e.

$$(k_i - \langle\alpha|\alpha_i^\vee\rangle)(\alpha_i|\alpha_i) \in (\alpha|\alpha)\mathbb{Z},$$

or

$$2(\alpha|\alpha_i) \in (\alpha|\alpha)\mathbb{Z},$$

which follows from (5.2):

$$2(\alpha|\alpha_i) = \sum_j \frac{2(\alpha_j|\alpha_i)}{(\alpha_j|\alpha_j)} k_j(\alpha_j|\alpha_j) = \sum_j a_{ji}k_j(\alpha_j|\alpha_j) \in (\alpha|\alpha)\mathbb{Z}.$$

□

Let A be an indecomposable symmetrizable and $(\cdot|\cdot)$ be a standard invariant bilinear form. Then for a real root α we have $(\alpha|\alpha) = (\alpha_i|\alpha_i)$, where α_i is one of the simple roots. We call α a *short* (resp. *long*) root if $(\alpha|\alpha) = \min_i(\alpha_i|\alpha_i)$ (resp. $(\alpha|\alpha) = \max_i(\alpha_i|\alpha_i)$). This definition is independent of the choice of the standard form since α is a linear combination of simple roots.

Note that if A is symmetric then all simple roots are of the same length (so they are both short and long). If A is not symmetric and $S(A)$ has m arrows directed in the same direction then A has $m+1$ different lengths, as the arrow is directed from a longer to a shorter root. Hence if A is not symmetric in Figure 4.1 then every root is either long or short. Moreover, if A is not symmetric and affine and its type is not $A_{2\ell}^{(2)}$ for $\ell > 1$, then every real root is either short or long. In the exceptional case there are three root lengths for real roots. We use notation

$$\Delta_s^{\mathrm{re}}, \quad \Delta_l^{\mathrm{re}}, \quad \Delta_i^{\mathrm{re}}$$

to denote the set of all short, long, and intermediate roots, repspectively.

Note that α is a short real root for $\mathfrak{g}(A)$ if and only if $\alpha^\vee$ is a long real root for $\mathfrak{g}(A^t)$. Indeed, by Proposition 5.1.1(iv)(b)

$$(\alpha^\vee|\alpha^\vee) = (\frac{2\nu^{-1}(\alpha)}{(\alpha|\alpha)}|\frac{2\nu^{-1}(\alpha)}{(\alpha|\alpha)}) = \frac{4}{(\alpha|\alpha)}. \tag{5.3}$$

Throughout this chapter: we normalize the form so that $(\alpha_i|\alpha_i)$ are mutually prime positive integers for each connected component of $S(A)$. In particular, if A is symmetric then $(\alpha_i|\alpha_i) = 1$ for all i.

5.2 Real roots for finite and affine types

Throughout this section we assume that A is finite or affine type.

If $\alpha = \sum_i k_i\alpha_i \in Q$ then $(\alpha|\alpha) = \sum_{i,j} k_ik_j(\alpha_i|\alpha_j)$. Now, $(\alpha_i|\alpha_j) \in \mathbb{Q}$ for all i, j. Thus there exists a positive integer d such that $(\alpha_i|\alpha_j) \in \frac{1}{d}\mathbb{Z}$ for all i, j. Thus if $(\alpha|\alpha) > 0$ then $(\alpha|\alpha) \geq \frac{1}{d}$. Hence there exists $m > 0$ such that

$$m = \min\{(\alpha|\alpha)|\alpha \in Q \text{ and } (\alpha|\alpha) > 0\}.$$

Lemma 5.2.1 *Let* $\alpha = \sum_i k_i\alpha_i \in Q$.

(i) *If* $(\alpha|\alpha) = m$ *then* $\pm\alpha \in Q_+$.

(ii) *If* $k_i(\alpha_i|\alpha_i) \in (\alpha|\alpha)\mathbb{Z}$ *for all* i *then* $\pm\alpha \in Q_+$.

Proof If $\pm\alpha \notin Q_+$, then $\alpha = \beta - \gamma$ for $\beta, \gamma \in Q_+$ and $\operatorname{supp}\beta \cap \operatorname{supp}\gamma = \varnothing$. Hence $(\beta|\gamma) \leq 0$ and

$$(\alpha|\alpha) = (\beta|\beta) + (\gamma|\gamma) - 2(\beta|\gamma) \geq (\beta|\beta) + (\gamma|\gamma).$$

All proper principal minors of A have finite type, so, considering connected components $\beta_1, \dots, \beta_r$ of β we have $(\beta|\beta) = (\beta_1|\beta_1) + \cdots + (\beta_r|\beta_r) > 0$. Hence $(\beta|\beta) \geq m$. Similarly $(\gamma|\gamma) \geq m$. Hence $(\alpha|\alpha) \geq 2m$, which proves (i).

Next, (ii) is clear if $(\alpha|\alpha) = 0$, so assume that $(\alpha|\alpha) > 0$. Then

$$\begin{aligned} \frac{(\beta|\beta)}{(\alpha|\alpha)} &= \frac{1}{(\alpha|\alpha)} \left(\sum_i k_i^2(\alpha_i|\alpha_i) + \sum_{i<j} 2k_ik_j(\alpha_i|\alpha_j) \right) \\ &= \sum_i k_i \frac{k_i(\alpha_i|\alpha_i)}{(\alpha|\alpha)} + \sum_{i<j} a_{ij}k_j\left(\frac{k_i(\alpha_i|\alpha_i)}{(\alpha|\alpha)}\right) \in \mathbb{Z}, \end{aligned}$$

where all indices in the summations are assumed to belong to $\operatorname{supp}\beta$. Since $(\beta|\beta) > 0$, it follows that $(\beta|\beta) \geq (\alpha|\alpha)$. Similarly $(\gamma|\gamma) \geq (\alpha|\alpha)$. So $(\alpha|\alpha) \geq (\beta|\beta) + (\gamma|\gamma) \geq 2(\alpha|\alpha)$. This contradiction yields (ii). □

Proposition 5.2.2

$$\Delta^{\mathrm{re}} = \{\alpha = \sum_i k_i\alpha_i \in Q \mid (\alpha|\alpha) > 0,\ \textit{and}\ k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}\ \textit{for all}\ i\}.$$

Proof "$\subset$" is obvious for the short roots, follows from Proposition 5.2.4 for long roots, and from (5.4) for intermediate roots. Conversely, let α be as in the right hand side. Then $w(\alpha) \in \pm Q_+$ for any $w \in W$ be Lemmas 5.1.2 and 5.2.1(ii). We may assume that $\alpha \in Q_+$, and let $\beta = \sum_{i=0}^{\ell} k_i'\alpha_i$ be an element of $\{w(\alpha) \mid w \in W\} \cap Q_+$ with minimal possible height. Since $(\beta|\beta) > 0$, we have $\sum_{i=0}^{\ell} k_i'(\alpha_i|\beta) > 0$. As all $k_i' \geq 0$, there is i with $(\alpha_i|\beta) > 0$, and $\langle \beta, \alpha_i^\vee \rangle = 2\frac{(\alpha_i|\beta)}{(\alpha_i|\alpha_i)} > 0$. So $r_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$ has smaller height, and $r_i(\beta) \notin Q_+$. But by Lemmas 5.2.1(ii) and 5.1.2, $\pm r_i(\beta) \in Q_+$, so $r_i(\beta) \in -Q_+$. Hence $\beta = k\alpha_i$ for some positive integer k. Thus $m' = (\beta|\beta) = k^2(\alpha_i|\alpha_i)$. So $k_i'\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} = \frac{1}{k}$, whence $k = 1$, and we are done. □

Proposition 5.2.3 *Let A be an indecomposable GCM of finite or affine type. Then*

$$\Delta_s^{\mathrm{re}} = \{\alpha \in Q \mid (\alpha|\alpha) = m\}.$$

Proof Suppose $\alpha \in Q$ satisfies $(\alpha|\alpha) = m$. By Lemma 5.2.1(i), we may assume that $\alpha \in Q_+$. Consider the set

$$\{w(\alpha) \mid w \in W\} \cap Q_+.$$

We choose an element $\beta = \sum k_i\alpha_i$ in this set with ht β minimal. Since $(\beta|\beta) = (\alpha|\alpha) = m$, we have

$$\sum_i k_i(\alpha_i|\beta) = m.$$

Since $k_i \geq 0$ and $m > 0$ there exists i with $(\alpha_i|\beta) > 0$. Then $\langle \beta, \alpha_i^\vee \rangle > 0$. So $r_i(\beta)$ has smaller height than β, whence $s_i(\beta) \in -Q_+$, using the previous lemma. It follows that $\beta = r\alpha_i$ for some positive integer r. Since $(r\alpha_i|r\alpha_i) \geq r^2m$, we have $r = 1$. Hence $\beta \in \Delta_s^{\rm re}$ and $\alpha \in \Delta_s^{\rm re}$ also.

Conversely, if $\alpha \in \Delta_s^{\rm re}$ then $\alpha = w(\alpha_i)$ for some i and $(\alpha|\alpha) = (\alpha_i|\alpha_i)$. However, we have seen in the previous paragraph that the short simple roots have $(\alpha_i|\alpha_i) = m$, so $(\alpha|\alpha) = m$ also. □

Note from Proposition 5.2.3 that m is achieved on simple roots, so m is just $\min_i(\alpha_i|\alpha_i)$. The following easier result follows immediately from Proposition 5.2.2.

Proposition 5.2.4 *Let A be an indecomposable GCM of finite or affine type, and*

$$M := \max\{(\alpha|\alpha) \mid \alpha \in \Delta^{\rm re}\}.$$

Then

$$\Delta_l^{\rm re} = \{\alpha = \sum_i k_i\alpha_i \in Q \mid (\alpha|\alpha) = M,\ k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z} \textit{ for all } i\}.$$

Proposition 5.2.5 *Let $A = A_{2\ell}^{(2)}$ and $m' = (\alpha_i|\alpha_i)$ for $1 \leq i < \ell$. Then*

$$\Delta_i^{\rm re} = \{\alpha \in Q \mid (\alpha|\alpha) = m'\}.$$

Proof Let $\alpha = \sum_{i=0}^{\ell} k_i\alpha_i \in Q$ satisfy $(\alpha|\alpha) = m'$. We just need to check that

$$k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z} \quad \text{for all } i. \tag{5.4}$$

Indeed the condition $k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ is obvious for $i \neq 0$ since $(\alpha_i|\alpha_i) = m'$

for $i = 1, \dots, \ell - 1$ and $2m'$ for $i = \ell$. It just remains to show that k_0 is even. We have

$$\begin{aligned}(\alpha|\alpha) &= k_0^2(\alpha_0|\alpha_0) + 2k_0k_1(\alpha_0|\alpha_1) + (\sum_{i=1}^{\ell}\alpha_i|\sum_{i=1}^{\ell}\alpha_i) \\ &= k_0^2(\alpha_0|\alpha_0) + k_0k_1a_{10}(\alpha_1|\alpha_1) + \sum_{i=1}^{\ell}k_i^2(\alpha_i|\alpha_i) \\ &\quad + \sum_{1\le i<j\le\ell}k_ik_ja_{ij}(\alpha_i|\alpha_i).\end{aligned}$$

Thus $(\alpha|\alpha) \in k_0^2(\alpha_0|\alpha_0) + \mathbb{Z}m'$. But $(\alpha|\alpha) = m$, so $k_0^2(\alpha_0|\alpha_0) \in \mathbb{Z}m'$. Since $(\alpha_0|\alpha_0) = m'/2$, we have $k_0^2/2 \in \mathbb{Z}$, whence k_0 is even as required. □

5.3 Imaginary roots

If a root is not real it is called *imaginary*. Denote by Δ^{im} and Δ_+^{im} the sets of the imaginary and positive imaginary roots respectively.

Proposition 5.3.1

(i) *The set Δ_+^{im} is W-invariant.*

(ii) *For $\alpha \in \Delta_+^{\text{im}}$ there exists a unique (positive) root $\beta \in -C^\vee$ which is W-conjugate to α.*

(iii) *If A is symmetrizable then the root α is imaginary if and only if $(\alpha|\alpha) \le 0$.*

Proof (i) As $\Delta_+^{\text{im}} \subset \Delta\setminus\Pi$ and the set $\Pi\setminus\{\alpha_i\}$ is r_i-invariant, it follows that Δ_+^{im} is W-invariant.

(ii) Let $\alpha \in \Delta_+^{\text{im}}$ and β be the element of minimal height in $W\cdot\alpha \subset \Delta_+$. Then $\beta \in -C^\vee$. Indeed, if $\langle\beta,\alpha_i^\vee\rangle > 0$ then $r_i\beta \in \Delta_+$ has smaller height. Uniqueness of β follows from Proposition 3.4.1(ii).

(iii) If $\alpha \in \Delta^{\text{im}}$. Since the form is W-invariant, as in (ii), we may assume that $\alpha = \sum_i k_i\alpha_i \in -C^\vee$ and $k_i \in \mathbb{Z}_+$. Then

$$(\alpha|\alpha) = \sum_i k_i(\alpha|\alpha_i) = \sum_i \frac{k_i}{2}|\alpha_i|^2\langle\alpha,\alpha_i^\vee\rangle \le 0.$$

The converse follows from Proposition 5.1.1(iv)(a). □

For $\alpha = \sum_i k_i \alpha_i \in Q$ define the *support* of α, denoted $\operatorname{supp} \alpha$, as the subdiagram of $S(A)$ of the vertices i such that $k_i \neq 0$ and all edges connecting them. By Lemma 1.4.7, $\operatorname{supp} \alpha$ is connected. Set

$$K = \{\alpha \in Q_+ \setminus \{0\} \mid \langle \alpha, \alpha_i^\vee \rangle \leq 0 \text{ for all } i \text{ and } \operatorname{supp} \alpha \text{ is connected}\}.$$

Lemma 5.3.2 $K \subset \Delta_+^{\mathrm{im}}$.

Proof Let $\alpha = \sum_i k_i \alpha_i \in K$. Set

$$\Omega_\alpha = \{\gamma \in \Delta_+ \mid \gamma \leq \alpha\}.$$

The set Ω_α is finite, and it is non-empty since the simple roots appearing in decomposition of α belong to Ω_α. Let $\beta = \sum_i m_i \alpha_i$ be an element of maximal height in Ω_α. Note by definition

$$\beta + \alpha_i \notin \Delta_+ \quad \text{if } k_i > m_i. \tag{5.5}$$

Next,

$$\operatorname{supp} \beta = \operatorname{supp} \alpha.$$

Indeed, if some $i \in \operatorname{supp} \alpha \setminus \operatorname{supp} \beta$, we may assume that $\langle \beta, \alpha_i^\vee \rangle < 0$, whence $\beta + \alpha_i \in \Omega_\alpha$ by Proposition 3.1.5(v), giving a contradiction.

Let A_1 be the principal minor of A corresponding to the subset $\operatorname{supp} \alpha$. If A_1 is of finite type then $\langle \alpha, \alpha_i^\vee \rangle \leq 0$ for all i implies $\alpha = 0$ giving a contradiction (see the argument in the proof of Proposition 4.3.2). If A_1 is not of finite type, then by Proposition 4.3.2(vi),

$$P := \{j \in \operatorname{supp} \alpha \mid k_j = m_j\} \neq \varnothing.$$

We aim to first show that $P = \operatorname{supp} \alpha$, and so $\alpha = \beta \in \Delta_+$. Let R be a connected component of subdiagram $\operatorname{supp} \alpha \setminus P$. By (5.5) and Proposition 3.1.5(v),

$$\langle \beta, \alpha_i^\vee \rangle \geq 0 \quad \text{for all } i \in R. \tag{5.6}$$

Set $\beta' = \sum_{i \in R} m_i \alpha_i$. Then

$$\langle \beta', \alpha_i^\vee \rangle = \langle \beta, \alpha_i^\vee \rangle - \sum_{j \in \operatorname{supp} \alpha \setminus R} m_j a_{ij}.$$

Now (5.6) implies $\langle \beta', \alpha_i^\vee \rangle \geq 0$ for all $i \in R$ and $\langle \beta', \alpha_j^\vee \rangle > 0$ for some $j \in R$.

Let A_R be the principal minor corresponding to the subset R, and u be the column vector with entries m_j, $j \in R$. Since

$$\langle \beta', \alpha_i^\vee \rangle = \sum_{j \in R} a_{ij} m_j \qquad (i \in R),$$

we have $u > 0$, $A_M u \geq 0$, and $A_M u \neq 0$. It follows that A_M is not affine or indefinite type, hence it is finite type. Now let

$$\alpha' = \sum_{i \in R} (k_i - m_i) \alpha_i.$$

We have $k_i - m_i > 0$ for all $i \in R$, and $\alpha - \beta = \sum_{i \in \operatorname{supp} \alpha \setminus P} (k_i - m_i) \alpha_i$. Thus for $i \in R$ we have

$$\langle \alpha - \beta, \alpha_i^\vee \rangle = \sum_{j \in \operatorname{supp} \alpha \setminus P} (k_j - m_j) a_{ij} = \sum_{j \in R} (k_i - m_i) a_{ij} = \langle \alpha', \alpha_i^\vee \rangle,$$

since R is a connected component of $\operatorname{supp} \alpha \setminus P$. Thus

$$\langle \alpha', \alpha_i^\vee \rangle = \langle \alpha, \alpha_i^\vee \rangle - \langle \beta, \alpha_i^\vee \rangle \qquad (i \in R).$$

Now $\langle \alpha, \alpha_i^\vee \rangle \leq 0$ since $\alpha \in K$ and $\langle \beta, \alpha_i^\vee \rangle \geq 0$ by (5.6), so $\langle \alpha', \alpha_i^\vee \rangle \leq 0$ for all $i \in R$. Now let u be the column vector with coordinates $k_i - m_i$ for $i \in R$. Then we have $u > 0$ and $A_M u \leq 0$. Since A_M has finite type $A_M(-u) \geq 0$ implies $-u > 0$ or $-u = 0$, giving a contradiction. This completes the proof of the fact that $\alpha \in \Delta_+$.

Finally, 2α satisfies all the assumptions of the lemma, so $\alpha \in \Delta_+$, and by Proposition 5.1.1(ii), $\alpha \in \Delta_+^{\mathrm{im}}$. □

Theorem 5.3.3 $\Delta_+^{\mathrm{im}} = \bigcup_{w \in W} w(K)$.

Proof "⊃" follows from Lemma 5.3.2 and Proposition 5.3.1(i). The converse embedding holds in view of Proposition 5.3.1(i),(ii) and the fact that $\operatorname{supp} \alpha$ is connected for every root α. □

Proposition 5.3.4 *If $\alpha \in \Delta_+^{\mathrm{im}}$ and r a non-zero rational number such that $r\alpha \in Q$, then $r\alpha \in \Delta^{\mathrm{im}}$.*

Proof In view of Proposition 5.3.1(i),(ii) we may assume that $\alpha \in -C^\vee \cap Q_+$. Since α is a root, its support is connected, so $\alpha \in K$. Then $r\alpha \in K$ for any $r > 0$ as in the assumption. By Lemma 5.3.2, $r\alpha \in \Delta_+^{\mathrm{im}}$. □

Theorem 5.3.5 *Let A be indecomposable.*

(i) *If A is finite type then $\Delta^{\mathrm{im}} = \varnothing$.*
(ii) *If A is affine type then $\Delta^{\mathrm{im}}_{+} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$, where $\delta = \sum_{i=0}^{\ell} a_i\alpha_i$ and a_i are the marks in the Dynkin diagram.*
(iii) *If A is indefinite type then there exists a positive imaginary root $\alpha = \sum_i k_i\alpha_i$ such that $k_i > 0$ and $\langle\alpha, \alpha_i^\vee\rangle < 0$ for all $i = 1, \dots, n$.*

Proof By the definition of types and Remark 4.1.2, the set

$$\{\alpha \in Q_+ \mid \langle\alpha, \alpha_i\rangle \le 0\}$$

is $\{0\}$ if A is finite type, is $\mathbb{Z}\delta$ is A is affine type, and there exists $\alpha = \sum_i k_i\alpha_i$ such that $k_i > 0$ and $\langle\alpha, \alpha_i^\vee\rangle < 0$ for all $i = 1, \dots, n$, if A is indefinite type. Now apply Theorem 5.3.3. □

We call a root α *null-root* if $\alpha|_{\mathfrak{h}'} = 0$, or equivalently $\langle\alpha, \alpha_i^\vee\rangle = 0$ for all i. It follows from Theorem 4.1.12 that if α is a null-root if and only if $\operatorname{supp}\alpha$ is affine type which represents a connected component of the diagram A and $\alpha = k\delta$ for $k \in \mathbb{Z}$. We call a root α *isotropic* if $(\alpha|\alpha) = 0$.

Proposition 5.3.6 *Let A be symmetrizable. A root α is isotropic if and only if it is W-conjugate to an imaginary root β such that* $\operatorname{supp}\beta$ *is a subdiagram of affine type in $S(A)$.*

Proof Let α be an isotropic root. We may assume that $\alpha > 0$. Then $\alpha \in \Delta^{\mathrm{im}}_{+}$ by Proposition 5.1.1(iv)(a), and α is W-conjugate to an imaginary root $\beta \in K$ such that $\langle\beta, \alpha_i^\vee\rangle \le 0$ for all i, thanks to Proposition 5.3.1(ii). Let $\beta = \sum_{i\in P} k_i\alpha_i$ and $P = \operatorname{supp}\beta$. Then

$$(\beta|\beta) = \sum_{i\in P} k_i(\beta|\alpha_i) = 0,$$

where $k_i > 0$ and

$$(\beta|\alpha_i) = \frac{1}{2}|\alpha_i|^2\langle\beta, \alpha_i^\vee\rangle \le 0$$

for all $i \in P$. So $\langle\beta, \alpha_i^\vee\rangle = 0$ for all $i \in P$, and P is an affine diagram.

Conversely, let $\beta = k\delta$ be an imaginary root for an affine diagram. Then

$$(\beta|\beta) = k^2(\delta|\delta) = k^2\sum_i a_i(\delta|\alpha_i) = 0$$

since $\langle\delta, \alpha_i^\vee\rangle = 0$ for all i. □

6
Affine Algebras

6.1 Notation

Throughout we use the following notation in the affine case:

- A is an indecomposable GCM of affine type of order $\ell + 1$ and rank ℓ.
- $a_0, a_1, \dots, a_\ell$ are the marks of the diagram $S(A)$ (note that $a_0 = 1$, unless $A = A_{2\ell}^{(2)}$ in which case $a_0 = 2$).
- $a_0^\vee, a_1^\vee, \dots, a_\ell^\vee$ are the marks of the dual diagram $S(A^t)$ (this diagram is obtained from $S(A)$ by changing direction of all arrows and preserving the labels of the vertices). Note that in all cases $a_0^\vee = 1$.
- The numbers
$$h := \sum_{i=0}^{\ell} a_i, \qquad h^\vee := \sum_{i=0}^{\ell} a_i^\vee$$
are *Coxeter* and *dual Coxeter* numbers.
- $r \in \{1, 2, 3\}$ refers to the number r in the type $X_N^{(r)}$.
- $c = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee$ is the *canonical central element*. By Proposition 1.4.6, the center $\mathfrak{c}$ of $\mathfrak{g}$ is $\mathbb{C}c$.
- $\delta = \sum_{i=0}^{\ell} a_i \alpha_i$. Then $\Delta^{\mathrm{im}} = \{\pm\delta, \pm 2\delta, \dots\}$, $\Delta_+^{\mathrm{im}} = \{\delta, 2\delta, \dots\}$, see Theorem 5.3.5.

6.2 Standard bilinear form

We know that A is symmetrizable. Moreover,

$$A = \mathrm{diag}(\frac{a_0}{a_0^\vee}, \dots, \frac{a_\ell}{a_\ell^\vee})B \tag{6.1}$$

for a symmetric matrix B. Indeed let $\delta = (a_0, \ldots, a_\ell)^t$ and $\delta^\vee = (a_0^\vee, \ldots, a_\ell^\vee)^t$. If $A = DB$ where D is a diagonal invertible matrix and B is a symmetric matrix then $B\delta = 0$, and hence $\delta^t B = 0$. On the other hand, $(\delta^\vee)^t A = 0$ implies $(\delta^\vee)^t DB = 0$, whence $BD\delta^\vee = 0$, and since $\dim\ker B = 1$, we get $D\delta^\vee$ is proportional to δ.

Fix an element $d \in \mathfrak{h}$ such that

$$\langle \alpha_i, d\rangle = 0 \quad \text{for } i = 1, \ldots, \ell, \qquad \langle \alpha_0, d\rangle = 1.$$

d is defined up to a summand proportional to c and is called *energy element*. Note that $\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee, d\}$ is a basis of $\mathfrak{h}$. Indeed, we must show that d is not a linear combination of $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee$. Otherwise $d = \sum_{i=0}^{\ell} u_i \alpha_i^\vee$, and $A^t u \geq 0$, $A^t u \neq 0$, giving a contradiction with the affine type of A^t.

Note that

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}d.$$

Following §2.2, define the non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ by

$$\begin{aligned} (\alpha_i^\vee|\alpha_j^\vee) &= \frac{a_j}{a_j^\vee} a_{ij} \qquad (0 \leq i, j \leq \ell); \\ (\alpha_i^\vee|d) &= \delta_{i,0} a_0 \qquad (0 \leq i \leq \ell); \\ (d|d) &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} (c|\alpha_i^\vee) &= 0 \qquad (0 \leq i \leq \ell); \\ (c|c) &= 0; \\ (c|d) &= a_0. \end{aligned}$$

By Theorem 2.2.3, this form can be uniquely extended to $\mathfrak{g}$ so that all conditions of that theorem hold. The extended form $(\cdot|\cdot)$ will be referred to as the *normalized invariant form*.

Next define $\Lambda_0 \in \mathfrak{h}^*$ by

$$\langle \Lambda_0, \alpha_i^\vee \rangle = \delta_{i0}, \quad \langle \Lambda_0, d\rangle = 0.$$

Then

$$\{\alpha_0, \ldots, \alpha_1, \Lambda_0\}$$

is a basis of $\mathfrak{h}^*$.

The isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by the form $(\cdot|\cdot)$ is given by

$$\nu : \alpha_i^\vee \mapsto \frac{a_i}{a_i^\vee}\alpha_i,$$
$$\nu : d \mapsto a_0\Lambda_0.$$

We also have that

$$\nu : c \mapsto \delta.$$

The transported form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ has the following properties:

$$\begin{aligned}
(\alpha_i|\alpha_j) &= \frac{a_i^\vee}{a_i}a_{ij} \qquad (0 \le i, j \le \ell);\\
(\alpha_i|\Lambda_0) &= \delta_{i0}a_0^{-1} \qquad (0 \le i \le \ell);\\
(\Lambda_0|\Lambda_0) &= 0;\\
(\delta|\alpha_i) &= 0 \qquad (0 \le i \le \ell);\\
(\delta|\delta) &= 0;\\
(\delta|\Lambda_0) &= 1.
\end{aligned}$$

It follows that there is an isometry of lattices

$$Q^\vee(A) \cong Q(A^t). \tag{6.2}$$

Denote by $\overset{\circ}{\mathfrak{h}}$ (resp. $\overset{\circ}{\mathfrak{h}}_{\mathbb{R}}$) the $\mathbb{C}$-span (resp. $\mathbb{R}$-span) of $\alpha_1^\vee, \dots, \alpha_\ell^\vee$. The dual notions $\overset{\circ}{\mathfrak{h}}{}^*$ and $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$ are defined as similar linear combinations of $\alpha_1, \dots, \alpha_\ell$. Then we have decompositions into orthogonal direct sums

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus (\mathbb{C}c + \mathbb{C}d), \quad \mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}{}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0).$$

Set

$$\mathfrak{h}_{\mathbb{R}} := \overset{\circ}{\mathfrak{h}}_{\mathbb{R}} + \mathbb{R}c + \mathbb{R}d, \qquad \mathfrak{h}^*_{\mathbb{R}} = \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}} + \mathbb{R}\Lambda_0 + \mathbb{R}\delta.$$

By Theorem 4.2.4, the restriction of the bilinear form $(\cdot|\cdot)$ to $\overset{\circ}{\mathfrak{h}}_{\mathbb{R}}$ and $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$ (resp. $\overset{\circ}{\mathfrak{h}}_{\mathbb{R}} + \mathbb{R}c$ and $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}} + \mathbb{R}\delta$) is positive definite (resp. positive semidefinite with kernels $\mathbb{R}c$ and $\mathbb{R}\delta$).

For a subset $S \subset \mathfrak{h}^*$ denote by $\bar{S}$ the orthogonal projection of S onto $\overset{\circ}{\mathfrak{h}}{}^*$. We have

$$\lambda - \bar{\lambda} = \langle \lambda, c\rangle\Lambda_0 + \frac{|\lambda|^2 - |\bar{\lambda}|^2}{2\langle \lambda, c\rangle}\delta \qquad (\lambda \in \mathfrak{h}^*,\ \langle \lambda, c\rangle \neq 0). \tag{6.3}$$

Indeed, $\lambda - \bar{\lambda} = b_1\Lambda_0 + b_2\delta$. Applying $(\cdot|\delta)$, we deduce that $b_1 = (\lambda|\delta) =$

$\langle\lambda, c\rangle$. Now, $|\lambda|^2 = |\bar{\lambda}|^2 + 2b_1b_2$, which implies the required expression for b_2. The following closely related formula is proved similarly:

$$\lambda = \bar{\lambda} + \langle\lambda, c\rangle\Lambda_0 + (\lambda|\Lambda_0)\delta. \tag{6.4}$$

Define $\rho \in \mathfrak{h}^*$ by

$$\langle\rho, d\rangle = 0, \quad \langle\rho, \alpha_i^\vee\rangle = 1 \qquad (0 \le i \le \ell).$$

Then (6.4) gives

$$\rho = \bar{\rho} + h^\vee\Lambda_0. \tag{6.5}$$

6.3 Roots of affine algebra

Denote by $\mathring{\mathfrak{g}}$ the subalgebra of $\mathfrak{g}$ generated by e_i and f_i for $i = 1, \dots, \ell$. This subalgebra is isomorphic to $\mathfrak{g}(\mathring{A})$ where $\mathring{A}$ is obtained from A by removing 0th row and 0th column. This is a finite dimensional simple Lie algebra whose Dynkin diagram comes from $S(A)$ by deleting the 0th vertex, see Proposition 4.3.2.

Indeed, let

$$\mathring{\Pi} = \{\alpha_1, \dots, \alpha_\ell\}, \quad \mathring{\Pi}^\vee = \{\alpha_1^\vee, \dots, \alpha_\ell^\vee\}.$$

Then $\mathring{\mathfrak{h}}, \mathring{\Pi}, \mathring{\Pi}^\vee$ is a realization of $\mathring{A}$, and since $[e_i, f_i] = \alpha_i^\vee$, $\mathring{\mathfrak{g}}$ is generated by e_i, f_i for $i = 1, \dots, \ell$ and $\mathring{\mathfrak{h}}$, and the relations (1.12)-(1.15) hold. So there is a homomorphism from $\tilde{\mathfrak{g}}(\mathring{A})$ onto $\mathring{\mathfrak{g}}$. We claim that $\mathring{\mathfrak{g}}$ has no non-trivial ideals which have trivial intersection with $\mathring{\mathfrak{h}}$. Otherwise, if $\mathfrak{i}$ is such an ideal let $x \in \mathfrak{i}$ be a non-zero element of weight $\alpha \ne 0$. Then $\alpha \in \mathring{\Delta}$, where

$$\mathring{\Delta} = \Delta \cap \mathring{\mathfrak{h}}^*. \tag{6.6}$$

We may assume that α is the smallest positive root for which such x exists. Then

$$[f_i, x] = 0 \qquad (i = 1, \dots, \ell).$$

But it is also clear from the relations that $[f_0, x] = 0$. By Lemma 1.4.5, $x = 0$. This contradiction proves that there is a homomorphism from $\mathfrak{g}(\mathring{A})$ onto $\mathring{\mathfrak{g}}$. Since $\mathfrak{g}(\mathring{A})$ is simple by Proposition 1.4.8(i), this homomorphism must be an isomorphism.

We will also use the notations

$$\mathring{\Delta}_+ := \mathring{\Delta} \cap \Delta_+, \quad \mathring{Q} = \mathbb{Z}\,\mathring{\Delta}, \quad \mathring{Q}^\vee = \mathbb{Z}\mathring{\Delta}^\vee.$$

$\overset{\circ}{\Delta}_s$ and $\overset{\circ}{\Delta}_l$ for the sets of short and long roots in $\overset{\circ}{\Delta}$, respectively, and $\overset{\circ}{W}$ for the Weyl group for $\overset{\circ}{\Delta}$.

Note that $a_0^\vee = 1$ implies

$$Q^\vee = \overset{\circ}{Q}{}^\vee \oplus \mathbb{Z}c \qquad \text{(orthogonal direct sum)}. \tag{6.7}$$

We denote by Δ_s^{re} and Δ_l^{re} the sets of short and long real roots, respectively. For type $A_{2\ell}^{(2)}$ we denote by Δ_i^{re} the set of real roots of intermediate length.

Proposition 6.3.1

(i) *If $r = 1$ then $\Delta^{\mathrm{re}} = \{\alpha + n\delta \mid \alpha \in \overset{\circ}{\Delta},\ n \in \mathbb{Z}\}$, and $\alpha + n\delta \in \Delta^{\mathrm{re}}$ is short if and only if $\alpha \in \overset{\circ}{\Delta}$ is short.*

(ii) *If $r = 2$ or 3 and $A \neq A_{2\ell}^{(2)}$ then*

$$\begin{aligned} \Delta_s^{\mathrm{re}} &= \{\alpha + n\delta \mid \alpha \in \overset{\circ}{\Delta}_s,\ n \in \mathbb{Z}\}, \\ \Delta_l^{\mathrm{re}} &= \{\alpha + nr\delta \mid \alpha \in \overset{\circ}{\Delta}_l,\ n \in \mathbb{Z}\}. \end{aligned}$$

(iii) *If $A = A_{2\ell}^{(2)}$ for $\ell > 1$ then*

$$\begin{aligned} \Delta_s^{\mathrm{re}} &= \{\frac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in \overset{\circ}{\Delta}_l,\ n \in \mathbb{Z}\}, \\ \Delta_i^{\mathrm{re}} &= \{\alpha + n\delta \mid \alpha \in \overset{\circ}{\Delta}_s,\ n \in \mathbb{Z}\}, \\ \Delta_l^{\mathrm{re}} &= \{\alpha + 2n\delta \mid \alpha \in \overset{\circ}{\Delta}_l,\ n \in \mathbb{Z}\}. \end{aligned}$$

(iv) *If $A = A_2^{(2)}$ then*

$$\begin{aligned} \Delta_s^{\mathrm{re}} &= \{\frac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in \overset{\circ}{\Delta},\ n \in \mathbb{Z}\}, \\ \Delta_l^{\mathrm{re}} &= \{\alpha + 2n\delta \mid \alpha \in \overset{\circ}{\Delta},\ n \in \mathbb{Z}\}. \end{aligned}$$

(v) $\Delta^{\mathrm{re}} + r\delta = \Delta^{\mathrm{re}}$.

(vi) $\Delta_+^{\mathrm{re}} = \{\alpha \in \Delta^{\mathrm{re}}$ *with* $n > 0\} \cup \overset{\circ}{\Delta}_+$.

Proof (v),(vi) follow from (i)-(iv).

Suppose that $A \neq A_{2\ell}^{(2)}$. Then $\overset{\circ}{\Delta}_s \subset \Delta_s^{\mathrm{re}}$. Let $\alpha \in \overset{\circ}{\Delta}_s$. Then $(\alpha|\alpha) = m$. Hence for $n \in \mathbb{Z}$ we have $(\alpha + n\delta|\alpha + n\delta) = m$. By Proposition 5.2.3, $\alpha + n\delta \in \Delta_s^{\mathrm{re}}$.

Conversely, let $\beta = \sum_{i=0}^{\ell} k_i\alpha_i \in \Delta_s^{\mathrm{re}}$. By Proposition 5.2.3,

$$(\beta - k_0\delta|\beta - k_0\delta) = (\beta|\beta) = m.$$

Since $a_0 = 1$, we have $\beta - k_0\delta = \sum_{i=1}^{\ell}(k_i - k_0 a_i)\alpha_i$. So by (6.6) and Proposition 5.2.3 again, we deduce that $\beta - k_0\delta \in \overset{\circ}{\Delta}_s$, thus the short roots have the required form.

We now consider the long roots. Note that $\overset{\circ}{\Delta}_l \subset \Delta_l^{\mathrm{re}}$. Let $\alpha = \sum_{i=1}^{n} k_i\alpha_i \in \overset{\circ}{\Delta}_l$. Then $(\alpha + n\delta|\alpha + n\delta) = (\alpha|\alpha) = M$. By Proposition 5.2.4, we have $k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ for $i = 1, \dots, \ell$. By the same proposition $\alpha + n\delta \in \Delta_l^{\mathrm{re}}$ if and only $na_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ for $i = 0, \dots, \ell$. Now $(\alpha_i|\alpha_i) = \frac{2a_i^\vee}{a_i}$, so the condition is $n\frac{2a_i^\vee}{(\alpha|\alpha)} \in \mathbb{Z}$. Note also that $(\alpha_0|\alpha_0) = 2$.

First suppose that α_0 is a long root, i.e. we are in the case (i). Then $(\alpha|\alpha) = 2$, and so $n\frac{2a_i^\vee}{(\alpha|\alpha)} \in \mathbb{Z}$. Hence $\alpha + n\delta \in \Delta_l^{\mathrm{re}}$ for all $n \in \mathbb{Z}$.

Conversely, let $\beta = \sum_{i=0}^{\ell} k_i\alpha_i \in \Delta_l^{\mathrm{re}}$. By Proposition 5.2.4, $(\beta - k_0\delta|\beta - k_0\delta) = (\beta|\beta) = M$, and $k_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} \in \mathbb{Z}$ for $i = 0, \dots, \ell$. We have $k_0 a_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} \in \mathbb{Z}$ also since $(\alpha_i|\alpha_i) = \frac{2a_i^\vee}{a_i}$, and $(\beta|\beta) = 2$. We now conclude that have $\beta - k_0\delta = \sum_{i=1}^{\ell}(k_i - k_0 a_i)\alpha_i \in \overset{\circ}{\Delta}_l$ by (6.6) and Proposition 5.2.4 again, and so the long roots have the required form.

Now suppose that α_0 is a short root, i.e. we are in the case (ii). Note that $r = \frac{(\alpha|\alpha)}{(\alpha_0|\alpha_0)}$. Thus

$$n\frac{2a_i^\vee}{(\alpha|\alpha)} = n\frac{a_i^\vee}{r},$$

Since $a_0^\vee = 1$ this lies in $\mathbb{Z}$ for all $i = 0, \dots, \ell$ if and only if n is divisible by r. Thus by Proposition 5.2.4, $\alpha + rn\delta \in \Delta_l^{\mathrm{re}}$ for all $n \in \mathbb{Z}$.

Conversely, let $\beta = \sum_{i=0}^{\ell} k_i\alpha_i \in \Delta_l^{\mathrm{re}}$. By Proposition 5.2.4, $(\beta - k_0\delta|\beta - k_0\delta) = (\beta|\beta) = M$, and $k_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} \in \mathbb{Z}$ for $i = 0, \dots, \ell$. In particular, $k_0\frac{(\alpha_0|\alpha_0)}{(\beta|\beta)} = \frac{k_0}{r} \in \mathbb{Z}$. We have

$$k_0 a_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} = a_i^\vee\frac{k_0}{r} \in \mathbb{Z}$$

for $i = 1, \dots, \ell$, as $(\alpha_i|\alpha_i) = \frac{2a_i^\vee}{a_i}$ and $(\beta|\beta) = 2r$. Thus by Proposition 5.2.4, $\beta - k_0\delta = \sum_{i=1}^{\ell}(k_i - k_0 a_i)\alpha_i \in \overset{\circ}{\Delta}_l$ by (6.6) and Proposition 5.2.4 again, and so the long roots have the required form.

The proof of (iii) and (iv) is similar. □

Proposition 6.3.1 will also follow from explicit constructions of affine Lie algebras given in the next chapter.

Remark 6.3.2 $\bar{\Delta} = \overset{\circ}{\Delta} \setminus \{0\}$ in all cases, except $A_{2\ell}^{(2)}$, in which case the root system $\bar{\Delta}$ is not reduced, and $\overset{\circ}{\Delta}$ is the corresponding reduced root system.

Introduce the element

$$\theta := \delta - a_0\alpha_0 = \sum_{i=1}^{\ell} a_i\alpha_i \in \overset{\circ}{Q} . \tag{6.8}$$

We have

$$(\theta|\theta) = (\delta - a_0\alpha_0|\delta - a_0\alpha_0) = a_0^2(\alpha_0|\alpha_0) = 2a_0.$$

Thus $(\theta|\theta) = M$ if $r = 1$ or $A = A_{2\ell}^{(2)}$, and $(\theta|\theta) = m$ otherwise. In all cases it follows from Propositions 5.2.3 and 5.2.4 that $\theta \in \overset{\circ}{\Delta}_+$. Moreover,

$$\theta^\vee = 2\frac{\nu^{-1}(\theta)}{(\theta|\theta)} = \frac{1}{a_0}\nu^{-1}(\theta) = \frac{1}{a_0}\sum_{i=1}^{\ell} a_i^\vee \alpha_i^\vee .$$
$$(\theta^\vee|\theta^\vee) = \frac{2}{a_0},$$
$$\alpha_0^\vee = \nu^{-1}(\delta - \theta) = c - a_0\theta^\vee .$$

Proposition 6.3.3 *If $r = 1$ or $A = A_{2\ell}^{(2)}$, then $\theta \in (\overset{\circ}{\Delta}_+)_l$ and θ is the unique root in $\overset{\circ}{\Delta}$ of maximal height $(= h - a_0)$. Otherwise $\theta \in (\overset{\circ}{\Delta}_+)_s$ and θ is the unique root in $\overset{\circ}{\Delta}_s$ of maximal height $(= h - 1)$.*

Proof One checks that all simple roots in $\overset{\circ}{\Delta}$ of the same length are $\overset{\circ}{W}$-conjugate (this is essentially a type A_2 argument). Hence $\overset{\circ}{\Delta}_s$ and $\overset{\circ}{\Delta}_l$ are the orbits of $\overset{\circ}{W}$ on $\overset{\circ}{\Delta}$. Moreover,

$$\langle \theta, \alpha_i^\vee \rangle = \langle \delta - a_0\alpha_0, \alpha_i^\vee \rangle = -a_0 a_{i0} \geq 0 \qquad (1 \leq i \leq \ell).$$

Hence θ is in the fundamental domain of $\overset{\circ}{W}$, which determines the short or long root uniquely. The height of θ is easy to compute from the definition. Finally, if θ' is a maximal height root in the W-orbit of roots in $\overset{\circ}{\Delta}$ of the same length as θ, then a standard argument shows that θ' is in the fundamental chamber, hence $\theta' = \theta$. □

If A is a matrix of *finite* type, we assume that the standard invariant form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ is normalized by the condition $(\alpha|\alpha) = 2$ for $\alpha \in \Delta_l$, and call it the *normalized invariant form.*

Corollary 6.3.4 *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$. Then the ratio of the restriction to the subalgebra $\mathring{\mathfrak{g}}$ of the normalized invariant on $\mathfrak{g}$ to the normalized invariant form on $\mathring{\mathfrak{g}}$ is equal to r.*

6.4 Affine Weyl Group

Since $\langle \delta, \alpha_i^\vee \rangle = 0$ for all i, we have $w(\delta) = \delta$ for all $w \in W$. Denote by $\mathring{W}$ the subgroup of W generated by $r_1, \dots, r_\ell$. As $r_i(\Lambda_0) = \Lambda_0$ for $i \geq 1$, $\mathring{W}$ acts trivially on $\mathbb{C}\Lambda_0 + \mathbb{C}\delta$. It is also clear that $\mathring{\mathfrak{h}}^*$ is $\mathring{W}$-invariant. So the action of $\mathring{W}$ on $\mathring{\mathfrak{h}}^*$ is faithful, and we can identify $\mathring{W}$ with the Weyl group of of $\mathring{\mathfrak{g}}$ also acting on $\mathring{\mathfrak{h}}^*$. Hence $\mathring{W}$ is finite.

We have

$$r_0 r_\theta(\lambda) = \lambda + \langle \lambda, c \rangle \nu(\theta^\vee) - \big(\langle \lambda, \theta^\vee \rangle + \frac{1}{2}(\theta^\vee | \theta^\vee) \langle \lambda, c \rangle\big)\delta. \tag{6.9}$$

Indeed,

$$\begin{aligned}
r_0 r_\theta(\lambda) &= r_0(\lambda - \langle \lambda, \theta^\vee \rangle \theta) \\
&= \lambda - \langle \lambda, \alpha_0^\vee \rangle \alpha_0 - \langle \lambda, \theta^\vee \rangle (\theta - \langle \theta, \alpha_0^\vee \rangle \alpha_0) \\
&= \lambda - \langle \lambda, \alpha_0^\vee \rangle \frac{1}{a_0}(\delta - \theta) - \langle \lambda, \theta^\vee \rangle \theta + \langle \lambda, \theta^\vee \rangle \langle \theta, \alpha_0^\vee \rangle \frac{1}{a_0}(\delta - \theta) \\
&= \lambda + (\frac{\langle \lambda, \alpha_0^\vee \rangle}{a_0} - \langle \lambda, \theta^\vee \rangle - \frac{\langle \lambda, \theta^\vee \rangle \langle \theta, \alpha_0^\vee \rangle}{a_0})\theta \\
&\quad + (-\frac{\langle \lambda, \alpha_0^\vee \rangle}{a_0} + \frac{\langle \lambda, \theta^\vee \rangle \langle \theta, \alpha_0^\vee \rangle}{a_0})\delta \\
&= \lambda + (\langle \lambda, \alpha_0^\vee \rangle - a_0 \langle \lambda, \theta^\vee \rangle - \langle \lambda, \theta^\vee \rangle \langle \theta, \alpha_0^\vee \rangle) \frac{1}{a_0}\theta \\
&\quad - (\frac{\langle \lambda, \alpha_0^\vee \rangle}{a_0} - \frac{\langle \lambda, \theta^\vee \rangle \langle \theta, \alpha_0^\vee \rangle}{a_0})\delta \\
&= \lambda + (\langle \lambda, c - a_0\theta^\vee \rangle - a_0 \langle \lambda, \theta^\vee \rangle - \langle \lambda, \theta^\vee \rangle \langle \theta, c - a_0\theta^\vee \rangle)\nu(\theta^\vee) \\
&\quad - (\frac{\langle \lambda, c - a_0\theta^\vee \rangle}{a_0} - \frac{\langle \lambda, \theta^\vee \rangle \langle \theta, c - a_0\theta^\vee \rangle}{a_0})\delta,
\end{aligned}$$

which easily implies (6.9).

Set

$$t_\alpha(\lambda) = \lambda + \langle \lambda, c \rangle \alpha - \big((\lambda|\alpha) + \frac{1}{2}(\alpha|\alpha)\langle \lambda, c \rangle\big)\delta \qquad (\lambda \in \mathfrak{h}^*,\ \alpha \in \mathring{\mathfrak{h}}^*). \tag{6.10}$$

Then (6.9) is equivalent to $r_0 r_\theta = t_{\nu(\theta^\vee)}$.

Proposition 6.4.1 *Let* $\alpha, \beta \in \overset{\circ}{\mathfrak{h}}$, $w \in \overset{\circ}{W}$. *Then*

(i) $t_\alpha t_\beta = t_{\alpha+\beta}$.
(ii) $w t_\alpha w^{-1} = t_{w(\alpha)}$.

Proof The linear map $t_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$ is uniquely determined by the properties

$$t_\alpha(\lambda) = \lambda - (\lambda|\alpha)\delta \quad \text{if } \langle \lambda, c \rangle = 0, \tag{6.11}$$

$$t_\alpha(\Lambda_0) = \Lambda_0 + \alpha - \frac{1}{2}(\alpha|\alpha)\delta. \tag{6.12}$$

since $\langle \alpha_i, c \rangle = 0$ and $\langle \Lambda_0, c \rangle = 1$. If $\langle \lambda, c \rangle = 0$ then

$$\begin{aligned} t_\alpha t_\beta(\lambda) &= t_\alpha(\lambda - (\lambda|\beta)\delta) \\ &= \lambda - (\lambda|\alpha)\delta - (\lambda|\beta)(\delta - (\delta|\alpha)\delta) \\ &= \lambda - (\lambda|\alpha+\beta)\delta \\ &= t_{\alpha+\beta}(\lambda), \end{aligned}$$

since $(\delta|\alpha) = 0$, and

$$\begin{aligned} w t_\alpha w^{-1}(\lambda) &= w(w^{-1}(\lambda) - (w^{-1}(\lambda)|\alpha)\delta) \\ &= \lambda - (\lambda|w(\alpha))\delta \\ &= t_{w(\alpha)}(\lambda), \end{aligned}$$

since $\langle w^{-1}(\lambda), c \rangle = 0$ and $w(\delta) = \delta$. Also

$$\begin{aligned} t_\alpha t_\beta(\Lambda_0) &= t_\alpha(\Lambda_0 + \beta - \frac{1}{2}(\beta|\beta)\delta) \\ &= \Lambda_0 + \alpha - \frac{1}{2}(\alpha|\alpha)\delta + \beta - (\beta|\alpha)\delta - \frac{1}{2}(\beta|\beta)(\delta - (\delta|\alpha)\delta) \\ &= \Lambda_0 + (\alpha+\beta) - \frac{1}{2}(\alpha+\beta|\alpha+\beta)\delta \\ &= t_{\alpha+\beta}(\Lambda_0), \end{aligned}$$

using $(\delta|\alpha) = 0$ again, and

$$\begin{aligned} w t_\alpha w^{-1}(\Lambda_0) &= w(\Lambda_0 + \alpha - \frac{1}{2}(\alpha|\alpha)\delta) \\ &= \Lambda_0 + w(\alpha) - \frac{1}{2}(w(\alpha)|w(\alpha))\delta \\ &= t_{w(\alpha)}(\Lambda_0), \end{aligned}$$

since $w^{-1}(\Lambda_0) = w(\Lambda_0) = \Lambda_0$. This proves (i) and (ii). □

Now define the lattice M in $\mathring{\mathfrak{h}}^*_{\mathbb{R}}$. Let $\mathbb{Z}(\mathring{W}\cdot\theta^\vee)$ denote the lattice in $\mathring{\mathfrak{h}}_{\mathbb{R}}$ generated over Z by the finite set $\mathring{W}\cdot\theta^\vee$, and set

$$M = \nu(\mathbb{Z}(\mathring{W}\cdot\theta^\vee)).$$

Lemma 6.4.2

(i) *If A is symmetric or $r > a_0$ then $M = \bar{Q} = \mathring{Q}$.*

(ii) *In all other cases $M = \nu(\overline{Q^\vee}) = \nu(\mathring{Q}^\vee)$.*

Proof If $r = 1$ then $\theta^\vee$ is a short root in $\mathring{\Delta}^\vee$, see Proposition 6.3.3. So $\mathring{W}\cdot\theta^\vee = \mathring{\Delta}^\vee_s$. It is known (Exercise 6.9 in Kac) that for the finite type the short roots generate the root lattice, so we have $M = \nu(\mathring{Q}^\vee)$, which implies the result for $r = 1$.

Similarly if $a_0 r = 2$ or 3, then $\theta^\vee$ is a long root in $\mathring{\Delta}^\vee$, so $\mathring{W}\cdot\theta^\vee = \mathring{\Delta}^\vee_l$, whence $M = \mathring{Q}$. Finally, for $A^{(2)}_{2\ell}$ we have $\nu(\theta^\vee) = \frac{1}{2}\theta$. Hence $M = \frac{1}{2}\mathbb{Z}\mathring{\Delta}_l = \mathring{Q}$ again. □

Corollary 6.4.3

(i) *If A is not of types $B^{(1)}_\ell, C^{(1)}_\ell, F^{(1)}_4, G^{(1)}_2, A^{(2)}_{2\ell}$, then $M = \mathbb{Z}\mathring{\Delta} = \sum_{i=1}^{\ell}\mathbb{Z}\alpha_i$.*

(ii) *If A is of types $B^{(1)}_\ell, C^{(1)}_\ell, F^{(1)}_4, G^{(1)}_2$, then*

$$M = \mathbb{Z}\mathring{\Delta}_l = \sum_{\alpha_i\in\mathring{\Delta}_l}\mathbb{Z}\alpha_i + \sum_{\alpha_i\in\mathring{\Delta}_s}\mathbb{Z}p\alpha_i,$$

where $p = 3$ for $G^{(1)}_2$ and 2 in the other cases.

(iii) *If A is of type $A^{(2)}_{2\ell}$, then*

$$M = \frac{1}{2}\mathbb{Z}\mathring{\Delta}_l = \sum_{i=1}^{\ell-1}\mathbb{Z}\alpha_i + \mathbb{Z}\frac{1}{2}\alpha_\ell.$$

The lattice M considered as an abelian group acts on $\mathfrak{h}^*$ by the formula (6.10). This action is faithful in view of (6.11),(6.12).

Denote the corresponding subgroup of $GL(\mathfrak{h}^*)$ by T and call it the *group of translations*. In view of (6.9) and Proposition 6.4.1(ii), T is a subgroup of W.

Proposition 6.4.4 $W = T \rtimes \mathring{W}$.

Proof Since $\mathring{W}$ is finite and T is a free abelian group, we have $\mathring{W} \cap T = \{1\}$. Moreover, $r_0 = t_{\nu(\theta^\vee)} r_\theta \in T\, \mathring{W}$, so T and $\mathring{W}$ generate W. Finally T is normal in W by Proposition 6.4.1(ii). □

Observe that $t_{\nu(\theta^\vee)} = r_0 r_\theta$ has determinant 1, and since T is generated by the elements $w t_{\nu(\theta^\vee)} w^{-1}$, all elements of T have determinant 1.

For $s \in \mathbb{R}$ set

$$\mathfrak{h}_s^* := \{\lambda \in \mathfrak{h}_\mathbb{R}^* \mid \langle \lambda, c\rangle = 1\}.$$

Note that $\mathfrak{h}_s^*$ is W-invariant, so W acts on $\mathfrak{h}_s^*$ with affine transformations. The elements of $\mathfrak{h}_1^*$ are of the form

$$\sum_{i=1}^{\ell} c_i \alpha_i + b\delta + \Lambda_0 \qquad (b, c_i \in \mathbb{R}).$$

Since W acts trivially on δ, the action of W on $\mathfrak{h}_1^*$ factors through to give an action of W on $\mathfrak{h}_1^*/\mathbb{R}\delta$. Note that the last space can be identified with

$$\mathbb{R}\alpha_1 + \cdots + \mathbb{R}\alpha_\ell = \mathring{\mathfrak{h}}_\mathbb{R}^*$$

via

$$\sum_{i=1}^{\ell} c_i \alpha_i + b\delta + \Lambda_0 \mapsto \sum_{i=1}^{\ell} c_i \alpha_i.$$

We use this identification to get an *affine* action of W on $\mathring{\mathfrak{h}}_\mathbb{R}^*$. The affine transformation of $\mathring{\mathfrak{h}}_\mathbb{R}^*$ corresponding to $w \in W$ is denoted by $\mathrm{af}(w)$, so that

$$\mathrm{af}(w)(\bar{\lambda}) = \overline{w(\lambda)} \qquad (\lambda \in \mathfrak{h}_1^*).$$

Proposition 6.4.5 *Let* $w \in \mathring{W}$, $m \in M$, $\lambda \in \mathring{\mathfrak{h}}_\mathbb{R}^*$. *Then* $\mathrm{af}(w)(\lambda) = w(\lambda)$, $\mathrm{af}(t_m)(\lambda) = \lambda + m$.

Proof The first statement follows from $w(\delta) = \delta, w(\Lambda_0) = \Lambda_0$. For the second one, using (6.11) and (6.12), we get

$$\mathrm{af}(t_m)(\lambda) = \overline{t_m(\lambda + \Lambda_0)} = \overline{\lambda - (\lambda|m)\delta + \Lambda_0 + m - \frac{1}{2}(m|m)\delta} = \lambda + m.$$

□

Corollary 6.4.6 *The affine action of* W *on* $\mathring{\mathfrak{h}}_\mathbb{R}^*$ *is faithful.*

Proof Suppose $t_m w \in W$ for $m \in M, w \in \overset{\circ}{W}$ acts trivially on $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$. Then $\mathrm{af}(t_m w)(0) = 0$ implies $m = 0$, i.e. $t_m = 1$. But $\overset{\circ}{W}$ acts faithfully on $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$, so $w = 1$ also. □

Corollary 6.4.7 *s_0 acts on $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$ as the reflection in the affine hyperplane*

$$T_{\theta,1} := \{\lambda \in \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}} \mid (\lambda|\theta) = 1\}.$$

Proof For $\lambda \in \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$, we have

$$r_0(\lambda) = t_{\nu(\theta^\vee)} r_\theta(\lambda) = r_\theta(\lambda) + \nu(\theta^\vee) = \lambda - \langle \lambda, \theta^\vee \rangle \theta + \nu(\theta^\vee) = \lambda - ((\lambda|\theta) - 1)\nu(\theta^\vee),$$

and the result follows. □

Define the *fundamental alcove*

$$C_{\mathrm{af}} = \{\lambda \in \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}} \mid (\lambda|\alpha_i) \geq 0 \text{ for } 1 \leq i \leq \ell \text{ and } (\lambda|\theta) \leq 1\}.$$

Proposition 6.4.8 *C_{af} is the fundamental domain for the action of W on $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$.*

Proof Consider the projection $\pi : \mathfrak{h}_1^* \to \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}, \lambda \mapsto \bar{\lambda}$. It is surjective and $\mathrm{af}(w) \circ \pi = \pi \circ w$ for $w \in W$. Moreover

$$\pi^{-1}(C_{\mathrm{af}}) = C^\vee \cap \mathfrak{h}_1^*.$$

It remains to note that $C^\vee \cap \mathfrak{h}_1^*$ is the fundamental domain for the W-action on $\mathfrak{h}_1^*$. □

We complete this section with the list of explicit constructions related to the root systems of finite type. We identify Q and $Q^\vee$ via the non-degenerate form $(\cdot|\cdot)$. Let $\mathbb{R}^n$ be the standard Euclidean space with standard basis $\varepsilon_1, \dots, \varepsilon_n$.

6.4.1 A_ℓ

$$
\begin{aligned}
&Q = Q^\vee = \{\sum_i k_i\varepsilon_i \in \mathbb{R}^{\ell+1} \mid k_i \in \mathbb{Z},\ \sum k_i = 0\},\\
&\Delta = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le \ell+1\},\\
&\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \le \ell \le \ell\},\\
&\theta = \varepsilon_1 - \varepsilon_{\ell+1},\\
&W \cong S_{\ell+1} = \{\text{all permutations of the } \varepsilon_i\}.
\end{aligned}
$$

6.4.2 D_ℓ

$$
\begin{aligned}
&Q = Q^\vee = \{\sum_i k_i\varepsilon_i \in \mathbb{R}^{\ell} \mid k_i \in \mathbb{Z},\ \sum k_i \in 2\mathbb{Z}\},\\
&\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le \ell\},\\
&\Pi = \{\alpha_1 = \varepsilon_1 - \varepsilon_2, \ldots, \alpha_{\ell-1} = \varepsilon_{\ell-1} - \varepsilon_\ell, \alpha_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell\},\\
&\theta = \varepsilon_1 + \varepsilon_2.
\end{aligned}
$$

6.4.3 E_8

$$
\begin{aligned}
&Q = Q^\vee = \{\sum_i k_i\varepsilon_i \in \mathbb{R}^{8} \mid \text{all } k_i \in \mathbb{Z} \text{ or all } k_i \in \frac{1}{2} + \mathbb{Z},\ \sum k_i \in 2\mathbb{Z}\},\\
&\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le 8\}\\
&\qquad \cup\{\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_8) \text{ even number of minuses}\},\\
&\Pi = \{\alpha_i = \varepsilon_{i+1} - \varepsilon_{i+2} \mid 1 \le i \le 6\}\\
&\qquad \cup\{\alpha_7 = \frac{1}{2}(\varepsilon_1 - \varepsilon_2 - \cdots - \varepsilon_7 + \varepsilon_8), \alpha_8 = \varepsilon_7 + \varepsilon_8\},\\
&\theta = \varepsilon_1 + \varepsilon_2.
\end{aligned}
$$

6.4.4 E_7

$$Q = Q^\vee = \{\sum_i k_i\varepsilon_i \in \mathbb{R}^8 \mid \text{all } k_i \in \mathbb{Z} \text{ or all } k_i \in \frac{1}{2} + \mathbb{Z},\ \sum k_i = 0\},$$

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \le i < j \le 8\}$$
$$\cup\{\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_8) \text{ (four minuses)}\},$$
$$\Pi = \{\alpha_i = \varepsilon_{i+1} - \varepsilon_{i+2} \mid 1 \le i \le 6\}$$
$$\cup\{\alpha_7 = \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4 + \varepsilon_5 + \varepsilon_6 + \varepsilon_7 + \varepsilon_8)\},$$
$$\theta = \varepsilon_2 - \varepsilon_1.$$

6.4.5 E_6

$$Q = Q^\vee = \{\sum_{i=1}^{6} k_i\varepsilon_i + \sqrt{2}k_7\varepsilon_7 \in \mathbb{R}^7 \mid \text{all } k_i \in \mathbb{Z} \text{ or all } k_i \in \frac{1}{2} + \mathbb{Z},\ \sum_{i=1}^{6} k_i = 0\},$$

$$\Delta = \{\pm\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le 6\}$$
$$\cup\{\frac{1}{2}(\pm\varepsilon_1 \pm \cdots \pm \varepsilon_6) \pm \sqrt{2}\varepsilon_7 \text{ (three minuses)}\} \cup \{\pm\sqrt{2}\varepsilon_7\},$$
$$\Pi = \{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid 1 \le i \le 5\}$$
$$\cup\{\alpha_6 = \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4 + \varepsilon_5 + \varepsilon_6 +) + \sqrt{2}\varepsilon_7)\},$$
$$\theta = \sqrt{2}\varepsilon_7.$$

7

Affine Algebras as Central extensions of Loop Algebras

7.1 Loop Algebras

Affine algebras of type $X_\ell^{(1)}$ are called *untwisted*. In this chapter we will describe an explicit construction of untwisted affine algebras. Recall the material from §1.5. In particular, $\mathcal{L} = \mathbb{C}[t, t^{-1}]$, and let φ be a bilinear form on $\mathcal{L}$ defined by

$$\varphi(P, Q) = \operatorname{Res} \frac{dP}{dt} Q.$$

One checks that

$$\varphi(P, Q) = -\varphi(Q, P), \tag{7.1}$$

$$\varphi(PQ, R) + \varphi(QR, P) + \varphi(RP, Q) = 0 \qquad (P, Q, R \in \mathcal{L}). \tag{7.2}$$

Note that Cartan matrix A of type $X_\ell^{(1)}$ is the so-called *extended Cartan matrix* of the simple finite dimensional Lie algebra $\mathring{\mathfrak{g}} = \mathfrak{g}(\mathring{A})$, where the matrix $\mathring{A}$ obtained from A by removing the 0th row and column is of type X_ℓ. Consider the loop algebra

$$\mathcal{L}(\mathring{\mathfrak{g}}) := \mathcal{L} \otimes \mathring{\mathfrak{g}} .$$

Fix a non-degenerate invariant symmetric bilinear form $(\cdot|\cdot)$ on $\mathring{\mathfrak{g}}$. It can be extended to a $\mathcal{L}$-valued bilinear form $(\cdot|\cdot)_t$ on $\mathcal{L}(\mathring{\mathfrak{g}})$ via

$$(P \otimes x | Q \otimes y)_t = PQ(x|y).$$

Also set

$$\psi(a, b) = (\frac{da}{dt} | b)_t \qquad (a, b \in \mathcal{L}(\mathring{\mathfrak{g}})).$$

We know from Lemma 1.5.3 that ψ is a 2-cocycle on $\mathcal{L}(\overset{\circ}{\mathfrak{g}})$, and

$$\psi(t^i \otimes x, t^j \otimes y) = i\delta_{i,-j}(x|y).$$

As in §1.5, we have a central extension

$$\bar{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$$

corresponding to ψ. Moreover, $\bar{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ is graded with $\deg t^j \otimes x = j$, $\deg c = 0$. We then have the corresponding derivation

$$d : \bar{\mathcal{L}}(\overset{\circ}{\mathfrak{g}}) \to \bar{\mathcal{L}}(\overset{\circ}{\mathfrak{g}}),\ t^j \otimes x \mapsto jt^j \otimes x,\ c \mapsto 0.$$

Finally, by adjoining d to $\bar{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ we get the Lie algebra

$$\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}}) := \mathcal{L}(\overset{\circ}{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with operation

$$\begin{aligned} & [t^m \otimes x + \lambda c + \mu d, t^n \otimes y + \lambda' c + \mu' d] \\ = \quad & (t^{m+n} \otimes [x,y] + \mu n t^n \otimes y - \mu' m t^m \otimes x) + m\delta_{m,-n}(x|y)c. \end{aligned}$$

7.2 Realization of untwisted algebras

Let $\overset{\circ}{\omega}$ be the Chevalley involution of $\overset{\circ}{\mathfrak{g}}$, $\overset{\circ}{\Delta} \subset \overset{\circ}{\mathfrak{h}}{}^*$ be the root system of $\overset{\circ}{\mathfrak{g}}$, $\{\alpha_1, \dots, \alpha_\ell\}$ be a root base of $\overset{\circ}{\Delta}$ and

$$H_1, \dots, H_\ell$$

be the coroot base in $\overset{\circ}{\mathfrak{h}}$, $E_1, \dots, E_\ell$, $F_1, \dots, F_\ell$ be the Chevalley generators, θ be the highest root in $\overset{\circ}{\Delta}$, and let

$$\overset{\circ}{\mathfrak{g}} = \bigoplus_{\alpha \in \overset{\circ}{\Delta} \cup \{0\}} \overset{\circ}{\mathfrak{g}}_\alpha$$

be the root space decomposition. Choose $F_0 \in \overset{\circ}{\mathfrak{g}}_\theta$ so that

$$(F_0 | \overset{\circ}{\omega} (F_0)) = -\frac{2}{(\theta|\theta)},$$

and set

$$E_0 = -\overset{\circ}{\omega} (F_0).$$

Then by Theorem 2.2.3(v), we have

$$[E_0, F_0] = -\theta^\vee. \tag{7.3}$$

The elements $E_0, E_1, \ldots, E_\ell$ generate the algebra $\overset{\circ}{\mathfrak{g}}$ since in the adjoint representation we have $\overset{\circ}{\mathfrak{g}} = U(\overset{\circ}{\mathfrak{n}}_+)(E_0)$.

Return to the algebra $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$. It is clear that $\mathbb{C}c$ is the (1-dimensional) center of $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$, and the centralizer of d in $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ is the direct sum of Lie algebras $\mathbb{C}c \oplus \mathbb{C}d \oplus (1 \otimes \overset{\circ}{\mathfrak{g}})$. From now on we identify $\overset{\circ}{\mathfrak{g}}$ with the subalgebra $1 \otimes \overset{\circ}{\mathfrak{g}} \subset \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$. Further,

$$\mathfrak{h} := \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d$$

is an $(\ell+2)$-dimensional abelian subalgebra of $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$. Continue $\lambda \in \overset{\circ}{\mathfrak{h}}{}^*$ to a linear function on $\mathfrak{h}$ by setting $\langle \lambda, c \rangle = \langle \lambda, d \rangle = 0$, so $\overset{\circ}{\mathfrak{h}}{}^*$ gets identified with a subspace of $\mathfrak{h}^*$. Denote by δ the linear function on $\mathfrak{h}$ defined from

$$\delta|_{\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c} = 0, \quad \langle \delta, d \rangle = 1.$$

Set

$$\begin{aligned} e_0 &= t \otimes E_0, \\ f_0 &= t^{-1} \otimes F_0, \\ e_i &= 1 \otimes E_i \qquad (1 \le i \le \ell) \\ f_i &= 1 \otimes F_i \qquad (1 \le i \le \ell). \end{aligned}$$

From (7.3) we get

$$[e_0, f_0] = \frac{2}{(\theta|\theta)} c - \theta^\vee. \tag{7.4}$$

Note the following facts on the root decomposition

$$\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}}) = \bigoplus_{\alpha \in \Delta \cup \{0\}} \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})_\alpha$$

with respect to $\mathfrak{h}$:

$$\begin{aligned} \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})_0 &= \mathfrak{h}, \\ \Delta &= \{ j\delta + \gamma \mid j \in \mathbb{Z},\ \gamma \in \overset{\circ}{\Delta} \} \cup \{ j\delta \mid j \in \mathbb{Z} \setminus \{0\} \}, \\ \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})_{j\delta+\gamma} &= t^j \otimes \overset{\circ}{\mathfrak{g}}_\gamma, \\ \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})_{j\delta} &= t^j \otimes \overset{\circ}{\mathfrak{h}}. \end{aligned}$$

Set

$$\begin{aligned} \Pi &= \{\alpha_0 := \delta - \theta, \alpha_1, \ldots, \alpha_\ell\}, \\ \Pi^\vee &= \{\alpha_0^\vee := \frac{2}{(\theta|\theta)} c - \theta^\vee, \alpha_1^\vee := 1 \otimes H_1, \ldots, \alpha_\ell^\vee := 1 \otimes H_\ell\}. \end{aligned}$$

Note in view of Proposition 6.3.3(i) that our element θ agrees with the one introduced in (6.8), whence

$$A = (\langle \alpha_i^\vee, \alpha_j \rangle)_{0 \le i,j \le \ell}. \tag{7.5}$$

So $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of A.

Theorem 7.2.1 *$\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ is affine Kac-Moody algebra $\mathfrak{g}(A)$, $\mathfrak{h}$ is its Cartan matrix, Π and $\Pi^\vee$ are its root and coroot bases, and $e_0, \ldots, e_\ell, f_0, \ldots, f_\ell$ are Chevalley generators.*

Proof We apply Proposition 1.5.1. All relations are easy to check (or have already been checked).

Further, we will prove that $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ has no non-trivial ideals $\mathfrak{i}$ with $\mathfrak{i} \cap \mathfrak{h} = \{0\}$. Indeed, if $\mathfrak{i}$ is such an ideal, then by the Weight Lemma, $\mathfrak{i} \cap \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})_\alpha \neq \{0\}$ for some $\alpha = j\delta + \gamma \in \Delta$. So some $t^j \otimes x \in \mathfrak{i}$ for some $j \in \mathbb{Z}$ and $x \in \overset{\circ}{\mathfrak{g}}_\gamma$, $x \neq 0$, $\gamma \in \Delta \cup \{0\}$. By taking $y \in \overset{\circ}{\mathfrak{g}}_{-\gamma}$ such that $(x|y) \neq 0$, we get

$$[t^j \otimes x, t^{-j} \otimes y] = j(x|y)c + [x, y] \in \mathfrak{h} \cap \mathfrak{i}.$$

Hence $j(x|y)c + [x, y] = 0$. Since $[x, y] \in \overset{\circ}{\mathfrak{h}}$ we deduce that $j = 0$. Since $\alpha = j\delta + \gamma \neq 0$, we have $\gamma \neq 0$. Then $0 \neq [x, y] \in \overset{\circ}{\mathfrak{h}} \cap \mathfrak{i}$. Contradiction.

Finally we prove that the e_i, f_i and $\mathfrak{h}$ generate $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$. Let $\mathfrak{g}_1$ be the subalgebra in $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ generated by the e_i, f_i and $\mathfrak{h}$. Since $E_1, \ldots, E_\ell, F_1, \ldots, F_\ell$ generate $\overset{\circ}{\mathfrak{g}}$, we deduce $1 \otimes \overset{\circ}{\mathfrak{g}} \subset \mathfrak{g}_1$. Let

$$\mathfrak{i} = \{x \in \overset{\circ}{\mathfrak{g}} \mid t \otimes x \in \mathfrak{g}_1\}.$$

Then $e_0 = t \otimes E_0 \in \mathfrak{g}_1$, so $E_0 \in \mathfrak{i}$, and $\mathfrak{i} \neq 0$. Also, if $x \in \mathfrak{i}, y \in \overset{\circ}{\mathfrak{g}}$, then

$$t \otimes [x, y] = [t \otimes x, 1 \otimes y] \in \mathfrak{g}_1,$$

whence $\mathfrak{i}$ is an ideal of $\overset{\circ}{\mathfrak{g}}$. Since $\overset{\circ}{\mathfrak{g}}$ is simple we have $\mathfrak{i} = \overset{\circ}{\mathfrak{g}}$ or $t \otimes \overset{\circ}{\mathfrak{g}} \subset \mathfrak{g}_1$. We may now use the relation

$$[t \otimes x, t^{k-1} \otimes y] = t^k \otimes [x, y]$$

to deduce by induction on k that $t^k \otimes \overset{\circ}{\mathfrak{g}} \subset \mathfrak{g}_1$ for all $k > 0$. In an analogous way, starting with $f_0 = t^{-1} \otimes F_0$ we can show that $t^{-k} \otimes \overset{\circ}{\mathfrak{g}} \subset \mathfrak{g}_1$ for all $k > 0$. □

Corollary 7.2.2 *Let $\mathfrak{g}$ be a non-twisted affine Lie algebra of rank $\ell + 1$. Then the multiplicity of each imaginary root in $\mathfrak{g}$ is ℓ.*

Let $(\cdot|\cdot)$ be the normalized invariant form on $\overset{\circ}{\mathfrak{g}}$ (see the end of §6.3). Extend it to $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ by

$$\begin{aligned} (P \otimes x | Q \otimes y) &= (\operatorname{Res} t^{-1} PQ)(x|y) \qquad (x, y \in \overset{\circ}{\mathfrak{g}},\ P, Q \in \mathcal{L}), \\ (\mathbb{C}c + \mathbb{C}d | \mathcal{L}(\overset{\circ}{\mathfrak{g}})) &= 0, \\ (c|c) &= 0, \\ (d|d) &= 0, \\ (c|d) &= 1. \end{aligned}$$

The definition implies

$$(t^i \otimes x | t^j \otimes y) = \delta_{i,-j}(x|y).$$

We get a non-degenerate symmetric bilinear form. In order to check invariance, let us consider the only non-trivial case:

$$([d, P \otimes x] | Q \otimes y) = (d | [P \otimes x, Q \otimes y]).$$

The left hand side of this equality is

$$(t\frac{dP}{dt} \otimes x | Q \otimes y) = (\operatorname{Res} \frac{dP}{dt} Q)(x|y),$$

while the right hand side is

$$(d | PQ \otimes [x, y] + (\operatorname{Res} \frac{dP}{dt} Q)(x|y)c) = (\operatorname{Res} \frac{dP}{dt} Q)(x|y).$$

Finally, the restriction of $(\cdot|\cdot)$ to $\mathfrak{h}$ agrees with the form defined in §6.2.

Note that the element c is the canonical central element and d is the energy element.

Let $\overset{\circ}{\mathfrak{g}} = \overset{\circ}{\mathfrak{n}}_- \oplus \overset{\circ}{\mathfrak{h}} \oplus \overset{\circ}{\mathfrak{n}}_+$ be the canonical triangular decomposition of $\overset{\circ}{\mathfrak{g}}$. Then the triangular decomposition of $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$ is

$$\begin{aligned} \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}}) &= \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \text{where} \\ \mathfrak{n}_- &= (t^{-1}\mathbb{C}[t^{-1}] \otimes (\overset{\circ}{\mathfrak{n}}_+ \oplus \overset{\circ}{\mathfrak{h}})) \oplus \mathbb{C}[t^{-1}] \otimes \overset{\circ}{\mathfrak{n}}_-, \\ \mathfrak{n}_+ &= (t\mathbb{C}[t] \otimes (\overset{\circ}{\mathfrak{n}}_- \oplus \overset{\circ}{\mathfrak{h}})) \oplus \mathbb{C}[t] \otimes \overset{\circ}{\mathfrak{n}}_+ . \end{aligned}$$

The Chevalley involution of $\mathfrak{g}$ can be written in terms of $\overset{\circ}{\omega}$ as follows

$$\omega(P(t)\otimes x+\lambda c+\mu d)=P(t^{-1})\otimes \overset{\circ}{\omega}(x)-\lambda c-\mu d.$$

Set

$$\mathfrak{t}=\mathbb{C}c+\sum_{s\in\mathbb{Z}\setminus\{0\}}\mathfrak{g}_{s\delta}.$$

Then $\mathfrak{t}$ is isomorphic to the infinite dimensional Heisenberg algebra with center $\mathbb{C}c$. Indeed, $\mathfrak{t}=\mathbb{C}c\oplus_{s\in\mathbb{Z}\setminus\{0\}}t^s\otimes\mathfrak{h}$, and the only non-trivial commutation is

$$[t^s\otimes h,t^{-s}\otimes h']=s(h|h').$$

7.3 Explicit Construction of Finite Dimensional Lie Algebras

Let Q be the root lattice of type A_ℓ, D_ℓ, or E_ℓ, and let $(\cdot|\cdot)$ be the normalized form on Q, i.e.

$$\Delta=\{\alpha\mid(\alpha|\alpha)=2\}.$$

We then also have $(\alpha|\alpha)\in 2\mathbb{Z}$ for all $\alpha\in Q$ (explicit check). Let

$$\varepsilon:Q\times Q\to\{\pm1\}$$

be a function satisfying the "bilinearity" condition for all $\alpha,\alpha'\beta,\beta'\in Q$:

$$\varepsilon(\alpha+\alpha',\beta)=\varepsilon(\alpha,\beta)\varepsilon(\alpha',\beta),\quad \varepsilon(\alpha,\beta+\beta')=\varepsilon(\alpha,\beta)\varepsilon(\alpha,\beta'),\tag{7.6}$$

and the condition

$$\varepsilon(\alpha,\alpha)=(-1)^{(\alpha|\alpha)/2}\qquad(\alpha\in Q).\tag{7.7}$$

We call such ε an *asymmetry function*. Substituting $\alpha+\beta$ to the last equation we get

$$\varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)=(-1)^{(\alpha|\beta)}\qquad(\alpha,\beta\in Q).\tag{7.8}$$

An asymmetry function can be constructed as follows: choose an orientation of the Dynkin diagram, and let

$$\begin{aligned}\varepsilon(\alpha_i,\alpha_j)&=-1\quad\text{if } i=j \text{ or if } \overset{i}{\circ}\!\rightarrow\!\overset{j}{\circ}\\ \varepsilon(\alpha_i,\alpha_j)&=1\quad\text{otherwise, i.e. if } \overset{i}{\circ}\!\leftarrow\!\overset{j}{\circ}\text{ or }\overset{i}{\circ}\ \ \overset{j}{\circ},\end{aligned}$$

and extend by bilinearity. An easy check shows that the required conditions are satisfied.

Now let us $\mathfrak{h}$ be the complex hull of Q and extend $(\cdot|\cdot)$ to $\mathfrak{h}$. Take the direct sum of $\mathfrak{h}$ with 1-dimensional vector spaces $\mathbb{C}E_\alpha$, one for each $\alpha \in \Delta$:

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha\in\Delta} \mathbb{C}E_\alpha).$$

Define the bracket on $\mathfrak{g}$ as follows:

$$\begin{cases} [h,h'] = 0 & \text{if } h,h' \in \mathfrak{h} \\ [h,E_\alpha] = (h|\alpha)E_\alpha & \text{if } h \in \mathfrak{h},\ \alpha \in \Delta \\ [E_\alpha, E_{-\alpha}] = -\alpha & \text{if } \alpha \in \Delta \\ [E_\alpha, E_\beta] = 0 & \text{if } \alpha,\beta \in \Delta,\ \alpha+\beta \notin \Delta \cup \{0\} \\ [E_\alpha, E_\beta] = \varepsilon(\alpha,\beta)E_{\alpha+\beta} & \text{if } \alpha,\beta,\alpha+\beta \in \Delta \end{cases} \tag{7.9}$$

Define the symmetric bilinear form on $\mathfrak{g}$ extending it from $\mathfrak{h}$ as follows:

$$\begin{cases} (h|E_\alpha) = 0 & \text{if } h \in \mathfrak{h},\ \alpha \in \Delta \\ (E_\alpha|E_\beta) = -\delta_{\alpha,-\beta} & \text{if } \alpha,\beta \in \Delta \end{cases} \tag{7.10}$$

Proposition 7.3.1 $\mathfrak{g}$ *is the simple Lie algebra of type A_ℓ, D_ℓ or E_ℓ, respectively, the form $(\cdot|\cdot)$ being the normalized invariant form.*

Proof To check the skew-commutativity it suffices to prove that $[E_\alpha, E_\beta] = -[E_\beta, E_\alpha]$ when $\alpha, \beta, \alpha+\beta \in \Delta$. Note that

$$\alpha \pm \beta \in \Delta \ \Leftrightarrow\ (\alpha|\beta) = \mp 1 \qquad (\alpha,\beta \in \Delta). \tag{7.11}$$

Now the required equality follows from (7.8).

Next we check Jacobi identity for three basis elements x, y, z. If one of these elements is in $\mathfrak{h}$, the Jacobi identity trivially holds. So let $x = E_\alpha, y = E_\beta, z = E_\gamma$. If $\alpha+\beta, \alpha+\gamma, \beta+\gamma \notin \Delta \cup \{0\}$, the identity holds trivially, so we may assume that $\alpha + \beta \in \Delta \cup \{0\}$.

If $\alpha + \beta = 0$, consider four cases:

(1) $\alpha \pm \gamma \notin \Delta \cup \{0\}$;

(2) $\alpha + \gamma$ or $\alpha - \gamma = 0$;

(3) $\alpha + \gamma \in \Delta$;

(4) $\alpha - \gamma \in \Delta$.

The Jacobi identity holds in cases (1) and (2) in view of (7.11). In case (3) it reduces to $\varepsilon(-\alpha, \alpha+\gamma)\varepsilon(\alpha,\gamma) = (\alpha|\gamma)$, which follows from the bilinearity and (7.7). The case (4) is similar.

Thus we may assume that $\alpha+\beta, \alpha+\gamma, \beta+\gamma \in \Delta$, for the remaining cases follow either trivially or from bilinearity of ε. So $(\alpha|\beta) = (\alpha|\gamma) =$

$(\beta|\gamma) = -1$, whence $|\alpha+\beta+\gamma|^2 = 0$, so $\alpha+\beta+\gamma = 0$ using positive definiteness of the form. So Jacobi identity boils down to

$$\varepsilon(\beta,\gamma)(\beta+\gamma) = -\varepsilon(\alpha,\beta)(\alpha+\beta) + \varepsilon(\alpha,\gamma)(\alpha+\gamma),$$

which holds by bilinearity again.

Thus $\mathfrak{g}$ is a Lie algebra. Let

$$\Pi = \Pi^\vee = \{\alpha_1, \ldots, \alpha_\ell\},\ e_i = E_{\alpha_i},\ f_i = -E_{-\alpha_i}.$$

We can now apply Proposition 1.5.1. To check that the form is invariant is straightforward. $\square$

8

Twisted Affine Algebras and Automorphisms of Finite Order

8.1 Graph Automorphisms

Let A be of finite type X_N. Let σ be a permutation of $\{1, \dots, N\}$ such that $a_{\sigma(i)\sigma(j)} = a_{ij}$. Such σ can be thought of as a graph automorphism of the Dynkin diagram of A. Let $\mathfrak{g} = \mathfrak{g}(A)$. It is clear that such graph automorphism defines an automorphism, denoted by the same letter σ and called *graph automorphism* of $\mathfrak{g}$:

$$\sigma : \mathfrak{g} \to \mathfrak{g},\ e_i \mapsto e_{\sigma(i)},\ f_i \mapsto f_{\sigma(i)},\ \alpha_i^\vee \mapsto \alpha_{\sigma(i)}^\vee.$$

The interesting graph automorphisms are listed in Figure 8.1:

Our first main goal is to determine the fixed point subalgebra $\mathfrak{g}^\sigma$. We consider the linear action of σ on $V := \mathfrak{h}_{\mathbb{R}}^*$ given from

$$\sigma(\alpha_i) = \alpha_{\sigma(i)} \qquad (1 \le i \le N).$$

For each orbit J of σ on $\{1, \dots, N\}$ define

$$\alpha_J := \frac{1}{|J|} \sum_{j \in J} \alpha_j. \tag{8.1}$$

Then the α_J form a basis of V^σ as J runs over the σ-orbits on $\{1, \dots, N\}$. Note that α_J is the orthogonal projection of α_j onto V^σ. We see from Figure 8.1 that the following possible situations for the orbits J are possible:

(1) $|J| = 1$;
(2) $|J| = 2$, $J = \{j, j'\}$, $\alpha_j + \alpha_{j'} \notin \Delta$;
(3) $|J| = 3$, $J = \{j, j', j''\}$, $\alpha_j + \alpha_{j'}, \alpha_j + \alpha_{j''}, \alpha_{j'} + \alpha_{j''} \notin \Delta$;
(4) $|J| = 2$, $J = \{j, j'\}$, $\alpha_j + \alpha_{j'} \in \Delta$.

These are referred to as orbits of types A_1, $A_1 \times A_1$, $A_1 \times A_1 \times A_1$, and A_2, respectively.

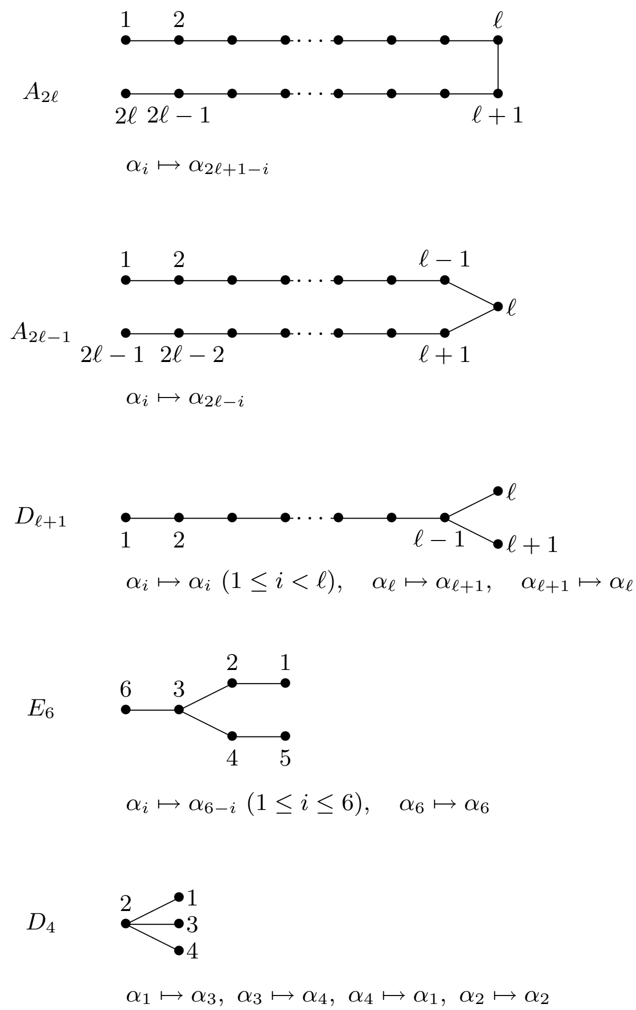
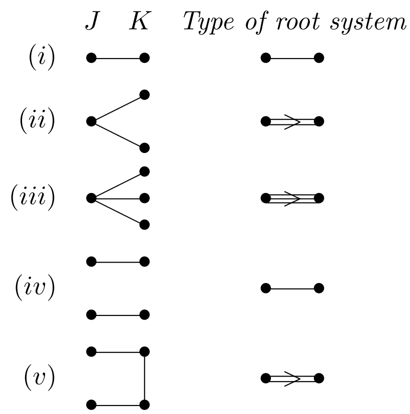


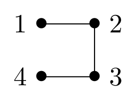
Fig. 8.1. Graph automorphisms

Lemma 8.1.1 *The vectors α_J, α_K for distinct σ-orbits J, K form a base* *of a root system of rank 2 as follows:*



Finally, if no node in J is connected to any node in K then the type of the root system is $A_1 \times A_1$.

Proof This is an easy calculation. Suppose for example that we have case (v) with roots numbered



Then

$$\alpha_J = \frac{\alpha_1 + \alpha_4}{2}, \alpha_K = \frac{\alpha_2 + \alpha_3}{2}.$$

So

$$\begin{aligned} (\alpha_J|\alpha_J) &= \frac{1}{2}(\alpha_1|\alpha_1), \\ (\alpha_K|\alpha_K) &= \frac{1}{4}(\alpha_1|\alpha_1), \\ (\alpha_J|\alpha_K) &= -\frac{1}{4}(\alpha_1|\alpha_1), \end{aligned}$$

This is what was claimed. □

Corollary 8.1.2 *Let Π^σ be the set of vectors α_J for all σ-orbits J on*

$\{1, \ldots, N\}$. *Then* Π^σ *is a base of a root system of the following type:*

Type Π	Order of σ	Type Π^σ
$A_{2\ell}$	2	B_ℓ
$A_{2\ell-1}$	2	C_ℓ
$D_{\ell+1}$	2	B_ℓ
D_4	3	G_2
E_6	2	F_4

Now let Δ^σ be the root system in V^σ with base Π^σ and Weyl group W^σ and Cartan matrix A^σ. In view of Lemma 8.1.1 and Corollary 8.1.2, we know the type of this Cartan matrix and:

Lemma 8.1.3 *Let* I, J *be distinct* σ*-orbits* σ*-orbits on* $\{1, \ldots, N\}$. *Then*

$$a^\sigma_{IJ} = \begin{cases} \sum_{i\in I} a_{ij} & \text{for any } j \in J \text{ if } I \text{ has type } A_1,\ A_1 \times A_1 \text{ or } A_1 \times A_1 \times A_1 \\ 2\sum_{i\in I} a_{ij} & \text{for any } j \in J \text{ if } I \text{ has type } A_2 \end{cases}$$

Lemma 8.1.4 *There is an isomorphism* $W^\sigma \to W^1 := \{w \in W \mid w\sigma = \sigma w\}$ *under which the fundamental reflection* $r_J \in W^\sigma$ *corresponding to* α_J *maps to* $(w_0)_J \in W$, *the element of maximal length in the Weyl group* W_J *generated by the* $r_i, i \in J$.

Proof Observe first that W^1 acts on V^σ. Next, we claim that $(w_0)_J \in W^1$ for each J. Indeed, $\sigma r_j \sigma^{-1} = r_{\sigma(j)}$ implies $\sigma W_J \sigma^{-1} = W_J$, so σ induces the length preserving automorphism of W_J, hence $(w_0)_J$ is invariant.

Now, note that $(w_0)_J|_{V^\sigma} = r_J$. Indeed, using the defining property of the longest element we have

$$(w_0)_J(\alpha_J) = (w_0)_J(\frac{1}{|J|}\sum_{j\in J}\alpha_j) = -\frac{1}{|J|}\sum_{j\in J}\alpha_j = -\alpha_J$$

Moreover, if $v \in V^\sigma$ and $(\alpha_J|v) = 0$ then $(\alpha_j|v) = 0$ for all $j \in J$, whence $(w_0)_J(v) = v$.

Next, we show that the elements $(w_0)_J$ generate W^1. Take $w \neq 1$ in W^1. Then there exists a simple root α_j with $w(\alpha_j) < 0$. Let J be the σ-orbit of j. Then $w(\alpha_i) < 0$ for all $i \in J$. Now $(w_0)_J$ changes the signs of all roots in Δ_J but of none in $\Delta \setminus \Delta_J$. Hence $\ell(w(w_0)_J) < \ell(w)$. Now apply induction on the length.

We may now define a homomorphism $W^1 \to W^\sigma$ by restricting the action of $w \in W^1$ from V to V^σ, which maps $(w_0)_J$ to r_J and so is surjective. To see that the homomorphism is injective, take $w \neq 1$ in W^1. We saw that there exists a σ-orbit J such that $w(\alpha_i) < 0$ for all $i \in J$, whence $w(\alpha_J) \neq \alpha_J$. □

From now on we identify W^σ and W^1.

For each $\alpha \in \Delta$ denote by α^σ its orthogonal projection into V^σ.

Lemma 8.1.5

(i) *For each $\alpha \in \Delta$, α^σ is a positive multiple of a root in Δ^σ.*

(ii) *Let $\sim$ be the equivalence relation on Δ given by $\alpha \sim \beta \Leftrightarrow \alpha^\sigma$ is a positive multiple of β^σ. Then the equivalence classes are the subsets of Δ of the form $w(\Delta_J^+)$ where $w \in W^\sigma$ and J is a σ-orbit on $\{1, \ldots, N\}$.*

(iii) *There is a bijection between equivalence classes in Δ and roots in Δ^σ given by $w(\Delta_J^+) \mapsto w(\alpha_J)$.*

Proof We first show that each $\alpha \in \Delta$ lies in $w(\Delta_J^+)$ for some $w \in W^\sigma$ and some σ-orbit J. We have $\sigma w_0 \sigma^{-1} = w_0$, so $w_0 \in W^\sigma$. By Lemma 8.1.4 the elements $(w_0)_J$ generate W^σ, and so we can write $w_0 = (w_0)_{J_1} \ldots (w_0)_{J_r}$. Let $\alpha \in \Delta^+$. Then $w_0(\alpha) \in \Delta^-$. Thus there exists i such that

$$(w_0)_{J_{i+1}} \ldots (w_0)_{J_r}(\alpha) \in \Delta^+,$$

but

$$(w_0)_{J_i}(w_0)_{J_{i+1}} \ldots (w_0)_{J_r}(\alpha) \in \Delta^-.$$

Hence

$$(w_0)_{J_{i+1}} \ldots (w_0)_{J_r}(\alpha) \in \Delta_{J_i}^+,$$

that is

$$\alpha \in (w_0)_{J_r} \ldots (w_0)_{J_{i+1}}(\Delta_{J_i}^+),$$

and

$$-\alpha \in (w_0)_{J_r} \ldots (w_0)_{J_{i+1}}(w_0)_{J_i}(\Delta_{J_i}^+).$$

Now consider the projection α^σ for $\alpha \in \Delta_J^+$. If J has type $A_1, A_1 \times A_1$ or $A_1 \times A_1 \times A_1$ then $\Delta_J^+ = \Pi_J$, so $\alpha^\sigma = \alpha_J$. If J has type A_2, then $\Pi_J = \{\alpha_j, \alpha_{j'}\}$ and $\Delta_J^+ = \{\alpha_j, \alpha_{j'}, \alpha_j + \alpha_{j'}\}$, and

$$\alpha^\sigma = \begin{cases} \alpha_J & \text{if } \alpha = \alpha_j \text{ or } \alpha_{j'}, \\ 2\alpha_J & \text{if } \alpha = \alpha_j + \alpha_{j'}. \end{cases}$$

Thus for $\alpha \in \Delta_J^+$, we know that α^σ is a positive multiple of α_J. Hence for $\alpha \in w(\Delta_J^+)$ with $w \in W^\sigma$ we know that α^σ is a positive multiple of $w(\alpha_J) \in \Delta^\sigma$, proving (i).

We now know that the elements of each set $w(\Delta_J^+)$ for $w \in W^\sigma$ lie in the same equivalence class. Suppose $w(\Delta_J^+)$ and $w'(\Delta_K^+)$ lie in the same class for $w, w' \in W^\sigma$ and orbits J, K. Then $w(\alpha_J) = w'(\alpha_K) \in \Delta^\sigma$ or $w'^{-1}w(\alpha_J) = \alpha_K$. Consider the root $w'^{-1}w(\alpha_j) \in \Delta$ for $j \in J$. The root has the property that $(w'^{-1}w(\alpha_j))^\sigma = \alpha_K$. So $w'^{-1}w(\alpha_j)$ is a non-negative linear combination of the α_k for $k \in K$. Hence $w'^{-1}w(\Pi_J) \subset \Delta_K^+$, and so $w'^{-1}w(\Delta_J^+) \subset \Delta_K^+$. By symmetry we also have $w'w^{-1}(\Delta_K^+) \subset \Delta_J^+$. Hence we have equality, that is $w(\Delta_J^+) = w'(\Delta_K^+)$, which completes the proof of (ii).

Now, any root in Δ^σ has form $w(\alpha_J)$ for some $w \in W^\sigma$ and some σ-orbit J. The set of the roots $\alpha \in \Delta$ such that α^σ is a positive multiple of $w(\alpha_J)$ is $w(\Delta_J^+)$, as shown above. Thus $w(\Delta_J^+) \mapsto w(\alpha_J)$ is a bijection between equivalence classes of Δ and elements of Δ^σ, giving (iii). □

Theorem 8.1.6 *Let A be of finite type, and σ be a graph automorphism of $\mathfrak{g} = \mathfrak{g}(A)$. Then $\mathfrak{g}^\sigma$ is isomorphic to $\mathfrak{g}(A^\sigma)$.*

Proof For each σ-orbit J on $\{1, \dots, N\}$ we define elements $e_J, f_J, \alpha_J^\vee$ of $\mathfrak{g}^\sigma$ by

$$e_J = \sum_{j \in J} e_j, \quad f_J = \sum_{j \in J} f_j, \quad \alpha_J^\vee = \sum_{j \in J} \alpha_j^\vee$$

if J is of type $A_1, A_1 \times A_1$ or $A_1 \times A_1 \times A_1$, and

$$e_J = \sqrt{2} \sum_{j \in J} e_j, \quad f_J = \sqrt{2} \sum_{j \in J} f_j, \quad \alpha_J^\vee = 2 \sum_{j \in J} \alpha_j^\vee$$

if J is of type A_2. Then the $\alpha_J^\vee$ form a basis of $\mathfrak{h}^\sigma$. One checks using Lemma 8.1.3 that $\mathfrak{h}^\sigma$ together with $\Pi = \{\alpha_J\}, \Pi^\vee = \{\alpha_J^\vee\}$ give a realization of A^σ and the relations (1.12-1.15) hold.

Thus the subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}^\sigma$ generated by the elements $e_J, f_J, \alpha_J^\vee$ is a quotient of $\tilde{\mathfrak{g}}(A^\sigma)$. Since the dimension of $\mathfrak{h}^\sigma$ is the same as the dimension of the Cartan subalgebra of $\tilde{\mathfrak{g}}(A^\sigma)$, we deduce that $\mathfrak{g}_1$ is the quotient of $\tilde{\mathfrak{g}}(A^\sigma)$ by an ideal whose intersection with the Cartan subalgebra is trivial. But we know that among such ideals there is the largest one $\mathfrak{r}$ so that $\tilde{\mathfrak{g}}(A^\sigma)/\mathfrak{r} \cong \mathfrak{g}(A^\sigma)$, the simple finite dimensional Lie algebra of type A^σ. Moreover, the root spaces of $\mathfrak{g}(A^\sigma)$ are 1-dimensional. Now it follows that $\mathfrak{g}_1 = \mathfrak{g}^\sigma$ and it is isomorphic to $\mathfrak{g}(A^\sigma)$ by dimension considerations. Indeed, consider the decomposition of Δ into equivalence classes given by Lemma 8.1.5. For each equivalence class S let

$$\mathfrak{g}_S = \oplus_{\alpha\in S}\mathfrak{g}_\alpha.$$

Then $\sigma(\mathfrak{g}_S) = \mathfrak{g}_S$, and $\mathfrak{g} = \mathfrak{h}^\sigma \oplus \sum_S \mathfrak{g}_S^\sigma$. Now $\dim \mathfrak{g}_S^\sigma \leq 1$ for each equivalence class S. This is clear if S has type A_1, $A_1 \times A_1$ or $A_1 \times A_1 \times A_1$. Suppose S has type A_2. Then $S = \{\alpha, \beta, \alpha+\beta\}$ with $\sigma(\mathfrak{g}_\alpha) = \mathfrak{g}_\beta$, $\sigma(\mathfrak{g}_\beta) = \mathfrak{g}_\alpha$, $\sigma(\mathfrak{g}_{\alpha+\beta}) = \mathfrak{g}_{\alpha+\beta}$. Take non-zero $e_\alpha \in \mathfrak{g}_\alpha, e_\beta \in \mathfrak{g}_\beta$. Then $\sigma(e_\alpha) = \lambda e_\beta, \sigma(e_\beta) = \lambda^{-1}e_\alpha$. Hence $\sigma([e_\alpha, e_\beta]) = [\lambda e_\beta, \lambda^{-1}e_\alpha] = -[e_\alpha, e_\beta]$. It follows that $\mathfrak{g}_S^\sigma = \mathbb{C}(e_\alpha + \lambda e_\beta)$. We thus have

$$\dim \mathfrak{g}^\sigma \leq \dim \mathfrak{h}^\sigma + |\Delta/\sim| = \dim \mathfrak{h}^\sigma + |\Delta^\sigma| = \dim \mathfrak{g}(A^\sigma),$$

which completes the proof. □

If σ is of order r (recall that $r = 2$ or 3) set $\eta = e^{2\pi i/r}$, and $\mathfrak{g}^{(i)}$ be the η^i-eigenspace of σ on $\mathfrak{g}$ for $0 \leq i < r$. Note that $\mathfrak{g}^{(0)} = \mathfrak{g}^\sigma$, and

$$\mathfrak{g} = \bigoplus_{0\leq i<r} \mathfrak{g}^{(i)},$$

is a $\mathbb{Z}/r\mathbb{Z}$ grading. In particular, each $\mathfrak{g}^{(i)}$ is a $\mathfrak{g}^\sigma$-module.

Proposition 8.1.7 *$\mathfrak{g}^{(i)}$ is an irreducible $\mathfrak{g}^\sigma$-module.*

Proof If $i = 0$ this is clear. Let $i \neq 0$. Suppose first that $A = A_{2\ell-1}, D_{\ell+1}$ or E_6 (and so $r = 2$, $i = 1$). Let $\{\alpha, \beta\}$ be a 2-element orbit of σ on Δ and E_α, E_β be the corresponding root elements such that $\sigma(E_\alpha) = E_\beta$. Then $E_\alpha - E_\beta \in \mathfrak{g}^{(1)}$, and, moreover, such elements yield a basis of $\mathfrak{g}^{(1)}$ as we run through all 2-element orbits. The roots $\alpha, \beta \in \mathfrak{h}^*$ have the same restriction to $\mathfrak{h}^\sigma$, and this restriction is the weight of $E_\alpha - E_\beta$ with respect to $\mathfrak{h}^\sigma$. The highest weight of the $\mathfrak{g}^\sigma$-module $\mathfrak{g}^{(1)}$ thus comes from the highest 2-element orbit. Explicit check shows that the highest 2-element orbits are:

for $A_{2\ell-1}$: $(\alpha_1 + \cdots + \alpha_{2\ell-2}, \alpha_2 + \cdots + \alpha_{2\ell-1})$;
for $D_{\ell+1}$: $(\alpha_1 + \cdots + \alpha_{\ell-1} + \alpha_\ell, \alpha_1 + \cdots + \alpha_{\ell-1} + \alpha_\ell)$;
for E_6: $(\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6)$.

Moreover, in this three cases the subalgebra $\mathfrak{g}^\sigma$ has type C_ℓ, B_ℓ, or F_4, respectively. For the standard labellings of the corresponding Dynkin diagrams, the highest weights for the $\mathfrak{g}^\sigma$-module $\mathfrak{g}^{(1)}$ are:

$$\text{for } C_\ell\text{: } \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{\ell-1} + \alpha_\ell = \omega_2;$$
$$\text{for } B_\ell\text{: } \alpha_1 + \alpha_2 + \cdots + \alpha_{\ell-1} + \alpha_\ell = \omega_1;$$
$$\text{for } F_4\text{: } \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 = \omega_4.$$

Note that in all cases we get the highest short root θ_0 as the highest weight. Now

$$\dim \mathfrak{g}^{(1)} = \dim \mathfrak{g} - \dim \mathfrak{g}^\sigma = \begin{cases} 2\ell^2 - \ell - 1, & \text{for } C_\ell, \\ 2\ell + 1, & \text{for } B_\ell, \\ 26, & \text{for } F_4, \end{cases}$$

which according to Weyl's dimension formula is the dimension of the irreducible module with the highest weight θ_0. The argument for D_4 and $A_{2\ell}$ is similar. □

To get a precise multiplication table for the non-simply-laced finite dimensional Lie algebras, it is convenient to change our notation. Roughly speaking we drop indices σ from objects related to the fixed points of σ (so Δ^σ becomes Δ) and use primes $'$ to distinguish the objects corresponding the big Lie algebra $\mathfrak{g}$ (so Δ becomes Δ'. To be more precise, Δ' is the root system of type $(X_N, r) = (D_{\ell+1}, 2)$, $(A_{2\ell-1}, 2)$, $(E_6, 2)$, $(D_4, 3)$ with roots $\alpha' \in \Delta'$, simple roots $\alpha'_1 \ldots, \alpha'_N$, etc. Let $\mathfrak{g}' = \mathfrak{g}(X_N^{(r)})$ be the corresponding Lie algebra, and σ the graph automorphism of $\mathfrak{g}'$ as before. We already know that $\mathfrak{g} := \mathfrak{g}'^\sigma$ is a simple Lie algebra of type

$$B_\ell,\ C_\ell,\ F_4,\ G_2,$$

respectively. In all four cases fix an orientation of the Dynkin diagram X_N which is σ-invariant, and let $\varepsilon(\alpha, \beta)$ be the corresponding asymmetry function, which is then also σ-invariant. This gives us an explicit realization

$$\mathfrak{g}' = \mathfrak{h}' \oplus \bigoplus_{\alpha' \in \Delta'} \mathbb{C} E'_{\alpha'}$$

as in (7.9). It is easy to see that

$$\mu : \alpha' \mapsto \sigma(\alpha'), \quad E'_{\alpha'} \mapsto E'_{\sigma(\alpha')} \quad (\alpha' \in \Delta') \tag{8.2}$$

is an automorphism of $\mathfrak{g}'$ which agrees with the graph automorphism σ on the generators, so $\sigma = \mu$. Note that there are no σ-orbits of type A_2, since we are staying away from type $A_{2\ell}$. Moreover, there is a bijection between σ-orbits on Δ' and the root system $\Delta := \Delta^\sigma$, given by mapping an orbit $J = \{\alpha', \dots\}$ to $\alpha'^\sigma = \frac{1}{|J|}\sum_{\alpha \in J}\alpha$. So we can (and will) identify the σ-orbits on Δ' with elements of Δ.

For $\alpha \in \Delta$ denote $E_\alpha = \sum_{\alpha' \in \alpha} E'_{\alpha'}$ (note we have identified elements $\alpha \in \Delta$ with σ-orbits on Δ'). Similarly, for simple roots $\alpha_1, \dots, \alpha_\ell \in \Delta$ we have $\alpha_i = \frac{1}{|\alpha_i|}(\sum_{\alpha' \in \alpha_i}\alpha')$—this is just the formula (8.1) in our new notation. Let $\mathfrak{h} = \mathfrak{h}'^\sigma$, the subspace with basis $\alpha_1, \dots, \alpha_\ell$. Then

$$\mathfrak{g} = \mathfrak{g}'^\sigma = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathbb{C}E_\alpha.$$

Moreover, the normalized invariant form $(\cdot|\cdot)'$ on $\mathfrak{g}'$ is σ-invariant and so it restricts to the invariant form $(\cdot|\cdot)$ on $\mathfrak{g}$, which is non-degenerate and invariant. Moreover $(\cdot|\cdot)$ is normalized since we already know that the single orbit elements α' correspond to the long roots $\alpha \in \Delta$, and so $(\alpha|\alpha) = (\alpha'|\alpha')' = 2$ for $\alpha \in \Delta_l$.

Proposition 8.1.8 *Let $\Delta = \Delta_s \cup \Delta_l$ be a non-simply-laced root system of finite type in Euclidean space $\mathfrak{h}_\mathbb{R}$ with root lattice Q. Set $r = 2$ if $\Delta = B_\ell$, C_ℓ, or F_4, and $r = 3$ if $\Delta = G_2$. Let us $\mathfrak{h}$ be the complex hull of $\mathfrak{h}_\mathbb{R}$ and extend $(\cdot|\cdot)$ to $\mathfrak{h}$. Let*

$$\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathbb{C}E_\alpha).$$

Define the bracket on $\mathfrak{g}$ as follows:

$$\begin{cases} [h,h'] = 0 & \text{if } h,h' \in \mathfrak{h} \\ [h,E_\alpha] = (h|\alpha)E_\alpha & \text{if } h \in \mathfrak{h},\ \alpha \in \Delta \\ [E_\alpha,E_{-\alpha}] = -\alpha & \text{if } \alpha \in \Delta_l \\ [E_\alpha,E_{-\alpha}] = -r\alpha & \text{if } \alpha \in \Delta_s \\ [E_\alpha,E_\beta] = 0 & \text{if } \alpha,\beta \in \Delta,\ \alpha+\beta \notin \Delta \cup \{0\} \\ [E_\alpha,E_\beta] = (p+1)\varepsilon(\alpha',\beta')E_{\alpha+\beta} & \text{if } \alpha,\beta,\alpha+\beta \in \Delta \text{ where } p \in \mathbb{Z}_{\geq 0} \\ & \text{is maximal with } \alpha - p\beta \in \Delta \text{ and} \\ & \alpha' \in \alpha, \beta' \in \beta \text{ are representatives} \\ & \text{such that } \alpha' + \beta' \in \alpha + \beta \end{cases}$$

The normalized bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ is given by extending $(\cdot|\cdot)$ as follows:

$$(h,E_\alpha) = 0, \quad (E_\alpha,E_\beta) = \begin{cases} -\delta_{\alpha,-\beta} & \text{if } \alpha,\beta \in \Delta_l \\ -r\delta_{\alpha,-\beta} & \text{if } \alpha,\beta \in \Delta_s \end{cases}$$

Proof We just need to check the relations. The first one is obvious. For the second, working in $\mathfrak{g}'$ we get

$$[h, E_\alpha] = \sum_{\alpha' \in \alpha} (h|\alpha') E'_{\alpha'} = (h|\alpha) E_\alpha,$$

since orthogonal projection of every $\alpha' \in \alpha$ to $\mathfrak{h}$ equals α. The third and fourth relations follow from

$$[E_\alpha, E_{-\alpha}] = [\sum_{\alpha' \in \alpha} E'_{\alpha'}, \sum_{\beta' \in \alpha} E'_{-\beta'}] = \sum_{\alpha' \in \alpha} [E'_{\alpha'}, E'_{-\alpha'}] = -\sum_{\alpha' \in \alpha} \alpha'.$$

The fifth relation comes from the following (easy) fact: if $\alpha+\beta \notin \Delta \cup \{0\}$ then $\alpha' + \beta' \notin \Delta' \cup \{0\}$.

For the last relation, we have

$$[E_\alpha, E_\beta] = [\sum_{\alpha' \in \alpha} E'_{\alpha'}, \sum_{\beta' \in \beta} E'_{\beta'}] = \sum_{\alpha' \in \alpha, \beta' \in \beta, \alpha'+\beta' \in \Delta'} \varepsilon(\alpha', \beta') E'_{\alpha'+\beta'}.$$

Now note that $\alpha' + \beta' \in \alpha + \beta$ and $\varepsilon(\alpha', \beta')$ is the same for any representatives $\alpha' \in \alpha, \beta' \in \beta$ such that $\alpha' + \beta' \in \Delta'$. Next check explicitly that each $E_{\alpha'+\beta'}$ appears $(p+1)$ times. □

8.2 Construction of Twisted Affine Algebras

In this section we construct explicit realization of affine algebras of types $X_N^{(r)}$ with $r > 1$, referred to as *twisted affine algebras*. Let $\mathring{\mathfrak{g}}$ be a finite dimensional Lie algebra of type X_N, and σ be its graph automorphism of order r. Then σ extends to a *graph* automorphism of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ denoted again by σ and given by

$$\sigma : \ c \mapsto c, \quad d \mapsto d, \quad t^i \otimes x \mapsto t^i \otimes \sigma(x) \qquad (x \in \mathring{\mathfrak{g}}).$$

Set $\eta = e^{2\pi i/r}$ and define a *twisted graph automorphism* of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})$ by

$$\tau : \ c \mapsto c, \quad d \mapsto d, \quad t^i \otimes x \mapsto \eta^{-i} t^i \otimes \sigma(x) \qquad (x \in \mathring{\mathfrak{g}}).$$

Proposition 8.2.1 *We have*

$$\begin{aligned} \hat{\mathcal{L}}(\mathfrak{g}(A_{2\ell-1}))^\tau &\cong \mathfrak{g}(A_{2\ell-1}^{(2)}), \\ \hat{\mathcal{L}}(\mathfrak{g}(A_{2\ell}))^\tau &\cong \mathfrak{g}(A_{2\ell}^{(2)}), \\ \hat{\mathcal{L}}(\mathfrak{g}(D_{\ell+1}))^\tau &\cong \mathfrak{g}(D_{\ell+1}^{(2)}), \\ \hat{\mathcal{L}}(\mathfrak{g}(E_6))^\tau &\cong \mathfrak{g}(E_6^{(2)}), \\ \hat{\mathcal{L}}(\mathfrak{g}(D_4))^\tau &\cong \mathfrak{g}(D_4^{(3)}). \end{aligned}$$

Proof We will skip the proof for $A_{2\ell}^{(2)}$. Let $\mathring{\mathfrak{g}}= \mathfrak{g}(\mathring{A})$, where $\mathring{A}$ is of type $A_{2\ell-1}, D_{\ell+1}, E_6$ or D_4. If $r = 2$ then we have

$$\hat{\mathcal{L}}(\mathring{\mathfrak{g}})^\tau = \sum_{n\in\mathbb{Z}}(t^{2n} \otimes (\mathring{\mathfrak{g}})^\sigma) \oplus \sum_{k\in\mathbb{Z}}(t^{2n+1} \otimes (\mathring{\mathfrak{g}})^{(1)}) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

whereas if $r = 3$, then

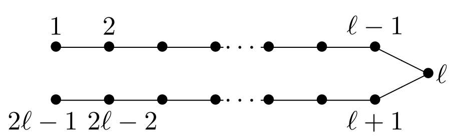
$$\hat{\mathcal{L}}(\mathring{\mathfrak{g}})^\tau = \sum_{n\in\mathbb{Z}}(t^{3n}\otimes(\mathring{\mathfrak{g}})^\sigma)\oplus\sum_{n\in\mathbb{Z}}(t^{3n+1}\otimes(\mathring{\mathfrak{g}})^{(1)})\oplus\sum_{n\in\mathbb{Z}}(t^{3n+2}\otimes(\mathring{\mathfrak{g}})^{(2)})\oplus\mathbb{C}c\oplus\mathbb{C}d.$$

Let $E_1,\dots,E_N,F_1,\dots,F_N,H_1,\dots,H_N$ be the standard generators of $\mathring{\mathfrak{g}}$. Pick a representative $\theta_0 \in\mathring{\Delta}$ of the highest 2- or 3-element σ-orbit on $\mathring{\Delta}$, cf. the proof of Proposition 8.1.7. Specifically we pick

for $A_{2\ell-1}$: $\theta_0 = \alpha_1 + \alpha_2 + \cdots + \alpha_{2\ell-2}$;
for $D_{\ell+1}$: $\theta_0 = \alpha_1 + \cdots + \alpha_{\ell-1} + \alpha_\ell$;
for E_6: $\theta_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$;
for D_4: $\theta_0 = \alpha_2 + \alpha_1 + \alpha_3$.

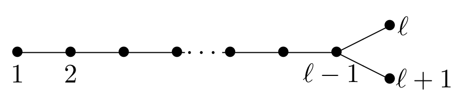
Choose elements $E_{\theta_0} \in\mathring{\mathfrak{g}}_{\theta_0}, F_{\theta_0} \in\mathring{\mathfrak{g}}_{-\theta_0}$ so that $[E_{\theta_0}, F_{\theta_0}] = \theta_0^\vee$, and similarly $E_{\sigma(\theta_0)}, F_{\sigma(\theta_0)}$. Now choose the elements $e_i, f_i, \alpha_i^\vee \in \hat{\mathcal{L}}(\mathring{\mathfrak{g}})^\tau$ as follows:

$A_{2\ell-1}$:



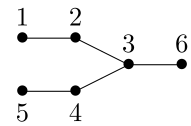
$$e_i = 1 \otimes (E_i + E_{2\ell-i}),\ f_i = 1 \otimes (F_i + F_{2\ell-i}),\ \alpha_i^\vee = 1 \otimes (H_i + H_{2\ell-i}) \qquad (1 \le i < \ell),$$
$$e_\ell = 1 \otimes E_\ell,\ f_\ell = 1 \otimes F_\ell,\ \alpha_\ell^\vee = 1 \otimes H_\ell$$
$$e_0 = t \otimes (F_{\theta_0} - F_{\sigma(\theta_0)}),\ f_0 = t^{-1} \otimes (E_{\theta_0} - E_{\sigma(\theta_0)}),\ \alpha_0^\vee = 1 \otimes (-\theta_0^\vee - (\sigma(\theta_0))^\vee) + 2c.$$

$D_{\ell+1}$:



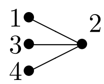
$$e_i = 1 \otimes E_i,\ f_i = 1 \otimes F_i,\ \alpha_i^\vee = 1 \otimes H_i \qquad (1 \le i < \ell),$$
$$e_\ell = 1 \otimes (E_\ell + E_{\ell+1}),\ f_\ell = 1 \otimes (F_\ell + F_{\ell+1}),\ \alpha_\ell^\vee = 1 \otimes (H_\ell + H_{\ell+1})$$
$$e_0 = t \otimes (F_{\theta_0} - F_{\sigma(\theta_0)}),\ f_0 = t^{-1} \otimes (E_{\theta_0} - E_{\sigma(\theta_0)}),\ \alpha_0^\vee = 1 \otimes (-\theta_0^\vee - (\sigma(\theta_0))^\vee) + 2c.$$

E_6:



$$e_i = 1 \otimes (E_i + E_{6-i}),\ f_i = 1 \otimes (F_i + F_{6-i}),\ \alpha_i^\vee = 1 \otimes (H_i + H_{6-i}) \qquad (1 \le i \le 2),$$
$$e_3 = 1 \otimes E_3,\ f_3 = 1 \otimes F_3,\ \alpha_3^\vee = 1 \otimes H_3$$
$$e_4 = 1 \otimes E_6,\ f_4 = 1 \otimes F_6,\ \alpha_4^\vee = 1 \otimes H_6$$
$$e_0 = t \otimes (F_{\theta_0} - F_{\sigma(\theta_0)}),\ f_0 = t^{-1} \otimes (E_{\theta_0} - E_{\sigma(\theta_0)}),\ \alpha_0^\vee = 1 \otimes (-\theta_0^\vee - (\sigma(\theta_0))^\vee) + 2c.$$

D_4:



$$e_1 = 1 \otimes E_2,\ f_1 = 1 \otimes F_2,\ \alpha_1^\vee = 1 \otimes H_2,$$
$$e_2 = 1 \otimes (E_1 + E_3 + E_4),\ f_2 = 1 \otimes (F_1 + F_3 + F_4),\ \alpha_2^\vee = 1 \otimes (H_1 + H_3 + H_4)$$
$$e_0 = t \otimes (F_{\theta_0} + \eta^2 F_{\sigma(\theta_0)} + \eta F_{\sigma^2(\theta_0)}),\ f_0 = t^{-1} \otimes (E_{\theta_0} + \eta E_{\sigma(\theta_0)} + \eta^2 E_{\sigma^2(\theta_0)}),$$
$$\alpha_0^\vee = 1 \otimes (-\theta_0^\vee - (\sigma(\theta_0))^\vee - (\sigma^2(\theta_0)^\vee) + 3c.$$

Let

$$\mathfrak{h} = 1 \otimes \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Note that

$$\mathfrak{h}^\sigma = \mathfrak{h}^\tau = \operatorname{span}(\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee) \oplus \mathbb{C}c \oplus \mathbb{C}d.$$

Define the elements $\alpha_1, \ldots, \alpha_\ell \in (\mathfrak{h}^\sigma)^*$ to be the restriction from $\mathfrak{h}^*$ of the roots $\alpha_1, \ldots, \alpha_\ell \in \mathfrak{h}^*$, respectively, in types $A_{2\ell-1}$ and $D_{\ell+1}$. Define the elements $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in (\mathfrak{h}^\sigma)^*$ to be the restriction from $\mathfrak{h}^*$ of the roots $\alpha_1, \alpha_2, \alpha_3, \alpha_6 \in \mathfrak{h}^*$, respectively, in type E_6. Define the elements $\alpha_1, \alpha_2 \in (\mathfrak{h}^\sigma)^*$ to be the restriction from $\mathfrak{h}^*$ of the roots $\alpha_2, \alpha_1 \in \mathfrak{h}^*$, respectively in type D_4. Also, in all cases, we define α_0 to be the restriction from $\mathfrak{h}^*$ of $\delta - \theta_0$.

We next claim that $(\mathfrak{h}^\sigma, \Pi, \Pi^\vee)$ is a realization of the Cartan matrix A' of type $X_N^{(r)}$. We know from Theorem 8.1.6 that

$$\langle \alpha_i^\vee, \alpha_j \rangle = a'_{ij}$$

for $i, j \geq 1$. The $\ell \times \ell$ matrix with entries $\langle \alpha_i^\vee, \alpha_j \rangle$ for $1 \leq i, j \leq \ell$ is non-singular, so $\alpha_1^\vee, \ldots, \alpha_\ell^\vee$ and $\alpha_1, \ldots, \alpha_\ell$ are linearly independent. We have $\langle d, \alpha_i \rangle = \delta_{i0}$, whence $\alpha_0, \alpha_1, \ldots, \alpha_\ell$ are linearly independent. Also c appears in $\alpha_0^\vee$, whence $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee$ are linearly independent. For the remaining entries of the Cartan matrix, note that

$$\sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee = rc,$$

where a_i are the marks of the diagram $X_N^{(r)}$. We also note that

$$\sum_{i=0}^{\ell} a_i \alpha_i = \delta|_{(\overset{\circ}{\mathfrak{h}}{}^\sigma)}.$$

Now

$$\langle \alpha_i^\vee, \alpha_0 \rangle = \langle \alpha_i^\vee, \delta - \sum_{j=1}^{\ell} a_j \alpha_j \rangle = -\sum_{j=1}^{\ell} a'_{ij} a_j = a_0 a'_{i0} = a'_{i0}$$

for $i = 1, \ldots, \ell$. Moreover,

$$\langle \alpha_0^\vee, \alpha_j \rangle = \langle rc - \sum_{i=1}^{\ell} a_i^\vee \alpha_i^\vee, \alpha_j \rangle = -\sum_{i=1}^{\ell} a'_{ij} a_i^\vee = a_0^\vee a'_{0j} = a'_{0j}$$

for $j = 1, \ldots, \ell$. Finally,

$$\langle \alpha_0^\vee, \alpha_0 \rangle = \langle -1 \otimes \theta_0^\vee - 1 \otimes \sigma(\theta_0)^\vee - \cdots + rc, -\theta_0 + \delta \rangle = \langle \theta_0^\vee, \theta_0 \rangle = 2,$$

since $\langle \sigma(\theta_0)^\vee, \theta_0 \rangle = 0$.

We next verify relations (1.12-1.15). The relation (1.12) is easy and the relation (1.13) is obvious. For (1.14,1.15), if $i, j \geq 1$, then we already know that

$$[\alpha_i^\vee, e_j] = a'_{ij} e_j, \quad [\alpha_i^\vee, f_j] = -a'_{ij} f_j \tag{8.3}$$

For $j = 1, \ldots, \ell$ we have

$$[\alpha_0^\vee, e_j] = [rc - \sum_{i=1}^{\ell} a_i^\vee \alpha_i^\vee, e_j] = -\sum_{i=1}^{\ell} a_i^\vee [\alpha_i^\vee, e_j] = -\sum_{i=1}^{\ell} a_i^\vee a'_{ij} e_j = a'_{0j} e_j,$$

and similarly we get $[\alpha_0^\vee, f_j] = -a'_{0j} f_j$ for $j = 1, \ldots, \ell$. Also, for $i = 1, \ldots, \ell$ we have

$$\begin{aligned}[][\alpha_i^\vee, e_0] &= [\alpha_i^\vee, t \otimes (F_{\theta_0} + \eta^{-1} F_{\sigma(\theta_0)} + \ldots)] \\ &= t \otimes \langle \alpha_i^\vee, -\theta_0 \rangle (F_{\theta_0} + \eta^{-1} F_{\sigma(\theta_0)} + \ldots) \\ &= \langle \alpha_i^\vee, -\sum_{j=1}^{\ell} a_j \alpha_j \rangle e_0 \\ &= -(\sum_{j=1}^{\ell} a_{ij} a_j) e_0 \\ &= a_{i0} e_0.\end{aligned}$$

Similarly we have $[\alpha_i^\vee, f_0] = -a_{i0} f_0$. We also have

$$\begin{aligned}[][\alpha_0^\vee, e_0] &= [1 \otimes (-\theta_0^\vee - \sigma(\theta_0)^\vee - \ldots), t \otimes (F_{\theta_0} + \eta^{-1} F_{\sigma(\theta_0)} + \ldots)] \\ &= 2t \otimes (F_{\theta_0} + \eta^{-1} F_{\sigma(\theta_0)} + \ldots)] \\ &= 2e_0.\end{aligned}$$

Similarly we have $[\alpha_0^\vee, f_0] = -2f_0$. Ffinally, for $h = c$ and d the relations $[h, e_i] = \langle h, \alpha_i \rangle e_i$ and $[h, f_i] = -\langle h, \alpha_i \rangle f_i$ are easy to check.

We next prove that the elements $e_0, e_1, \ldots, e_\ell, f_0, f_1, \ldots, f_\ell$ together with $\mathfrak{h}^\tau$ generate $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})^\tau$. Denote by $\mathfrak{g}_1$ the subalgebra generated by this elements. We know that $e_1, \ldots, e_\ell, f_1, \ldots, f_\ell$ generate $(\mathring{\mathfrak{g}})^\sigma$. So the degree 0 part of $\hat{\mathcal{L}}(\mathring{\mathfrak{g}})^\tau$ lies in $\mathfrak{g}_1$.

Suppose first that $r = 2$. We have $e_0 = t \otimes (F_{\theta_0} - F_{\sigma(\theta_0)}) \in \mathfrak{g}_1$ and $F_{\theta_0} - F_{\sigma(\theta_0)} \in (\mathring{\mathfrak{g}})^{(1)}$. Now, it is easy to see that the elements $y \in (\mathring{\mathfrak{g}})^{(1)}$ for which $t \otimes y \in \mathfrak{g}_1$ form a non-zero submodule of the $(\mathring{\mathfrak{g}})^\sigma$-module $(\mathring{\mathfrak{g}})^{(1)}$. Since this module is irreducible, we conclude that $t \otimes (\mathring{\mathfrak{g}})^{(1)} \subset \mathfrak{g}_1$. Now we can find elements $x, y \in (\mathring{\mathfrak{g}})^{(1)}$ such that $[x, y] \neq 0$. Then $[t \otimes x, t \otimes y] = t^2 \otimes [x, y]$ is a non-zero element of $\mathfrak{g}_1$. Now the set of all $z \in (\mathring{\mathfrak{g}})^\sigma$ such that $t^2 \otimes z \in \mathfrak{g}_1$ is an ideal of $(\mathring{\mathfrak{g}})^\sigma$, and $(\mathring{\mathfrak{g}})^\sigma$ is simple, so $t^2 \otimes (\mathring{\mathfrak{g}})^\sigma \subset \mathfrak{g}_1$. The relations

$$\begin{aligned}[][t^2 \otimes x, t^{2k} \otimes y] &= t^{2k+2} \otimes [x, y] \qquad (x, y \in (\mathring{\mathfrak{g}})^\sigma) \\ [t^2 \otimes x, t^{2k+1} \otimes y] &= t^{2k+3} \otimes [x, y] \qquad (x \in (\mathring{\mathfrak{g}})^\sigma,\ y \in (\mathring{\mathfrak{g}})^{(1)})\end{aligned}$$

can then be used to show by induction on k that $t^{2k} \otimes (\mathring{\mathfrak{g}})^\sigma \subset \mathfrak{g}_1$ and $t^{2k+1} \otimes (\mathring{\mathfrak{g}})^{(1)} \subset \mathfrak{g}_1$ for $k > 0$. The argument for $k < 0$ and for $r = 3$ is similar.

Finally we must show that $\hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})^\tau$ has non non-trivial ideals $\mathfrak{i}$ with $\mathfrak{i} \cap \mathfrak{h}^\tau = (0)$. Decompose $\hat{L}(\overset{\circ}{\mathfrak{g}})^\tau$ into root spaces with respect to $\mathfrak{h}^\tau$. We first suppose that $r = 2$. Then $\hat{L}(\overset{\circ}{\mathfrak{g}})^\tau$ is the direct sum of $\mathfrak{h}^\tau$ and the following weight spaces:

$t^{2k} \otimes (\overset{\circ}{\mathfrak{h}})^\sigma$ with weight $2k\delta$;
$t^{2k+1} \otimes (\overset{\circ}{\mathfrak{h}})^{(1)}$ with weight $(2k+1)\delta$;
$\mathbb{C}t^{2k} \otimes E_\alpha$ with weight $\alpha + 2k\delta$ where $\{\alpha\}$ is a one-element orbit of σ on $\overset{\circ}{\Delta}$;
$\mathbb{C}t^{2k} \otimes (E_\alpha + E_{\sigma(\alpha)})$ with weight $\alpha + 2k\delta$ where $\{\alpha, \sigma(\alpha)\}$ is a two-element orbit of σ on $\overset{\circ}{\Delta}$;
$\mathbb{C}t^{2k+1} \otimes (E_\alpha - E_{\sigma(\alpha)})$ with weight $\alpha + (2k+1)\delta$ where $\{\alpha, \sigma(\alpha)\}$ is a two-element orbit of σ on $\overset{\circ}{\Delta}$.

By the Weight Lemma, if $i \neq 0$, then it has a non-zero element x in one of these root spaces. Then we can find an element y in the negative root space such that $[x, y]$ is a non-zero element of $\mathfrak{h}^\tau$.

When $r = 3$ a similar argument can be applied. This time the weight spaces are

$t^{3k} \otimes (\overset{\circ}{\mathfrak{h}})^\sigma$ with weight $3k\delta$;
$t^{3k+1} \otimes (\overset{\circ}{\mathfrak{h}})^{(1)}$ with weight $(3k+1)\delta$;
$t^{3k+2} \otimes (\overset{\circ}{\mathfrak{h}})^{(2)}$ with weight $(3k+2)\delta$;
$\mathbb{C}t^{3k} \otimes E_\alpha$ with weight $\alpha + 3k\delta$ where $\{\alpha\}$ is a one-element orbit of σ on $\overset{\circ}{\Delta}$;
$\mathbb{C}t^{3k} \otimes (E_\alpha + E_{\sigma(\alpha)} + E_{\sigma^2(\alpha)})$ with weight $\alpha + 3k\delta$ where $\{\alpha, \sigma(\alpha), \sigma^2(\alpha)\}$ is a three-element orbit of σ on $\overset{\circ}{\Delta}$;
$\mathbb{C}t^{3k+1} \otimes (E_\alpha + \eta^{-1}E_{\sigma(\alpha)} + \eta E_{\sigma^2(\alpha)})$ with weight $\alpha + (3k+1)\delta$ where $\{\alpha, \sigma(\alpha), \sigma^2(\alpha)\}$ is a three-element orbit of σ on $\overset{\circ}{\Delta}$.
$\mathbb{C}t^{3k+2} \otimes (E_\alpha + \eta E_{\sigma(\alpha)} + \eta^{-1}E_{\sigma^2(\alpha)})$ with weight $\alpha + (3k+2)\delta$ where $\{\alpha, \sigma(\alpha), \sigma^2(\alpha)\}$ is a three-element orbit of σ on $\overset{\circ}{\Delta}$.

□

Corollary 8.2.2 *The multiplicities of the imaginary roots are as follows:*

(i) *Type $A_{2\ell}^{(2)}$: the multiplicity of any $k\delta$ is ℓ;*
(ii) *Type $A_{2\ell-1}^{(2)}$: the multiplicity of any $2k\delta$ is ℓ and the multiplicity of any $(2k+1)\delta$ is $\ell - 1$;*

(iii) *Type* $D_{\ell+1}^{(2)}$*: the multiplicity of any* $2k\delta$ *is* ℓ *and the multiplicity of any* $(2k+1)\delta$ *is* 1*;*

(iv) *Type* $E_6^{(2)}$*: the multiplicity of any* $2k\delta$ *is* 4 *and the multiplicity of any* $(2k+1)\delta$ *is* 2*;*

(v) *Type* $D_4^{(3)}$*: the multiplicity of any* $3k\delta$ *is* 2 *and the multiplicity of any* $(3k+1)\delta$ *and* $(3k+2)\delta$ *is* 1.

Proof By the previous theorem, the multiplicity of $rk\delta$ equals $\dim(\mathring{\mathfrak{h}})^{\sigma}$, and the multiplicity of $(2k+1)\delta$ in cases (i)-(iv) equals $\dim(\mathring{\mathfrak{h}})^{(1)}$, etc. □

8.3 Finite Order Automorphisms

Let $\mathfrak{g}$ be a simple finite dimensional Lie algebra of type X_N. Let μ be a diagram automorphism of $\mathfrak{g}$ of order r. Let E_i, F_i, H_i $(i = 0, 1, \dots, N)$ be the elements of $\mathfrak{g}$ introduced in §7.2, and let $\alpha_0, \alpha_1, \dots \alpha_N$ be the roots attached to $E_0, E_1, \dots, E_N$, respectively. Recall that the elements $E_0, E_1, \dots, E_N$ generate $\mathfrak{g}$, and that there exists a unique linear dependence $\sum_{i=0}^{\ell} a_i\alpha_i = 0$ such that the a_i are positive relatively prime numbers. Recall also that the vertices of the diagram $X_N^{(r)}$ are in one-to-one correspondence with the E_i and that the a_i are the labels at this diagram.

Lemma 8.3.1 *Every ideal of the Lie algebra* $\mathcal{L}(\mathfrak{g}, \mu)$ *is of the form* $P(t^r)\mathcal{L}(\mathfrak{g}, \mu)$*, where* $P(t) \in \mathcal{L}$*. In particular, a maximal ideal is of the form* $(1 - (at)^r)\mathcal{L}(\mathfrak{g}, \mu)$*, where* $a \in \mathbb{C}^{\times}$*.*

Proof Let $\mathfrak{i}$ be a non-trivial ideal of $\mathcal{L}(\mathfrak{g}, \mu)$, and

$$x = \sum_{\bar{j},s} t^j P_{\bar{j},s}(t) \otimes a_{\bar{j},s} \in \mathfrak{i}$$

where $0 \le j < r$ is such that $\bar{j} \equiv j \pmod r$, $P_{\bar{j},s}(t) \in \mathcal{L}$, and $a_{\bar{j},s} \in \mathfrak{g}_{\bar{j}}$ are linearly independent. We show that

$$Q(t^r)P_{\bar{j},s}(t)\mathcal{L}(\mathfrak{g}, \mu) \subset \mathfrak{i}$$

for all $Q(t) \in \mathcal{L}$. Let $\mathfrak{h}_{\bar{0}} = \mathfrak{h}^{\mu}$ be the Cartan subalgebra of $\mathfrak{g}_{\bar{0}} = \mathfrak{g}^{\mu}$. We can assume that x is an eigenvector for $\mathfrak{h}_{\bar{0}}$ with weight $\alpha \in \mathfrak{h}_{\bar{0}}^*$. If $\alpha \neq 0$, taking $[x, t^j \otimes a_{-\bar{j}}]$ with $a_{-\bar{j}}$ of weight $-\alpha$, instead of x, we reduce the problem to the case $\alpha = 0$ and $\bar{j} = 0$, i.e. $a_{\bar{j},s} \in \mathfrak{h}_{\bar{0}}$. Let $\gamma \in \mathfrak{h}_{\bar{0}}^*$ be the root of $\mathfrak{g}_{\bar{0}}$ such that $\langle \gamma, a_{\bar{j},s} \rangle \neq 0$. Then the element y □

Theorem 8.3.2 *Let $s = (s_0, s_1, \ldots, s_\ell)$ be a sequence of non-negative relatively prime numbers; put $m = r \sum_{i=0}^{\ell} a_i s_i$. Then*

(i) *The formulas*

$$\sigma_{s;r} : E_j \mapsto e^{2\pi i s_j / m} E_j \qquad (0 \leq j \leq \ell)$$

define (uniquely) an mth order automorphism $\sigma_{s,r}$ of $\mathfrak{g}$.

(ii) *Up to conjugation by an automorphism of $\mathfrak{g}$, the automorphisms $\sigma_{s,r}$ all mth order automorphisms of $\mathfrak{g}$.*

(iii) *The elements $\sigma_{s,r}$ and $\sigma_{s',r'}$ are conjugate by an automorphism of $\mathfrak{g}$ if and only if $r = r'$ and the sequence s can be transformed to the sequence s' by an automorphism of the diagram $X_N^{(r)}$.*

Proof See Kac. □

9

Highest weight modules over Kac-Moody algebras

9.1 The category $\mathcal{O}$

For an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module V we denote by $P(V)$ the set of weights of V. For $\lambda \in \mathfrak{h}^*$ denote

$$D(\lambda) = \{\mu \in \mathfrak{h}^* \mid \mu \leq \lambda\}.$$

The *category* $\mathcal{O}$ is defined as follows. Its objects are $\mathfrak{g}$-modules V which are $\mathfrak{h}$-diagonalizable with finite dimensional weight spaces and such that there exists a finite number of elements $\lambda_1, \dots, \lambda_s \in \mathfrak{h}^*$ such that

$$P(V) \subset D(\lambda_1) \cup \dots \cup D(\lambda_s).$$

The morphisms in $\mathcal{O}$ are homomorphisms of $\mathfrak{g}$-modules. By the Weight Lemma, any submodule or quotient module of a module from category $\mathcal{O}$ is also in $\mathcal{O}$. Also, a sum and a tensor product of a finite number of modules from $\mathcal{O}$ is again in $\mathcal{O}$. Finally, every module from $\mathcal{O}$ is restricted.

A *highest weight vector* of weight Λ is a Λ-weight vector v in a $\mathfrak{g}$-module V such that $\mathfrak{n}_+ v = 0$. A $\mathfrak{g}$-module is a *highest weight module with highest weight* $\Lambda \in \mathfrak{h}^*$ if it is generated by a highest weight vector of weight Λ. If V is such a module and $v_\Lambda \in V$ is a highest weight vector of weight Λ, then

$$V = U(\mathfrak{n}_-)v_\Lambda, \quad V = \bigoplus_{\lambda \leq \Lambda} V_\lambda, \quad V_\Lambda = \mathbb{C} \cdot v_\Lambda, \quad \dim V_\lambda < \infty \quad (\lambda \in \mathfrak{h}^*).$$

In particular $V \in \mathcal{O}$. Now form the *Verma module*

$$M(\Lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\Lambda,$$

where $\mathfrak{b}_+ = \mathfrak{n}_+ \oplus \mathfrak{h}$ and $\mathbb{C}_\Lambda$ is the 1-dimensional $\mathfrak{b}_+$-module with the trivial action of $\mathfrak{n}_+$ and the action of $\mathfrak{h}$ with the weight Λ. Let us write

$$v_\Lambda := 1 \otimes 1 \in M(\Lambda).$$

This is a highest weight vector and $M(\Lambda)$ is a highest weight module with highest weight Λ. By PBW, $M(\Lambda)$ restricted to $U(n_-)$ is a free module of rank 1 on basis v_Λ.

Lemma 9.1.1 *Suppose V is a highest weight module with highest weight Λ. Then there is a unique up to scalars surjective homomorphism from $M(\Lambda)$ onto V.*

Proof By adjointness of tensor and Hom,

$$\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), V) \cong \mathrm{Hom}_{\mathfrak{b}_+}(\mathbb{C}_\Lambda, V).$$

It is clear that the last Hom-space is 1-dimensional. □

Proposition 9.1.2 *$M(\Lambda)$ has a unique maximal submodule $M'(\Lambda)$, and the quotient*

$$L(\Lambda) := M(\Lambda)/M'(\Lambda)$$

is an irreducible $\mathfrak{g}$-module. Moreover, every irreducible module in the category $\mathcal{O}$ is isomorphic to one and only one $L(\Lambda)$, $\Lambda \in \mathfrak{h}^*$. Finally, $\mathrm{End}_{\mathfrak{g}}(L(\Lambda)) = \mathbb{C} \cdot I_{L(\Lambda)}$.

Proof If M is a proper submodule of $M(\Lambda)$ then $M_\Lambda = 0$, hence the sum of all proper submodules of $M(\Lambda)$ still has the trivial Λ-weight space, so is still proper. This proves the existence of a unique maximal submodule, whence $L(\Lambda)$ is irreducible. Next, let L be an irreducible module in $\mathcal{O}$. Pick a maximal weight Λ of L, an let $v \in L_\Lambda$. It follows that v generates L, so by Lemma 9.1.1, L is a quotient of $M(\Lambda)$, whence $L \cong L(\Lambda)$. The last claim is easy to check using the fact that $\dim L(\Lambda)_\Lambda = 1$. □

A vector v in a $\mathfrak{g}$-module V is called *primitive* (of weight λ) if v is a weight vector (of weight λ) and there exists a submodule $U \subset V$ such that $v + U$ is a highest weight vector in V/U. Every module $V \in \mathcal{O}$ is generated by its primitive vectors. Indeed, let V' be the submodule of V generated by the primitive vectors. If $V' \neq V$, then V/V' contains a highest weight vector, any preimage of which is a primitive vector. Actually, even more is true:

Lemma 9.1.3 *Any module $V \in \mathcal{O}$ is generated by its primitive vectors as an $\mathfrak{n}_-$-module.*

Proof Note first that a weight vector $v \in V$ is not primitive if and only if $v \in U(\mathfrak{n}_-)U_0(\mathfrak{n}_+)v$, where $U_0(\mathfrak{g})$ stands for the augmentation ideal of $U(\mathfrak{g})$. Indeed, for a weight vector v

$$U(\mathfrak{n}_-)U_0(\mathfrak{n}_+)v = U(\mathfrak{n}_-)U(\mathfrak{n}_+)\mathfrak{n}_+v = U(\mathfrak{n}_-)U(\mathfrak{n}_+)U(\mathfrak{h})\mathfrak{n}_+v = U(\mathfrak{g})\mathfrak{n}_+v$$

is the $\mathfrak{g}$-submodule generated by $\mathfrak{n}_+v$.

Now it follows that every non-primitive vector is obtained by application of some elements from $\mathfrak{n}_-$ to elements of higher weights. This implies the lemma using boundedness from above of $P(V)$. □

9.2 Formal Characters

Unfortunately, a module V in $\mathcal{O}$ need not have a composition series. So we cannot define things like multiplicities $[V : L(\Lambda)]$ in the usual way. The following provide a substitute for this:

Lemma 9.2.1 *Let $V \in \mathcal{O}$ and $\lambda \in \mathfrak{h}^*$. Then there exists a filtration*

$$V = V_t \supset V_{t-1} \supset \cdots \supset V_0 = 0$$

and a subset $J \subset \{1, \dots, t\}$ such that

(i) *if $j \in J$ then $V_j/V_{j-1} \cong L(\lambda_j)$ for some $\lambda_j \geq \lambda$;*
(ii) *if $j \notin J$ then $(V_j/V_{j-1})_\mu = 0$ for every $\mu \geq \lambda$.*

Proof Let

$$a(V, \lambda) = \sum_{\mu \geq \lambda} \dim V_\mu.$$

This is a well-defined non-negative integer. We prove the lemma by induction on $a(V, \lambda)$. If $a(V, \lambda) = 0$ then $0 = V_0 \subset V_1 = V$ is the required filtration, with $J = \varnothing$. If $a(V, \lambda) > 0$, let μ be a maximal weight of M such that $\mu \geq \lambda$. Choose a non-zero weight vector $v \in V_\mu$ and let $U = U(\mathfrak{g}) \cdot v$. Clearly U is a highest weight module. Hence it has a unique maximal submodule U'. Now we have

$$0 \subset U' \subset U \subset V$$

with $U/U' \cong L(\mu)$ and $\mu \geq 0$. Since $a(U', \lambda)$ and $a(V/U, \lambda)$ are both less than $a(V, \lambda)$, we now can proceed by induction. □

Lemma 9.2.2 *Let $V \in \mathcal{O}$, $\mu \in \mathfrak{h}^*$ and let λ be such that $\lambda \leq \mu$. Consider the corresponding filtration from Lemma 9.2.1. Then the number of times μ appears among the $\{\lambda_j \mid j \in J\}$ is independent of the choice of filtration and also the choice if λ.*

Proof We first observe that a filtration with respect to λ is also a filtration with respect to μ when $\mu \geq \lambda$. Also, the multiplicity of $L(\mu)$ in such filtration is the same whether it is regarded as a filtration with respect to λ or μ. Thus to prove the lemma it will be sufficient to take two filtrations with respect to μ and show that $L(\mu)$ has the same multiplicity in each. The following variant of the proof of the Jordan-Holder theorem achieves this. Let

$$V = V_0 \supset V_1 \supset \cdots \supset V_{l_1} = 0, \tag{9.1}$$

$$V = V_0' \supset V_1' \supset \cdots \supset V_{l_2}' = 0 \tag{9.2}$$

be two such filtrations of lengths l_1 and l_2. We use induction on $\min(l_1, l_2)$. If $\min(l_1, l_2) = 1$ then either V is irreducible and the two filtrations are identical or μ is not a weight of μ and $L(\mu)$ does not appear in both filtrations. Thus suppose $\min(l_1, l_2) > 1$.

Assume first that $V_1 = V_1'$. Then consider two filtrations

$$V_1 \supset \cdots \supset V_{l_1} = 0,$$
$$V_1' \supset \cdots \supset V_{l_2}' = 0$$

of V_1. By induction they give the same multiplicity of $L(\mu)$, and the filtrations for V are obtained by adding the additional factor V/V_1, which is the same for both. So we are done in this case.

Next, assume that $V_1 \neq V_1'$. Suppose first that one contains the other, say $V_1 \subset V_1'$. Then V/V_1 is not irreducible, so μ is not a weight of V/V_1. Thus neither V/V_1 nor V/V_1' is isomorphic to $L(\mu)$. Let

$$V \supset U_1 \supset \cdots \supset U_m = 0,$$

be a filtration of V_1 of the required type with respect to μ. We then consider the filtrations

$$V \supset V_1 \supset U_1 \cdots \supset U_m = 0, \tag{9.3}$$

$$V \supset V_1' \supset V_1 \supset U_1 \cdots \supset U_m = 0. \tag{9.4}$$

These are filtrations of the required type with respect to μ. Moreover, $L(\mu)$ has the same multiplicity in (9.1) and (9.3), since they have the same leading term V_1. Similarly $L(\mu)$ has the same multiplicity in (9.2) and (9.2). Finally, $L(\mu)$ has the same multiplicity in (9.3) and (9.4), since since none of $V/V_1, V/V_1', V_1'/V_1$ is isomorphic to $L(\mu)$. Thus $L(\mu)$ has the same multiplicity in (9.1) and (9.2), as required.

Finally, we assume that neither of V_1, V_1' is contained in the other. Let $U = V_1 \cap V_1'$ and choose a filtration of U of the required type with respect to μ:

$$U \supset U_1 \supset \cdots \supset U_m = 0.$$

We then consider the filtrations

$$V \supset V_1 \supset U \supset U_1 \cdots \supset U_m = 0, \tag{9.5}$$

$$V \supset V_1' \supset U \supset U_1 \cdots \supset U_m = 0. \tag{9.6}$$

These are filtrations of the required type with respect to μ. This is clear since

$$V_1/U \cong (V_1 + V_1')/V_1', \quad V_1'/U \cong (V_1 + V_1')/V_1.$$

Now $L(\mu)$ has the same multiplicity in (9.1), (9.5) and in (9.2), (9.6), since they have the same leading term. It is therefore sufficient to show that $L(\mu)$ has the same multiplicity in (9.5) and (9.6). These filtrations differ only in the two first factors. If $V_1 + V_2 = V$ then we have

$$V/V_1 \cong V_1'/U, \quad V/V_1' \cong V_1/U,$$

and we are done. If $V_1 + V_1' \neq V$ then V/V_1 and V/V_1' are not irreducible. In this case μ is not a weight of V/V_1 and V/V_1', so it is not a weight of V_1/U. Thus none of V/V_1, V_1/U, V/V_1', V_1'/U is isomorphic to $L(\mu)$. This completes the proof. □

Now, let $V \in \mathcal{O}$ and $\mu \in \mathfrak{h}^*$. Fix $\lambda \in \mathfrak{h}^*$ such that $\lambda \leq \mu$ and construct a filtration as in Lemma 9.2.1. Denote by $[V : L(\mu)]$ the number of times μ appears among the $\{\lambda_j \mid j \in J\}$ and call it the *multiplicity* of $L(\mu)$ in V. In view of Lemma 9.2.2, the multiplicity is well-defined.

Given a module $V \in \mathcal{O}$, we have by definition that all its weight spaces are finite dimensional. The idea of the formal character is to record the dimensions of each of these weight spaces in one "book-keeping devise" or a generating function. Since there may be infinitely many weights in $P(V)$, we are going to have to work with certain formal infinite sums.

So let $\mathcal{E}$ be the $\mathbb{C}$-algebra whose elements are series of the form

$$\sum_{\lambda \in \mathfrak{h}^*} c_\lambda e(\lambda)$$

where $c_\lambda \in \mathbb{C}$ and $c_\lambda = 0$ for λ outside the union of a finite number of sets of the form $D(\mu)$. The sum of two such series and the multiplication by a scalar are defined in the usual way. The product of two such series also makes sense if we use the rule $e(\lambda)e(\mu) = e(\lambda+\mu)$ and note that to calculate the coefficient of a given $e(\nu)$ in the product of two elements in $\mathcal{E}$ only involves calculating a finite sum. Under these operations $\mathcal{E}$ becomes a commutative associative $\mathbb{C}$-algebra with identity $e(0)$.

Given a module $V \in \mathcal{O}$, define its *formal character* to be

$$\mathrm{ch}\, V := \sum_{\lambda \in \mathfrak{h}^*} (\dim V_\lambda) e(\lambda) \in \mathcal{E}.$$

By definition, we have

$$\mathrm{ch}\,(V \oplus W) = \mathrm{ch}\, V + \mathrm{ch}\, W, \quad \mathrm{ch}\,(V \otimes W) = \mathrm{ch}\, V \mathrm{ch}\, W.$$

Proposition 9.2.3 *Let $V \in \mathcal{O}$. Then*

$$\mathrm{ch}\, V = \sum_{\lambda \in \mathfrak{h}^*} [V : L(\lambda)] \mathrm{ch}\, L(\lambda).$$

Proof Let

$$\varphi(V) = \mathrm{ch}\, V - \sum_{\lambda \in \mathfrak{h}^*} [V : L(\lambda)] \mathrm{ch}\, L(\lambda).$$

Note that $\varphi(V) \in \mathcal{E}$. Moreover, $\varphi(L(\lambda)) = 0$ and, given a SES of modules

$$0 \to V_1 \to V_2 \to V_3 \to 0$$

we have $\varphi(V_2) = \varphi(V_1) + \varphi(V_3)$. Now let us focus on a particular λ and V. By Lemma 9.2.1, there exists a filtration

$$V = V_t \supset V_{t-1} \supset \cdots \supset V_0 = 0$$

such that, setting $W_i := V_i / V_{i-1}$, we have either $W_i \cong L(\lambda_i)$ for some $\lambda_i \geq \lambda$ or that $(W_i)_\lambda = 0$. In the former case $\varphi(W_i) = 0$. In the latter case $\varphi(W_i)$ has $e(\lambda)$-coefficient 0. This was for all λ, so $\varphi(V) = 0$. □

Lemma 9.2.4 *For any $\lambda \in \mathfrak{h}^*$ the formal character of the Verma module is given by*

$$\mathrm{ch}\, M(\lambda) = e(\lambda) \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{-\operatorname{mult} \alpha}.$$

Proof Follows from PBW-theorem and freeness of $M(\lambda)$ over $U(\mathfrak{n}_-)$. □

Assume that A is symmetrizable and let $(\cdot|\cdot)$ be the standard form on $\mathfrak{g}$. Recall from Corollary 2.3.6 that if V is a $\mathfrak{g}$-module with highest weight Λ, then

$$\Omega = (|\Lambda+\rho|^2 - |\rho|^2)I_V.$$

Proposition 9.2.5 *Let V be a $\mathfrak{g}$-module with highest weight Λ. Then*

$$\mathrm{ch}\, V = \sum_{\lambda \leq \Lambda,\ |\lambda+\rho|^2 = |\Lambda+\rho|^2} c_\lambda \mathrm{ch}\, M(\lambda), \tag{9.7}$$

where $c_\lambda \in \mathbb{Z}$, $c_\Lambda = 1$.

Proof In view of Proposition 9.2.3, we may assume that $V = L(\Lambda)$. Using the same proposition, for any μ we deduce

$$\mathrm{ch}\, M(\mu) = \sum_{\nu \leq \mu} c_{\mu,\nu} \mathrm{ch}\, L(\nu),$$

for some non-negative integers $c_{\mu,\nu}$ with $c_{\mu,\mu} = 1$. We know that $c_{\mu,\nu} \neq 0$ if and only if $M(\mu)$ contains a primitive vector of weight ν. Using the action of the Casimir we deduce that $c_{\mu,\nu} = 0$ unless $|\mu+\rho|^2 = |\nu+\rho|^2$.

Set $B(\Lambda) = \{\lambda \leq \Lambda \mid |\lambda+\rho|^2 = |\Lambda+\rho|^2\}$, and order elements of this set, $\lambda_1, \lambda_2, \dots$ so that $\lambda_i \geq \lambda_j$ implies $i \leq j$. Then

$$\mathrm{ch}\, M(\lambda_i) = \sum_j c_{i,j} \mathrm{ch}\, L(\lambda_j),$$

with $c_{i,i} = 1$ and $c_{i,j} = 0$ for $j > i$. So we can solve this system of linear equations to complete the proof of the lemma. □

9.3 Generators and relations

Recall from chapter 1 that $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r}$, and $\mathfrak{r} = \mathfrak{r}_+ \oplus \mathfrak{r}_-$ where $\mathfrak{r}_\pm$ are ideals of $\tilde{\mathfrak{g}}$ defined by $\mathfrak{r}_\pm = \mathfrak{r} \cap \tilde{\mathfrak{n}}_\pm$. Set $\mathfrak{r}_\alpha = \mathfrak{r} \cap \tilde{\mathfrak{g}}_\alpha$.

We deal with some general lemmas first.

Lemma 9.3.1 *Let $\theta : \mathfrak{g} \to \mathfrak{g}'$ be a surjective homomorphism of Lie algebras with kernel $\mathfrak{r}$. Let $\varphi : U(\mathfrak{g}) \to U(\mathfrak{g}')$ be the corresponding homomorphism between enveloping algebras. Then the kernel of φ is $\mathfrak{r}U(\mathfrak{g})$.*

Proof Since $\mathfrak{r}$ is an ideal of $\mathfrak{g}$, $\mathfrak{r}U(\mathfrak{g})$ is a two-sided ideal of $U(\mathfrak{g})$. Moreover, $\mathfrak{r}U(\mathfrak{g}) \subset \ker\varphi$. Conversely, we have a homomorphism

$$\alpha : U(\mathfrak{g})/\mathfrak{r}U(\mathfrak{g}) \to U(\mathfrak{g}')$$

induced by φ. We consider the Lie algebra $U(\mathfrak{g})/\mathfrak{r}U(\mathfrak{g})$ (with respect to the commutator bracket). Define a map

$$\mathfrak{g}' \to U(\mathfrak{g})/\mathfrak{r}U(\mathfrak{g})$$

as follows. Given $x' \in \mathfrak{g}'$, choose $x_1 \in \mathfrak{g}$ with $\theta(x_1) = x'$. Then x_1 gives rise to $\bar{x}_1 \in U(\mathfrak{g})/\mathfrak{r}U(\mathfrak{g})$. This map is well-defined and is a Lie algebra homomorphism. By the universal property there is a map

$$\beta : U(\mathfrak{g}') \to U(\mathfrak{g})/\mathfrak{r}U(\mathfrak{g})$$

extending the constructed homomorphism of Lie algebras. It is readily checked that α and β are inverse homomorphisms, and thus isomorphisms. □

The two-sided ideal $U_0(\mathfrak{g}) := \mathfrak{g}U(\mathfrak{g})$ of $U(\mathfrak{g})$ is called the *augmentation ideal* of $U(\mathfrak{g})$.

Lemma 9.3.2 $\mathfrak{g} \cap (U_0(\mathfrak{g})^2) = [\mathfrak{g}, \mathfrak{g}]$.

Proof Since $\mathfrak{g} \subset U_0(\mathfrak{g})$ and $[x, y] = xy - yx$ in $U(\mathfrak{g})$, the embedding $\mathfrak{g} \cap (U_0(\mathfrak{g})^2) \supset [\mathfrak{g}, \mathfrak{g}]$ is clear. Conversely, let $\bar{\mathfrak{g}} = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$. We have a natural homomorphism $U(\mathfrak{g}) \to U(\bar{\mathfrak{g}})$ under which $\mathfrak{g} \cap (U_0(\mathfrak{g})^2)$ maps to $\bar{\mathfrak{g}} \cap (U_0(\bar{\mathfrak{g}})^2)$. Now $\bar{\mathfrak{g}}$ is abelian, so $U(\bar{\mathfrak{g}})$ is a polynomial algebra. In such a polynomial algebra, it is evident that $\bar{\mathfrak{g}} \cap (U_0(\bar{\mathfrak{g}})^2) = 0$. It follows that $\mathfrak{g} \cap (U_0(\mathfrak{g})^2)$ lies in the kernel of $\mathfrak{g} \to \bar{\mathfrak{g}}$, and so $\mathfrak{g} \cap (U_0(\mathfrak{g})^2) \subset [\mathfrak{g}, \mathfrak{g}]$. □

Lemma 9.3.3 *Let* $\mathfrak{r}$ *be a subalgebra of the Lie algebra* $\mathfrak{g}$. *Then* $\mathfrak{r} \cap \mathfrak{r}U_0(\mathfrak{g}) = [\mathfrak{r}, \mathfrak{r}]$.

Proof Since $\mathfrak{r} \subset U_0(\mathfrak{r})$ and $[x, y] = xy - yx$ in $U(\mathfrak{r})$, we have

$$[\mathfrak{r}, \mathfrak{r}] \subset \mathfrak{r} \cap \mathfrak{r}U_0(\mathfrak{r}) \subset \mathfrak{r} \cap \mathfrak{r}U_0(\mathfrak{g}).$$

Conversely, let $\{r_i\}$ be a basis of $\mathfrak{r}$ and extend it to a basis $\{r_i, u_j\}$ of $\mathfrak{g}$. The monomials $\prod r_i^{m_i} u_j^{n_j}$ with $\sum m_i + \sum n_j > 0$ form a basis of $U_0(\mathfrak{g})$ and those with $\sum m_i + \sum n_j \geq 2$ and $\sum m_i \geq 1$ form a basis of $\mathfrak{r}U_0(\mathfrak{g})$. Hence, each element of $\mathfrak{r} \cap \mathfrak{r}U_0(\mathfrak{g})$ is a linear combination of monomials

$\prod r_i^{m_i} u_j^{n_j}$ with $\sum n_j = 0$ and $\sum m_i \geq 2$. Hence

$$\mathfrak{r} \cap \mathfrak{r}U_0(\mathfrak{g}) \subset \mathfrak{r} \cap U_0(\mathfrak{r})^2 = [\mathfrak{r}, \mathfrak{r}],$$

where the last equality comes from Lemma 9.3.2 □

Proposition 9.3.4 *The ideal* $\mathfrak{r}_+$ *(resp.* $\mathfrak{r}_-$*) is generated as an ideal in* $\tilde{\mathfrak{n}}_+$ *(resp.* $\tilde{\mathfrak{n}}_-$*) by those* $\mathfrak{r}_\alpha$ *(resp.* $\mathfrak{r}_{-\alpha}$*) for which* $\alpha \in Q_+ \setminus \Pi$ *and* $2(\rho|\alpha) = (\alpha|\alpha)$.

Proof We define a Verma module $\tilde{M}(\lambda)$ over $\tilde{\mathfrak{g}}$ by

$$\tilde{M}(\lambda) = U(\tilde{\mathfrak{g}}) \otimes_{U(\tilde{\mathfrak{b}}_+)} \mathbb{C}_\lambda,$$

where $\tilde{\mathfrak{b}}_+ = \tilde{\mathfrak{n}}_+ \oplus \mathfrak{h}$ and $\mathbb{C}_\lambda$ is the 1-dimensional $\tilde{\mathfrak{b}}_+$-module with the trivial action of $\tilde{\mathfrak{n}}_+$ and the action of $\mathfrak{h}$ with the weight λ. As for the usual Verma modules, one proves that $\tilde{M}(\lambda)$ has a unique maximal proper submodule $\tilde{M}'(\lambda)$ and that as a $U(\tilde{\mathfrak{n}}_-)$-module, $\tilde{M}(\lambda)$ is a free module on the generator $\tilde{v}_\lambda := 1 \otimes 1$.

Consider the special case $\lambda = 0$. Write $\tilde{v} := \tilde{v}_0 = 1 \otimes 1 \in \tilde{M}(0)$. Since $\tilde{\mathfrak{n}}_-$ is a free Lie algebra on $f_1, \dots, f_n$, $U(\tilde{\mathfrak{n}}_-)$ is a free associative algebra on $f_1, \dots, f_n$. So as vector spaces,

$$U(\tilde{\mathfrak{n}}_-) = \mathbb{C}1 \oplus U(\tilde{\mathfrak{n}}_-)f_1 \oplus \cdots \oplus U(\tilde{\mathfrak{n}}_-)f_n.$$

Thus $U(\tilde{\mathfrak{n}}_-)f_1 \oplus \cdots \oplus U(\tilde{\mathfrak{n}}_-)f_n$ is a $U(\tilde{\mathfrak{n}}_-)$-submodule of codimension 1 in $U(\tilde{\mathfrak{n}}_-)$. It corresponds to the subspace

$$\bigoplus_{i=1}^{n} U(\tilde{\mathfrak{n}}_-)f_i\tilde{v}$$

of codimension 1 in $\tilde{M}(0)$. Moreover, this subspace is a $\mathfrak{g}$-submodule isomorphic to $\bigoplus_{i=1}^n \tilde{M}(-\alpha_i)$ since the vectors $f_i\tilde{v}$ are easily checked to be highest weight vectors of weight $-\alpha_i$, $i = 1, \dots, n$. It follows that

$$\tilde{M}'(0) = \bigoplus_{i=1}^{n} U(\tilde{\mathfrak{n}}_-)f_i\tilde{v} \cong \bigoplus_{i=1}^{n} \tilde{M}(-\alpha_i).$$

Tensoring with $U(\mathfrak{g}) \otimes_{U(\tilde{\mathfrak{g}})}$ we get an isomorphism of $U(\mathfrak{g})$-modules

$$U(\mathfrak{g}) \otimes_{U(\tilde{\mathfrak{g}})} \tilde{M}'(0) \cong \bigoplus_{i=1}^{n} M(-\alpha_i). \tag{9.8}$$

Let $\pi : \tilde{\mathfrak{g}} \to \mathfrak{g}$ be the canonical homomorphism. We define a map

$$\lambda_1 : \mathfrak{r}_- \to U(\mathfrak{g}) \otimes_{U(\tilde{\mathfrak{g}})} \tilde{M}'(0), \;\; a \mapsto 1 \otimes a(\tilde{v}).$$

This is a $\tilde{\mathfrak{g}}$-module homomorphism, where $\tilde{\mathfrak{g}}$ acts on $\mathfrak{r}_-$ via the adjoint action. Indeed, for $x \in \tilde{\mathfrak{g}}, a \in \mathfrak{r}_-$, we have

$$\lambda_1([x,a]) = 1 \otimes (xa - ax)(\tilde{v}) = \pi(x) \otimes a(\tilde{v}) = x(\lambda_1(a))$$

since $\pi(a) = 0$. A similar calculation shows that $\lambda_1([\mathfrak{r}_-, \mathfrak{r}_-]) = 0$. So we have a $\mathfrak{g}$-module homomorphism

$$\lambda : \mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-] \to \bigoplus_{i=1}^{n} M(-\alpha_i) \tag{9.9}$$

by (9.8). More explicitly λ is described as follows: write $a \in \mathfrak{r}_-$ in the form $a = \sum_i u_i f_i$, where $u_i \in U_0(\tilde{\mathfrak{n}}_-)$, and the action of u_i on f_i is the adjoint action extended to the universal enveloping. Then

$$\lambda(a + [\mathfrak{r}_-, \mathfrak{r}_-]) = \sum_i \pi(u_i) v_i,$$

where v_i is the highest weight vector of $M(-\alpha_i)$.

We claim that λ is injective. Indeed, $\lambda(a + [\mathfrak{r}_-, \mathfrak{r}_-]) = 0$ implies $\pi(u_i) = 0$ for all i, hence $u_i \in \mathfrak{r}_- U(\mathfrak{n}_-)$, see Lemma 9.3.1. So $\sum u_i f_i \in \mathfrak{r}_- U_0(\mathfrak{n}_-)$. So $a \in \mathfrak{r}_- \cap \mathfrak{r}_- U_0(\mathfrak{n}_-) = [\mathfrak{r}_-, \mathfrak{r}_-]$ by Lemma 9.3.3. Thus we have an embedding (9.9) in the category $\mathcal{O}$.

Now let $-\alpha$ ($\alpha \in Q_+$) be a primitive weight of the $\mathfrak{g}$-module $\mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-]$. Note that $\alpha \notin \Pi$ since no f_i belongs to $\mathfrak{r}_-$. Using the embedding and the action of Casimir we deduce that $(-\alpha+\rho|-\alpha+\rho) = (-\alpha_i+\rho|-\alpha_i+\rho)$ for some i. Since $2(\rho|\alpha_i) = (\alpha_i|\alpha_i)$ by (2.18), we get $2(\rho|\alpha) = (\alpha|\alpha)$.

By Lemma 9.1.3, $\mathfrak{r}_-/[\mathfrak{r}_-, \mathfrak{r}_-]$ is generated as an $\mathfrak{n}_-$-module by the representatives of those $\mathfrak{r}_{-\alpha}$ for which $\alpha \in \Delta_+ \setminus \Pi$ and $2(\rho|\alpha) = (\alpha|\alpha)$. We want to deduce from it that $\mathfrak{r}_-$ is generated as an $\tilde{\mathfrak{n}}_-$-module by such $\mathfrak{r}_\alpha$ (equivalently the ideal of $\tilde{\mathfrak{n}}_-$ generated by such $\mathfrak{r}_{-\alpha}$). Let $\mathfrak{k}$ be the the $\tilde{\mathfrak{n}}_-$-submodule generated by such $\mathfrak{r}_{-\alpha}$. Then $\mathfrak{k} + [\mathfrak{r}_-, \mathfrak{r}_-] = \mathfrak{r}_-$. Suppose $\mathfrak{k} \neq \mathfrak{r}_-$. Then $\mathfrak{r}_-/\mathfrak{k}$ is an $\tilde{\mathfrak{n}}_-$-module. Consider the submodule $[\mathfrak{r}_-/\mathfrak{k}, \mathfrak{r}_-/\mathfrak{k}]$ of $\mathfrak{r}_-/\mathfrak{k}$. This is an $\tilde{\mathfrak{n}}_-$-module whose weights are of the form $\beta + \gamma$ where β, γ are weights of $\mathfrak{r}_-/\mathfrak{k}$. Thus if α is a weight of $\mathfrak{r}_-/\mathfrak{k}$ for which $|\operatorname{ht}\alpha|$ is minimal then α cannot be a weight of $[\mathfrak{r}_-/\mathfrak{k}, \mathfrak{r}_-/\mathfrak{k}]$. Thus $[\mathfrak{r}_-/\mathfrak{k}, \mathfrak{r}_-/\mathfrak{k}] \neq \mathfrak{r}_-/\mathfrak{k}$, and this gives $\mathfrak{k} + [\mathfrak{r}_-, \mathfrak{r}_-] \neq \mathfrak{r}_-$, a contradiction.

This completes the proof for $\mathfrak{r}_-$. The result for $\mathfrak{r}_+$ follows by applying the involution $\tilde{\omega}$. □

Theorem 9.3.5 *Let A be symmetrizable. Then the elements*

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j, \quad i \neq j \qquad (i,j = 1, \dots, n), \tag{9.10}$$

$$(\operatorname{ad} f_i)^{1-a_{ij}} f_j, \quad i \neq j \qquad (i,j = 1, \dots, n) \tag{9.11}$$

generate the ideals $\mathfrak{r}_+$ *and* $\mathfrak{r}_-$*, respectively.*

Proof Denote by $\bar{\mathfrak{g}}$ the quotient of $\tilde{\mathfrak{g}}$ by the ideal generated by all elements (9.10) and (9.11). The natural surjection $\tilde{\mathfrak{g}} \to \mathfrak{g}$ factors through surjections $\tilde{\mathfrak{g}} \to \bar{\mathfrak{g}} \to \mathfrak{g}$, thanks to Lemma 3.1.1. We have the induced Q-gradation of $\bar{\mathfrak{g}}$:

$$\bar{\mathfrak{g}} = \bigoplus_{\alpha \in Q} \bar{\mathfrak{g}}_\alpha.$$

Let $\bar{\mathfrak{r}}$ (resp. $\bar{\mathfrak{r}}_\pm$) denote the image of $\mathfrak{r}$ (resp. $\mathfrak{r}_\pm$) in $\bar{\mathfrak{g}}$. We just need to show that $\bar{\mathfrak{r}}_+ = 0$ (then $\bar{\mathfrak{r}}_- = 0$ too by applying $\tilde{\omega}$). Otherwise, choose the root α of minimal height among the roots $\alpha \in Q_+ \setminus \{0\}$ such that $(\bar{\mathfrak{r}}_+)_\alpha \neq 0$ and let $\alpha = \sum k_i \alpha_i$. It is clear that $(\mathfrak{r}_+)_\alpha$ must occur in any system of homogeneous generators of $\mathfrak{r}_+$ as an ideal of $\mathfrak{n}_+$. It follows from Proposition 9.3.4 that $(\alpha|\alpha) = 2(\rho|\alpha)$.

We know that the Weyl group W acts on the weights of $\mathfrak{g} = \tilde{\mathfrak{g}}/\mathfrak{r}$ and that weights in the same W-orbit have the same multiplicity. The same argument can be applied to $\bar{\mathfrak{g}}$ to give a similar result (the proofs in §3.2 relied only on Serre relations, which hold in $\bar{\mathfrak{g}}$). Since

$$\dim \bar{\mathfrak{g}}_\alpha = \dim \mathfrak{g}_\alpha + \dim \bar{\mathfrak{r}}_\alpha$$

we see that W acts on the weights of $\bar{\mathfrak{r}}$ and that weights in the same W-orbit have the same multiplicity. It follows using Lemma 3.2.2 that $(\bar{\mathfrak{r}}_+)_{r_i\alpha} \neq 0$ for any i. Now $\mathrm{ht}\,(r_i\alpha) \geq \mathrm{ht}\,(\alpha)$ implies $(\alpha_i|\alpha) \leq 0$, whence $(\alpha|\alpha) \leq 0$. But

$$2(\rho|\alpha) = 2\sum k_i(\rho|\alpha_i) = \sum k_i(\alpha_i|\alpha_i) > 0.$$

This contradicts $(\alpha|\alpha) = 2(\rho|\alpha)$. □

10

Weyl-Kac Character formula

10.1 Integrable highest weight modules and Weyl group

Set

$$\begin{aligned} P &= \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \ (i = 1, \dots, n)\}, \\ P_+ &= \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \ (i = 1, \dots, n)\}, \\ P_{++} &= \{\lambda \in P \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \ (i = 1, \dots, n)\}. \end{aligned}$$

The set P is called the *weight lattice*, elements from P (resp. P_+, resp. P_{++}) are called *integral weights* (resp. *dominant*, resp. *regular dominant weights*). Note that P contains the root lattice Q.

Let V be a highest weight module over $\mathfrak{g}$ with highest weight vector v. It follows from Lemmas 3.1.2(ii) and 3.1.3 that V is integrable if and only if $f_i^{N_i} v = 0$ for some $N_i > 0$ $(i = 1, \dots, n)$.

Lemma 10.1.1 *The $\mathfrak{g}$-module $L(\Lambda)$ is integrable if and only if $\Lambda \in P_+$.*

Proof Follows from the previous paragraph and representation theory of $\mathfrak{sl}_2$. □

Denote by $P(\Lambda)$ the set of weights of $L(\Lambda)$. It is clear that $P(\Lambda) \subset P$ if $\lambda \in P$. The following proposition follows from Lemma 10.1.1 and Proposition 3.2.3.

Proposition 10.1.2 *If $\Lambda \in P_+$, then for all $w \in W$ we have*

$$\operatorname{mult}_{L(\Lambda)} \lambda = \operatorname{mult}_{L(\Lambda)} w(\lambda).$$

In particular, $P(\Lambda)$ is W-invariant.

Corollary 10.1.3 *If $\Lambda \in P_+$ then any $\lambda \in P(\Lambda)$ is W-conjugate to a unique $\mu \in P_+ \cap P(\Lambda)$.*

Proof Follows from Proposition 3.4.1. □

We let W act on the complex vector space $\tilde{\mathcal{E}}$ of all (possibly infinite) formal linear combinations $\sum_\lambda c_\lambda e(\lambda)$ by

$$w(\sum_\lambda c_\lambda e(\lambda)) = \sum_\lambda c_\lambda e(w\lambda).$$

The space $\tilde{\mathcal{E}}$ contains $\mathcal{E}$ as a subspace. However, the product of two elements $P_1, P_2 \in \tilde{\mathcal{E}}$ doesn't always make sense. If it does, then $w(P_1 P_2) = w(P_1)w(P_2)$. Proposition 10.1.2 implies that

$$w(\operatorname{ch} L(\Lambda)) = \operatorname{ch} L(\Lambda) \qquad (w \in W,\ \Lambda \in P_+). \tag{10.1}$$

Consider now the element

$$R := \prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\operatorname{mult} \alpha} \in \mathcal{E}.$$

For $w \in W$ set

$$\varepsilon(w) := (-1)^{\ell(w)} = \det_{\mathfrak{h}^*} w.$$

We next claim that

$$w(e(\rho)R) = \varepsilon(w)e(\rho)R \qquad (w \in W). \tag{10.2}$$

Indeed, it is sufficient to check (10.2) for each fundamental reflection r_i. Recall that the set $\Delta_+ \setminus \{\alpha_i\}$ is r_i-invariant and $\operatorname{mult} r_i(\alpha) = \operatorname{mult} \alpha$ for $\alpha \in \Delta_+$. So

$$\begin{aligned} r_i(e(\rho)R) &= (r_i e(\rho))(r_i R) \\ &= e(r_i\rho) r_i(1 - e(-\alpha_i)) \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} r_i(1 - e(-\alpha))^{\operatorname{mult} \alpha} \\ &= e(\rho - \alpha_i)(1 - e(\alpha_i)) \prod_{\alpha \in \Delta_+ \setminus \{\alpha_i\}} (1 - e(-\alpha))^{\operatorname{mult} \alpha} \\ &= -e(\rho)R. \end{aligned}$$

10.2 The character formula

From now on we assume that A is symmetrizable and $(\cdot|\cdot)$ is the standard bilinear form on $\mathfrak{g}$.

Lemma 10.2.1 *Let* $\lambda, \Lambda \in P$*,* $\lambda \le \Lambda$*, and* $\Lambda + \lambda \in P_+$*. Then either* $\langle \Lambda + \lambda, \alpha_i^\vee \rangle = 0$ *for* $i \in \operatorname{supp}(\Lambda - \lambda)$ *or* $(\Lambda|\Lambda) > (\lambda|\lambda)$*. In particular, if* $\Lambda \in P_{++}$*,* $\lambda \in P_+$*, and* $\lambda < \Lambda$*, then* $(\Lambda|\Lambda) > (\lambda|\lambda)$*.*

Proof We have $\Lambda - \lambda = \sum_i k_i \alpha_i$, $k_i \in \mathbb{Z}_+$. Hence

$$(\Lambda|\Lambda) - (\lambda|\lambda) = (\Lambda + \lambda|\Lambda - \lambda) = \sum_i \frac{(\alpha_i|\alpha_i)}{2} k_i \langle \Lambda + \lambda, \alpha_i^\vee \rangle.$$

Since $(\alpha_i|\alpha_i) > 0$ the result follows. □

Theorem 10.2.2 (Weyl-Kac character formula) *Let* $\mathfrak{g}$ *be a symmetrizable Kac-Moody algebra, and let* $L(\Lambda)$ *be an irreducible* $\mathfrak{g}$*-module with highest weight* $\Lambda \in P_+$*. Then*

$$\operatorname{ch} L(\Lambda) = \frac{\sum_{w \in W} \varepsilon(w) e(w(\Lambda + \rho) - \rho)}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\operatorname{mult} \alpha}}. \tag{10.3}$$

Proof Multiplying both sides of (9.7) by $e(\rho)R$ and using Lemma 9.2.4, we get

$$e(\rho) R \operatorname{ch} L(\Lambda) = \sum_{\lambda \le \Lambda,\ |\lambda+\rho|^2 = |\Lambda+\rho|^2} c_\lambda e(\lambda + \rho), \tag{10.4}$$

for $c_\lambda \in \mathbb{Z}$ with $c_\Lambda = 1$. By (10.1) and (10.2), the LHS of the last equation is W-skew-invariant. Hence the coefficients in the RHS have the following property:

$$c_\lambda = \varepsilon(w) c_\mu \quad \text{if } w(\lambda + \rho) = \mu + \rho \text{ for some } w \in W. \tag{10.5}$$

Let λ be such that $c_\lambda \neq 0$. Then by (10.5) we have $c_{w(\lambda+\rho)-\rho} \neq 0$ for all $w \in W$. Hence it follows from (10.4) that $w(\lambda+\rho) \le \Lambda+\rho$ for all $w \in W$. Let $\mu \in \{w(\lambda + \rho) - \rho \mid w \in W\}$ be such that $\operatorname{ht}(\Lambda - \mu)$ is minimal. Then $\mu + \rho \in P_+$ and $|\mu + \rho|^2 = |\Lambda + \rho|^2$. Applying Lemma 10.2.1 to the elements $\Lambda + \rho \in P_{++}$ and $\mu + \rho$, we deduce that $\mu = \Lambda$. Thus $c_\lambda \neq 0$ implies $\lambda = w(\Lambda + \rho)$ for some $w \in W$, and in this case $c_\lambda = \varepsilon(w)$, see (10.5).

But $\Lambda + \rho \in P_{++}$, so by Proposition 3.4.1(ii), $w(\Lambda + \rho) = \Lambda + \rho$ implies $w = 1$. Hence finally we obtain

$$e(\rho) R \operatorname{ch} L(\Lambda) = \sum_{w \in W} \varepsilon(w) e(w(\Lambda + \rho) - \rho),$$

as required. □

Take $\Lambda = 0$ in the Weyl-Kac character formula. Since $L(0)$ is the trivial module, its character is $e(0) = 1_{\mathcal{E}}$. This gives the following *denominator identity*:

$$\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\mathrm{mult}\,\alpha} = \sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho). \tag{10.6}$$

Substituting into (10.3) we get another form of the Weyl-Kac character formula:

$$\mathrm{ch}\,L(\Lambda) = \frac{\sum_{w\in W}\varepsilon(w)e(w(\Lambda+\rho)-\rho)}{\sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho)}. \tag{10.7}$$

Remark 10.2.3 In the proof of the Weyl-Kac formula we never used the fact that $L(\Lambda)$ is irreducible, but only that $L(\Lambda)$ is integrable highest weight module with highest weight Λ. This happens if and only if $\Lambda \in P_+$ and

$$f_i^{\langle\Lambda,\alpha_i^\vee\rangle+1}(v_\Lambda) = 0 \qquad (i = 1,\dots,n). \tag{10.8}$$

Indeed, if $L(\Lambda)$ is integrable, then clearly $\Lambda \in P_+$. Moreover, if the (10.8) fails then one of the $f_i^{\langle\Lambda,\alpha_i^\vee\rangle+1}(v_\Lambda)$ is a non-zero highest weight vector of negative highest weight for the corresponding $\mathfrak{sl}_2$, which contradicts integrability again. The converse follows from Lemmas 3.1.2(ii) and 3.1.3. We make two conclusions: first, if $\Lambda \in P_+$ is dominant and V is an integrable module generated by highest weight vector of weight Λ then $V = L(\Lambda)$. Second,

$$L(\Lambda) = M(\Lambda)/\sum_i(U(\mathfrak{n}_-)f_i^{\langle\Lambda,\alpha_i^\vee\rangle+1}(v_\Lambda)) \qquad (\Lambda\in P_+). \tag{10.9}$$

Consider the expansion

$$\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{-\mathrm{mult}\,\alpha} = \sum_{\beta\in\mathfrak{h}^*}K(\beta)e(-\beta), \tag{10.10}$$

defining the function

$$K : \mathfrak{h}^* \to \mathbb{Z}_+$$

called the *(generalized Kostant) partition function*. Note that $K(\beta) = 0$ unless $\beta \in Q_+$, $K(0) = 1$, and for $\beta \in Q_+$, $K(\beta)$ is the number of partitions of β into a sum of positive roots. Now the character formula for Verma modules can be rewritten as follows:

$$\mathrm{mult}_{M(\Lambda)}\,\lambda = K(\Lambda-\lambda). \tag{10.11}$$

Now, substitute (10.10) to (10.3):

$$\begin{aligned}\sum_{\lambda\le\Lambda}(\mathrm{mult}_{L(\Lambda)}\,\lambda)e(\lambda) &= \sum_{w\in W}\varepsilon(w)e(w(\Lambda+\rho)-\rho)\sum_{\beta\in\mathfrak{h}^*}K(\beta)e(-\beta)\\ &= \sum_{w\in W}\sum_{\beta\in\mathfrak{h}^*}\varepsilon(w)K(\beta)e(-\beta+w(\Lambda+\rho)-\rho)\\ &= \sum_{w\in W}\sum_{\lambda\in\mathfrak{h}^*}\varepsilon(w)K(w(\Lambda+\rho)-(\lambda+\rho))e(\lambda).\end{aligned}$$

Comparing the coefficients at $e(\lambda)$, we obtain the (generalized) *Kostant multiplicity formula*:

$$\mathrm{mult}_{L(\Lambda)}\,\lambda = \sum_{w\in W}\varepsilon(w)K(w(\Lambda+\rho)-(\lambda+\rho)). \tag{10.12}$$

Assume now that A is of finite type and for any $\lambda\in P$ denote

$$\chi(\lambda)=\frac{\sum_{w\in W}\varepsilon(w)e(w(\lambda+\rho)-\rho)}{\sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho)}=\frac{\sum_{w\in W}\varepsilon(w)e(w(\lambda+\rho))}{\sum_{w\in W}\varepsilon(w)e(w(\rho))}.$$

In particular, for $\lambda\in P_+$, we have $\chi(\lambda)=\mathrm{ch}\,L(\lambda)$, but it makes sense to consider $\chi(\lambda)$ for any $\lambda\in P$. It is interesting to specialize $e(\alpha)$ to 1 for all α. The result of such specialization in the expression $\chi(\lambda)$ is denoted $d(\lambda)$. For example, if $\lambda\in P_+$, then $d(\lambda)=\dim L(\lambda)$. There is a nice fomula for $d(\lambda)$ called the *Weyl dimension formula*:

$$d(\lambda)=\frac{\prod_{\alpha\in\Delta_+}(\lambda+\rho|\alpha)}{\prod_{\alpha\in\Delta_+}(\rho|\alpha)}. \tag{10.13}$$

To prove the formula, let $\mathcal{E}$ be the ring for $\mathfrak{g}$ defined in §9.2 and denote by $\mathcal{E}_0$ the subring consisting of all finite sums $\sum_{\mu\in P}n_\mu e(\mu)$ with $n_\mu\in\mathbb{Z}$. Then we have in $\mathcal{E}_0$:

$$\Big(\sum_{w\in W}\varepsilon(w)e(w\rho)\Big)\chi(\lambda)=\sum_{w\in W}\varepsilon(w)e(w(\lambda+\rho)). \tag{10.14}$$

Let $A=\mathbb{R}[[t]]$. Then for each $\xi\in P$ we have a ring homomorphism

$$\theta_\xi:\mathcal{E}_0\to A,\quad e(\mu)\mapsto\exp((\xi|\mu)t).$$

We have

$$\begin{aligned}\theta_\xi\big(\sum_{w\in W}\varepsilon(w)e(w\mu)\big) &= \sum_{w\in W}\varepsilon(w)\exp((\xi|w\mu)t)\\ &= \sum_{w\in W}\varepsilon(w)\exp((\mu|w\xi)t)\\ &= \theta_\mu\big(\sum_{w\in W}\varepsilon(w)e(w\xi)\big).\end{aligned}$$

In particular we have

$$\begin{aligned}\theta_\rho\big(\sum_{w\in W}\varepsilon(w)e(w(\lambda+\rho)\big) &= \theta_{\lambda+\rho}\big(\sum_{w\in W}\varepsilon(w)e(w\rho)\big)\\ &= \theta_{\lambda+\rho}\big(e_{-\rho}\prod_{\alpha\in\Delta_+}(e(\alpha)-1)\big)\\ &= \exp((\lambda+\rho|-\rho)t)\prod_{\alpha\in\Delta_+}\exp((\lambda+\rho|\alpha)t)-1)\\ &= t^N(\prod_{\alpha\in\Delta_+}(\lambda+\rho|\alpha)+\dots),\end{aligned}$$

where $N=|\Delta_+|$. By putting $\lambda=0$ we obtain

$$\theta_\rho(\sum_{w\in W}\varepsilon(w)e(w\rho))=t^N(\prod_{\alpha\in\Delta_+}(\rho|\alpha)+\dots).$$

Thus by applying θ_ρ to (10.14), we obtain

$$t^N(\prod_{\alpha\in\Delta_+}(\rho|\alpha)+\dots)\theta_\rho(\chi(\lambda))=t^N(\prod_{\alpha\in\Delta_+}(\lambda+\rho|\alpha)+\dots).$$

By canceling t^N and taking the constant term we obtain

$$\prod_{\alpha\in\Delta_+}(\rho|\alpha)d(\lambda)=\prod_{\alpha\in\Delta_+}(\lambda+\rho|\alpha).$$

10.3 Example: $\hat{\mathcal{L}}(\mathfrak{s}l_2)$

Consider the denominator identity for the case $\mathfrak{g}=\hat{\mathcal{L}}(\mathfrak{s}l_2)$. Remember from Example 1.5.4 that the positive roots are of the form $\alpha_1+k\delta$ for $k\in\mathbb{Z}_+$ and $-\alpha_1+n\delta$, $n\delta$ for $n\in\mathbb{N}$, and that they all have multiplicity 1. Denote $e(-\delta)$ by q and $e(-\alpha_1)$ by z. Then the left hand side of (10.6) is

$$\prod_{n>0}(1-q^n)(1-q^{n-1}z)(1-q^nz^{-1}).$$

To compute the right hand side, remember from Example 3.4.2(ii) that $W \cong \mathbb{Z} \rtimes S_2$, where the generators $1 \in \mathbb{Z}$ and $s \in S_2$ act on weights by the following formulas

$$\begin{aligned} 1 &: \alpha_1 \mapsto \alpha_1 - 2\delta,\ \delta \mapsto \delta,\ \Lambda_0 \mapsto \alpha_1 - \delta + \Lambda_0, \\ s &: \alpha_1 \mapsto -\alpha_1,\ \delta \mapsto \delta,\ \Lambda_0 \mapsto \Lambda_0. \end{aligned}$$

Now, we can take $\rho = \alpha_1/2 + 2\Lambda_0$. So

$$\begin{aligned} 1 &: \rho \mapsto \rho + 2\alpha_1 - 3\delta, \\ s &: \rho \mapsto \rho - \alpha_1. \end{aligned}$$

We deduce that

$$\begin{aligned} m &: \rho \mapsto \rho + 2m\alpha_1 - (2m^2 + m)\delta, \\ sm &: \rho \mapsto \rho - (2m+1)\alpha_1 - (2m^2 + m)\delta. \end{aligned}$$

Now the right hand side of (10.6) is

$$\begin{aligned} \sum_{w \in W} \varepsilon(w) e(w(\rho) - \rho) &= \sum_{m \in \mathbb{Z}} e(2m\alpha_1 - (2m^2 + m)\delta) \\ &\quad - \sum_{m \in \mathbb{Z}} e(-(2m+1)\alpha_1 - (2m^2 + m)\delta) \\ &= \sum_{k \in \mathbb{Z}} (-1)^k e(-k\alpha_1 - \frac{k(k-1)}{2}\delta) \\ &= \sum_{k \in \mathbb{Z}} (-1)^k z^k q^{\frac{k(k-1)}{2}}. \end{aligned}$$

So the Weyl-Kac denominator identity for the easiest affine type $A_1^{(1)}$ becomes

$$\prod_{n>0} (1 - q^n)(1 - q^{n-1}z)(1 - q^n z^{-1}) = \sum_{m \in \mathbb{Z}} (-1)^m z^m q^{\frac{m(m-1)}{2}}.$$

This is a highly non-trivial *Jacobi's triple product identity.* Let us divide both sides by $(1 - z)$ to get an equivalent form

$$\prod_{n>0} (1 - q^n)(1 - q^n z)(1 - q^n z^{-1}) = \sum_{k \in \mathbb{Z}} \frac{z^{-2k} - z^{(2k+1)}}{1 - z} q^{k(2k+1)}.$$

By various specializations we get more famous identities. For example, take $z = 1$ in the second form to get

$$\varphi(q)^3 = \sum_{k \in \mathbb{Z}} (4k+1) q^{k(2k+1)},$$

where

$$\varphi(q) := \prod_{n>0}(1-q^n).$$

Another specialization is obtained by applying a homomorphism

$$\theta : \mathbb{C}[[e(-\alpha_0), e(-\alpha_1)]] \to \mathbb{C}[[q]],\ e(-\alpha_0) \mapsto q^{s_0},\ e(-\alpha_1) \mapsto q^{s_1}.$$

Then the first form specializes to

$$\begin{aligned} & \prod_{n>0}(1-q^{(s_0+s_1)n})(1-q^{s_0(n-1)+s_1 n})(1-q^{s_0 n+s_1(n-1)}) \\ = & \sum_{m\in\mathbb{Z}}(-1)^m q^{s_0\frac{m(m-1)}{2}+s_1\frac{m(m+1)}{2}}. \end{aligned}$$

Let us take $(s_0, s_1) = (1, 1)$. We obtain

$$\frac{\varphi(q)^2}{\varphi(q^2)} = \sum_{m\in\mathbb{Z}}(-1)^m q^{m^2}$$

or

$$\begin{aligned} & (1-q)^2(1-q^2)(1-q^3)^2(1-q^4)(1-q^5)^2(1-q^6)\dots \\ = & 1-2q+2q^4-2q^9+2q^{16}-\dots. \end{aligned}$$

This is a classical *Gauss identity.* Next take $(s_0, s_1) = (2, 1)$. We obtain

$$\varphi(q) = \sum_{m\in\mathbb{Z}}(-1)^m q^{\frac{m(3m-1)}{2}}$$

or

$$\begin{aligned} & (1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)(1-q^6)\dots \\ = & 1-q-q^2+q^5+q^7-q^{12}-q^{15}+q^{22}+q^{26}-\dots. \end{aligned}$$

This is a classical *Euler identity.*

10.4 Complete reducibility

Lemma 10.4.1 *Let A be symmetrizable and $V \in \mathcal{O}$. Assume that for any primitive weights λ, μ of V such that $\lambda > \mu$ one has*

$$2(\lambda+\rho|\lambda-\mu) \neq (\lambda-\mu|\lambda-\mu). \qquad (10.15)$$

Then V is completely reducible.

Proof Every module in $\mathcal{O}$ is locally finite over Ω (this follows from the fact that Ω preserves weight spaces). It follows that V is a direct sum of generalized eigenspaces for Ω. We may assume that V is one such eigenspace, i.e. $\Omega - aI$ acts locally nilpotently on V for some $a \in \mathbb{C}$. Now, let v be a primitive vector of weight λ. Then there is a submodule U such that $\Omega(v) = (|\lambda + \rho|^2 - |\rho|^2)v \pmod U$ (see Corollary 2.3.6). Hence $|\lambda+\rho|^2 - |\rho|^2 = a$, whence $|\lambda+\rho|^2 = |\mu+\rho|^2$ for any two primitive weight λ and μ, which is equivalent to $2(\lambda + \rho|\lambda - \mu) \neq (\lambda - \mu|\lambda - \mu)$, which contradicts (10.15).

So we have proved that for two primitive weights λ and μ of V, the inequality $\lambda \geq \mu$ implies $\lambda = \mu$. This property actually implies complete reducibility. Indeed, let $V^0 = \oplus_{\lambda \in L} V^0_\lambda$ be the space of singular vectors in V (i.e. vectors killed by $\mathfrak{n}_+$), where L is the set of singular weights. Let v be a nonzero vector from V^0_λ. Then $U(\mathfrak{g})v$ is irreducible. Indeed, if this is not the case, there is a non-trivial submodule $U \subsetneq U(\mathfrak{g})v$, whose maximal μ weight would be singular and $\mu < \lambda$. Therefore the $U(\mathfrak{g})$-submodule V' generated by V^0 is completely reducible. It remains to show that $V' = V$. If this is not the case, consider a singular vector $v + V'$ of weight μ in V/V'. We have $e_i(v) \in V'$ for all i, and $e_i(v) \neq 0$ for some i. But then, in view of Lemma 9.1.3, there exists a primitive weight $\lambda \geq \mu + \alpha_i > \mu$, giving a contradiction. □

Theorem 10.4.2 *Let $\mathfrak{g}$ be symmetrizable. Then every integrable module from the category $\mathcal{O}$ is a direct sum of modules $L(\Lambda)$, $\Lambda \in P_+$.*

Proof In view of Lemma 10.4.1, it suffices to check that if $\lambda > \mu$ are primitive weights and $\beta := \lambda - \mu$, then

$$2\langle \lambda + \rho, \nu^{-1}(\beta)\rangle \neq (\beta|\beta).$$

Since the module is integrable, we have

$$\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_+ \qquad (i = 1, \dots, n)$$

for every primitive weight λ. But then

$$\begin{aligned} 2\langle \lambda + \rho, \nu^{-1}(\beta)\rangle - (\beta|\beta) &= \langle \lambda + (\lambda - \beta) + 2\rho, \nu^{-1}(\beta)\rangle \\ &= \langle \lambda + \mu + 2\rho, \nu^{-1}(\beta)\rangle > 0. \end{aligned}$$

□

Corollary 10.4.3

(i) *A $\mathfrak{g}$-module $V \in \mathcal{O}$ is integrable if and only if V is a direct sum of modules $L(\Lambda)$ with $\Lambda \in P_+$.*

(ii) *Tensor product of a finite number of integrable highest weight modules is a direct sum of modules $L(\Lambda)$ with $\Lambda \in P_+$.*

10.5 Macdonald's identities

From now on we assume that $\mathfrak{g}$ is affine.

Recall the Kac's denominator formula (10.6):

$$\prod_{\alpha\in\Delta_+}(1-e(-\alpha))^{\operatorname{mult}\alpha} = \sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho).$$

We first assume that $\mathfrak{g}$ is an untwisted affine algebra, i.e. $\mathfrak{g} = \hat{\mathcal{L}}(\overset{\circ}{\mathfrak{g}})$. Then

$$\begin{aligned}\Delta^{\mathrm{re}} &= \{\alpha + n\delta \mid \alpha \in \overset{\circ}{\Delta},\ n\in\mathbb{Z}\} \cup \{n\delta \mid n \in \mathbb{Z},\ n\neq 0\}.\\ \Delta_+ &= \{\alpha + n\delta \mid \alpha \in \overset{\circ}{\Delta},\ n>0\} \cup \overset{\circ}{\Delta}_+ \cup \{n\delta \mid n>0\}.\end{aligned}$$

The left hand side of the denominator formula can be expressed as

$$\prod_{\alpha\in\overset{\circ}{\Delta}_+}(1-e(-\alpha))\prod_{n>0}\Big(\big(1-e(-n\delta)\big)^{\ell}\prod_{\alpha\in\overset{\circ}{\Delta}}\big(1-e(-\alpha-n\delta)\big)\Big).$$

We also recall that

$$W = T\,\overset{\circ}{W}, \qquad T = \{t_\alpha \mid \alpha \in M\},$$

where

$$t_\alpha(\lambda) = \lambda + \langle\lambda, c\rangle\alpha - \big((\lambda|\alpha) + \frac{1}{2}(\alpha|\alpha)\langle\lambda,c\rangle\big)\delta,$$

and

$$M = \begin{cases} \sum_{i=1}^{\ell}\mathbb{Z}\alpha_i & \text{for types } A_\ell^{(1)},\ D_\ell^{(1)},\ E_\ell^{(1)};\\ \sum_{\alpha_i\ \mathrm{long}}\mathbb{Z}\alpha_i + \sum_{\alpha_i\ \mathrm{short}} p\mathbb{Z}\alpha_i & \text{for types } B_\ell^{(1)},\ C_\ell^{(1)},\ F_4^{(1)},\ G_2^{(1)}.\end{cases}$$

In calculating the right hand side of the denominator formula we recall that

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}c \oplus \mathbb{C}d, \qquad \mathfrak{h}^* = (\overset{\circ}{\mathfrak{h}})^* \oplus \mathbb{C}\delta \oplus \mathbb{C}\Lambda_0.$$

Accordingly $\lambda \in \mathfrak{h}^*$ can be written as

$$\lambda = \overset{\circ}{\lambda} + \langle\lambda, c\rangle\Lambda_0 + \langle\lambda, d\rangle\delta$$

where $\overset{\circ}{\lambda}\in (\overset{\circ}{\mathfrak{h}})^*$. We also recall that $\rho \in \mathfrak{h}^*$ satisfies

$$\langle \rho, d\rangle = 0, \quad \langle \rho, \alpha_i^\vee\rangle = 1 \qquad (0 \le i \le \ell).$$

Since $\langle \rho, c\rangle = \sum_{i=0}^{\ell} a_i^\vee = h^\vee$, we have

$$\rho = \overset{\circ}{\rho} + h^\vee \Lambda_0$$

where $\overset{\circ}{\rho}\in (\overset{\circ}{\mathfrak{h}})^*$ satisfies

$$\langle \overset{\circ}{\rho}, \alpha_i^\vee\rangle = 1 \qquad (1 \le i \le \ell).$$

We now consider the right hand side of the denominator formula. Let $w \in W$ have form $w = \overset{\circ}{w} t_\alpha$ where $\overset{\circ}{w}\in\overset{\circ}{W}$ and $\alpha \in M$. Then

$$\begin{aligned} w(\rho) - \rho &= \overset{\circ}{w} t_\alpha(\rho) - \rho \\ &= \overset{\circ}{w} \big(\rho + h^\vee\alpha - \big((\rho|\alpha) + \frac{1}{2}(\alpha|\alpha)h^\vee\big)\delta\big) - \rho \\ &= \overset{\circ}{w} (\rho) - \rho + h^\vee \overset{\circ}{w} (\alpha) - \big((\rho|\alpha) + \frac{1}{2}(\alpha|\alpha)h^\vee\big)\delta \\ &= \overset{\circ}{w} (\rho) - \rho + h^\vee \overset{\circ}{w} (\alpha) - \big((\overset{\circ}{\rho} |\alpha) + \frac{1}{2}(\alpha|\alpha)h^\vee\big)\delta. \end{aligned}$$

since $\overset{\circ}{w} (\Lambda_0) = \Lambda_0$ and $\langle \Lambda_0, \alpha\rangle = 0$ for $\alpha \in M$. Now the last expression equals

$$\overset{\circ}{w} (h^\vee\alpha + \overset{\circ}{\rho}) - \overset{\circ}{\rho} - \frac{(\overset{\circ}{\rho} + h^\vee\alpha | \overset{\circ}{\rho} + h^\vee\alpha) - (\overset{\circ}{\rho} \mid \overset{\circ}{\rho})}{2h^\vee}\delta.$$

Denote

$$c(\lambda) = (\lambda + \rho|\lambda + \rho) - (\rho|\rho).$$

If $\lambda \in (\overset{\circ}{\mathfrak{h}})^*$ then

$$c(\lambda) = (\lambda + \overset{\circ}{\rho} |\lambda + \overset{\circ}{\rho}) - (\overset{\circ}{\rho} \mid \overset{\circ}{\rho}).$$

We also write for $\lambda \in (\overset{\circ}{\mathfrak{h}})^*$,

$$\overset{\circ}{\chi} (\lambda) = \frac{\sum_{w\in \overset{\circ}{W}} \varepsilon(w) e(w(\lambda + \overset{\circ}{\rho}) - \overset{\circ}{\rho})}{\sum_{w\in \overset{\circ}{W}} \varepsilon(w) e(w(\overset{\circ}{\rho}) - \overset{\circ}{\rho})}.$$

When $\lambda \in (\overset{\circ}{\mathfrak{h}})^*$ is dominant integral, $\chi(\lambda)$ is the character of the irreducible $\overset{\circ}{\mathfrak{g}}$-module $L(\lambda)$. However $c(\lambda)$ and $\chi(z(\lambda)$ are defined for all

$\lambda \in (\overset{\circ}{\mathfrak{h}})^*$. Using the denominator formula for $\overset{\circ}{\mathfrak{g}}$ we have

$$\begin{aligned}\sum_{w\in W}\varepsilon(w)e(w(\rho)-\rho) &= \sum_{\alpha\in M}\sum_{\overset{\circ}{w}\in\overset{\circ}{W}}\varepsilon(\overset{\circ}{w})e(\overset{\circ}{w}(h^\vee\alpha+\overset{\circ}{\rho})-\overset{\circ}{\rho})e(\frac{-c(\mathfrak{h}^\vee\alpha)}{2h^\vee}\delta)\\ &= \sum_{\overset{\circ}{w}\in\overset{\circ}{W}}\varepsilon(\overset{\circ}{w})e(\overset{\circ}{w}(\overset{\circ}{\rho})-\overset{\circ}{\rho})\sum_{\alpha\in M}\overset{\circ}{\chi}(h^\vee\alpha)e(\frac{-c(\mathfrak{h}^\vee\alpha)}{2h^\vee}\delta)\\ &= \prod_{\alpha\in\overset{\circ}{\Delta}_+}(1-e(-\alpha))\sum_{\alpha\in M}\overset{\circ}{\chi}(h^\vee\alpha)e(\frac{-c(\mathfrak{h}^\vee\alpha)}{2h^\vee}\delta).\end{aligned}$$

We now put

$$q = e(-\delta)$$

and equate the left- and right-hand sides of Kac's denominator formula. We obtain:

Theorem 10.5.1 (Untwisted Macdonald's Identity)

$$\prod_{n>0}\left((1-q^n)^\ell\prod_{\alpha\in\overset{\circ}{\Delta}}(1-q^ne(-\alpha))\right)=\sum_{\alpha\in M}\overset{\circ}{\chi}(h^\vee\alpha)q^{c(h^\vee\alpha)/2h^\vee}.$$

We have seen that in the special case the Macdonald's identity gives Jacobi's triple product identity.

We next state Macdonald's identities for the twisted affine algebras. The right hand side looks the same as before, the only change being that the appropriate lattice M should be taken in each case, see Lemma 6.4.2.

Theorem 10.5.2 *We have $L = R$, where*

$$R=\sum_{\alpha\in M}\overset{\circ}{\chi}(h^\vee\alpha)q^{c(h^\vee\alpha)/2h^\vee}$$

and L is given as follows:

$A_{2\ell}^{(2)}$:

$$\prod_{n>0}\left[(1-q^n)^\ell\prod_{\alpha\in\overset{\circ}{\Delta}_s}(1-q^ne(-\alpha))\prod_{\alpha\in\overset{\circ}{\Delta}_l}\big(1-q^{\frac{2n-1}{2}}e(-\frac{\alpha}{2})\big)\big(1-q^{2n}e(-\alpha)\big)\right]$$

$A_{2\ell-1}^{(2)}$:

$$\prod_{n>0}\left[(1-q^{2n})^\ell(1-q^{2n-1})^{\ell-1}\prod_{\alpha\in\overset{\circ}{\Delta}_s}(1-q^ne(-\alpha))\prod_{\alpha\in\overset{\circ}{\Delta}_l}(1-q^{2n}e(-\alpha))\right]$$

$D_{\ell+1}^{(2)}$:

$$\prod_{n>0}\left[(1-q^{2n})^{\ell}(1-q^{2n-1})\prod_{\alpha\in\mathring{\Delta}_s}(1-q^n e(-\alpha))\prod_{\alpha\in\mathring{\Delta}_l}(1-q^{2n}e(-\alpha))\right]$$

$E_6^{(2)}$:

$$\prod_{n>0}\left[(1-q^{2n})^{4}(1-q^{2n-1})^2\prod_{\alpha\in\mathring{\Delta}_s}(1-q^n e(-\alpha))\prod_{\alpha\in\mathring{\Delta}_l}(1-q^{2n}e(-\alpha))\right]$$

$D_4^{(3)}$:

$$\prod_{n>0}\left[(1-q^{3n})^{2}(1-q^{3n-1})(1-q^{3n-2})\prod_{\alpha\in\mathring{\Delta}_s}(1-q^n e(-\alpha))\prod_{\alpha\in\mathring{\Delta}_l}(1-q^{3n}e(-\alpha))\right],$$

where

$$M=\begin{cases}\sum_{i=}^{\ell}\mathbb{Z}\alpha_i & \textit{for types } A_{2\ell-1}^{(2)}, D_{\ell+1}^{(2)}, E_6^{(2)}, D_4^{(3)}\\ \sum_{\alpha_i \text{ long}}\frac{1}{2}\mathbb{Z}\alpha_i+\sum_{\alpha_i \text{ short}}\mathbb{Z}\alpha_i & \textit{for type } A_{2\ell}^{(2)}\\ \frac{1}{2}\mathbb{Z}\alpha_1 & \textit{for type } A_2^{(2)}\end{cases}$$

Example 10.5.3 In the case $A_2^{(2)}$ we get the following *quintuple product identity*:

$$\begin{aligned}&\prod_{n>0}(1-q^n)(1-q^nz^{-1})(1-q^{n-1}z)(1-q^{2n-1}z^{-2})(1-q^{2n-1}z^2)\\ =\;&\sum_{n\in\mathbb{Z}}(z^{3n}-z^{-3n+1})q^{\frac{n(3n-1)}{2}}.\end{aligned}$$

10.6 Specializations of Macdonald's identities

One way to specialize is simply replace $e(\alpha)$ with 1 for all $\alpha\in\mathring{\Delta}$. The result of such specialization in the expression $\mathring{\chi}(\lambda)$ is denoted $\mathring{d}(\lambda)$, which is given essentially by the Weyl dimension formula (10.13):

$$\mathring{d}(\lambda)=\frac{\prod_{\alpha\in\mathring{\Delta}_+}(\lambda+\mathring{\rho}\,|\alpha)}{\prod_{\alpha\in\mathring{\Delta}_+}(\mathring{\rho}\,|\alpha)}.$$

Remember that

$$\varphi(q)=\prod_{n>0}(1-q^n)$$

is the Euler function. Now the specialization of the left hand side of the untwisted Macdonald identity is $\varphi(q)^{\ell+|\mathring{\Delta}|} = \varphi(q)^{\dim \mathring{\mathfrak{g}}}$. So we get

Theorem 10.6.1 (Macdonald's φ-function identity)

$$\varphi(q)^{\dim \mathring{\mathfrak{g}}} = \sum_{\alpha \in M} \mathring{d}\,(h^\vee \alpha) q^{c(h^\vee \alpha)/2h^\vee}.$$

Example 10.6.2

Type $A_1^{(1)}$:

$$\varphi(q)^3 = \sum_{n \in \mathbb{Z}} (4n+1) q^{n(2n+1)}.$$

Type $A_2^{(1)}$:

$$\varphi(q)^8 = \sum_{(n_1,n_2) \in \mathbb{Z}^2} \frac{1}{2}(6n_1 - 3n_2 + 1)(-3n_1 + 6n_2 + 1) \times (3n_1 + 3n_2 + 2) q^{3n_1^2 - 3n_1 n_2 + 3n_2^2 + n_1 + n_2)}.$$

Type $C_2^{(1)}$:

$$\varphi(q)^{10} = \sum_{(n_1,n_2) \in \mathbb{Z}^2} (12n_1 - 6n_2 + 1)(-6n_1 + 6n_2 + 1) \times (2n_2 + 1)(3n_1 + 1) q^{6n_1^2 - 6n_1 n_2 + 3n_2^2 + n_1 + n_2)}.$$

Type $G_2^{(1)}$:

$$\varphi(q)^{14} = \sum_{(n_1,n_2) \in \mathbb{Z}^2} \frac{1}{15}(8n_1 - 12n_2 + 1)(-12n_1 + 24n_2 + 1) \times (3n_1 - 3n_2 + 1)(12n_2 + 5)(-2n_1 + 6n_2 + 1) \times (4n_1 + 3) q^{4n_1^2 - 12n_1 n_2 + 12n_2^2 + n_1 + n_2)}.$$

Theorem 10.6.3 (Macdonald's twisted φ-function identities) *We have $L = R$ where*

$$R = \sum_{\alpha \in M} \mathring{d}\,(h^\vee \alpha) q^{c(h^\vee \alpha)/2h^\vee},$$

and L is given as follows:

$A_{2\ell}^{(2)}$: $\varphi(q^{\frac{1}{2}})^{2\ell} \varphi(q)^{2\ell^2 - 3\ell} \varphi(q^2)^{2\ell}$;
$A_{2\ell-1}^{(2)}$: $\varphi(q)^{2\ell^2 - \ell - 1} \varphi(q^2)^{2\ell+1}$;
$D_{\ell+1}^{(2)}$: $\varphi(q)^{2\ell+1} \varphi(q^2)^{2\ell^2 - \ell - 1}$;

$E_6^{(2)}$: $\varphi(q)^{26}\varphi(q^2)^{26}$;
$D_4^{(3)}$: $\varphi(q)^7\varphi(q^3)^7$.

Example 10.6.4
Type $A_2^{(2)}$:

$$\varphi(q^{\frac{1}{2}})^2\varphi(q)^{-1}\varphi(q^2)^2 = \sum_{n\in\mathbb{Z}}(3n+1)q^{\frac{1}{2}n(3n+2)}.$$

Type $D_4^{(2)}$:

$$\varphi(q)^5\varphi(q^2)^5 = \sum_{(n_1,n_2)\in\mathbb{Z}^2}\frac{1}{3}(8n_1-4n_2+1)(-8n_1+8n_2+1) \times(8n_1+3)(2n_2+1)q^{8n_1^2-8n_1n_2+4n_2^2+2n_1+n_2}.$$

10.7 On converegence of characters

If we replace $e(\lambda)$ in the formal character by the function

$$e^\lambda : \mathfrak{h} \to \mathbb{C},\ h \mapsto e^{\langle\lambda,h\rangle},$$

we will get the ("informal") *character*

$$\mathrm{ch}_V : \mathfrak{h} \to \mathbb{C}$$

of the module $V \in \mathcal{O}$. Of course, now the questions of convergence arise. Let $Y(V)$ be the set of elements $h \in \mathfrak{h}$ such that the series converges absolutely. Note that

$$\mathrm{ch}_V(h) = \mathrm{tr}_V e^h \qquad (h \in Y(V)).$$

Define the *complexified Tits cone* $X_{\mathbb{C}}$ by

$$X_{\mathbb{C}} = \{x+iy \mid x \in X, y \in \mathfrak{h}_{\mathbb{R}}\}.$$

Set

$$Y = \{h \in \mathfrak{h} \mid \sum_{\alpha\in\Delta_+}(\mathrm{mult}\,\alpha)|e^{-\langle\alpha,h\rangle}| < \infty\},$$
$$Y_N = \{h \in \mathfrak{h} \mid \mathrm{Re}\langle\alpha_i,h\rangle > N \text{ for all } i=1,2\dots,n\} \qquad (N \in \mathbb{R}_+).$$

Note by Proposition 3.4.1(iii) that $Y \subset X_C$. We also have

$$X_{\mathbb{C}} = \bigcup_{w\in W} w(\bar{Y}_0). \qquad (10.16)$$

Lemma 10.7.1 *Let V be a highest weight module over $\mathfrak{g}$. Then*

(i) $Y(V)$ *is a convex set.*
(ii) $Y(V) \supset Y \cap Y_0$.
(iii) $Y(V) \supset Y_{\ln n}$.

Proof (i) is clear from the convexity of the function $|e^{\lambda}|$ (a function f is called convex if its domain D is a convex set and $f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$ for any $x, y \in D$ and $t \in [0,1]$. Now each $|e^{\lambda}|$ is defined on $\mathfrak{h}$, and if the series ch_V converges at h_1 and h_2 then the convexity property guarantees that ch_V converges at $th_1 + (1-t)h_2$ (actually to a convex function)). Moreover, since V is a quotient of some $M(\Lambda)$, we have

$$\mathrm{mult}_V \lambda \leq K(\Lambda - \lambda),$$

which gives

$$\begin{aligned} \sum_{\lambda \in \mathfrak{h}^*} (\mathrm{mult}_V \lambda) |e^{\langle \lambda, h \rangle}| &\leq |e^{\langle \Lambda, h \rangle}| \sum_{\beta \in Q_+} K(\beta) |e^{-\langle \beta, h \rangle}| \\ &= |e^{\langle \Lambda, h \rangle}| \prod_{\alpha \in \Delta_+} (1 - |e^{-\langle \alpha, h \rangle}|)^{-\operatorname{mult} \alpha}, \end{aligned}$$

provided $h \in Y_0$. The product converges for $h \in Y$. This proves (ii). Now (iii) follows from (ii) since $Y_{\ln n} \subset Y_0$ and also $Y_{\ln n} \subset Y$ in view of (1.16). □

Lemma 10.7.2 *Let $T \subset X_{\mathbb{C}}$ be an open convex W-invariant set. Then*

$$T \subset \text{convex hull}\Big(\bigcup_{w \in W} w(T \cap Y_0) \Big).$$

Proof Note that $T_0 := \bigcup_{w \in W} w(\bar{Y}_0 \setminus Y_0)$ is nowhere dense (interior of closure is empty) in $X_{\mathbb{C}}$. Hence every $h \in T$ lies in the convex hull of $T \setminus T_0 = \bigcup_{w \in W} (T \cap Y_0)$ applied to $T = \operatorname{Int} X_{\mathbb{C}}$. □

For a convex set R in a real vector space denote by $\operatorname{Int} R$ the interior of R.

Proposition 10.7.3 *Let $\Lambda \in P_+$. Then*

(i) $Y(L(\Lambda))$ *is a solid (i.e. has non-empty interior) convex W-invariant set, which for every $x \in \operatorname{Int} X_{\mathbb{C}}$ contains tx for all sufficiently large $t \in \mathbb{R}$.*
(ii) $\mathrm{ch}_{L(\Lambda)}$ *is a holomorphic function on* $\operatorname{Int} Y(L(\Lambda))$.
(iii) $Y(L(\Lambda)) \supset \operatorname{Int} Y$.

(iv) *The series* $\sum_{w\in W}\varepsilon(w)e^{w(\Lambda+\rho)}$ *converges absolutely on* $\operatorname{Int}X_{\mathbb{C}}$ *to a holomorphic function, and diverges absolutely on* $\mathfrak{h}\setminus\operatorname{Int}X_{\mathbb{C}}$.

(v) *Provided that* A *is symmetrizable,* $\operatorname{ch}_{L(\Lambda)}$ *can be extended from* $Y(L(\Lambda))\cap X_{\mathbb{C}}$ *to a meromorphic function on* $\operatorname{Int}X_{\mathbb{C}}$.

Proof Set $T = \operatorname{Int}Y$. Then T is open, convex (see the proof of Lemma 10.7.1(i)), and W-invariant. By Lemma 10.7.1(ii), we have $Y(L(\Lambda))\supset Y\cap Y_0$. Furthermore, Lemma 10.7.1(i) and Proposition 10.1.2 imply that $Y(L(\Lambda)$ is a convex W-invariant set. Now (iii) follows from Lemma 10.7.2.

To finish the proof of (i), we have to show that $X' := \{x\in\operatorname{Int}X_{\mathbb{C}} \mid tx\in Y(L(\Lambda))$ for all sufficiently large $t\in\mathbb{R}\}$ coincides with $\operatorname{Int}X_{\mathbb{C}}$. But again X' is W-invariant, convex, and contains Y_0 by Lemma 10.7.1(iii). So X' contains the convex hull of $\bigcup_{w\in W}w(Y_0)=\operatorname{Int}X_{\mathbb{C}}$, the last equality being true by Lemma 10.7.2.

The convexity of $|e^{\lambda}|$ implies that the absolute convergence is uniform on compact sets. This implies (ii).

(iv) By Proposition 3.4.1(ii), all $w(\Lambda+\rho)-(\Lambda+\rho)$ are distinct, and also $w(\Lambda+\rho)-(\Lambda+\rho)\in -Q_+$. Hence we have for all $h\in Y_0$:

$$|\sum_{w\in W}\varepsilon(w)e^{\langle w(\Lambda+\rho)-(\Lambda+\rho),h\rangle}|\le\sum_{\alpha\in Q_+}|e^{-\langle\alpha,h\rangle}|<\infty.$$

Thus the region of absolute convergence of our series contains Y_0 and is convex and W-invariant, so it contains $\operatorname{Int}X_{\mathbb{C}}$, as above. On the other hand, let $h\in\mathfrak{h}\setminus X_{\mathbb{C}}$. Then the set $\Delta^0 := \{\alpha\in\Delta_+^{\mathrm{re}} \mid \operatorname{Re}\langle\alpha,h\rangle\le 0\}$ is infinite by Proposition 3.4.1(iii),(vi), and for every $\alpha\in\Delta^0$ we have $|e^{\langle r_\alpha(\Lambda+\rho),h\rangle}|>|e^{\langle\Lambda+\rho,h\rangle}|$, proving divergence at h.

(v) follows from (iv) and the Weyl character formula. □

11

Irreducible Modules for affine algebras

Throughout $\mathfrak{g}$ is affine.

11.1 Weights of irreducible modules

Let $\lambda \in \mathfrak{h}^*$. Since $\Lambda_0, \Lambda_1, \dots, \Lambda_\ell, \delta$ form a basis of $\mathfrak{h}^*$, we can write

$$\lambda = s_0\Lambda_0 + s_1\Lambda_2 + \cdots + s_\ell\Lambda_\ell + s\delta \qquad (c_i, c \in \mathbb{C}).$$

Note that $\lambda \in P$ if and only if all $c_i \in \mathbb{Z}$, and $\lambda \in P_+$ if and only if all $c_i \in \mathbb{Z}_{\geq 0}$.

Let $\lambda \in P_+$. Then every weight μ of $L(\Lambda)$ is of the form $\lambda - m_0\alpha_0 - m_1\alpha_1 - \cdots - m_\ell\alpha_\ell$ for some $m_i \in \mathbb{Z}_{\geq 0}$. Since $\langle \alpha_i, c\rangle = 0$ we have $\langle \mu, c\rangle = \langle \lambda, c\rangle$ for any weight μ of $L(\lambda)$. Now, $\langle \lambda, c\rangle = \sum_{i=0}^{\ell} a_i^\vee \langle \lambda, \alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}$. This non-negative integer $\langle \lambda, c\rangle$ is referred to as the *level* of the module $L(\lambda)$.

Proposition 11.1.1 *If $L(\lambda)$ has level zero, then $\lambda = s\delta$ for some $s \in \mathbb{C}$ and* $\dim L(\lambda) = 1$.

Proof The first statement is clear and from the Weyl-Kac character formula we get $\operatorname{ch} L(s\delta) = e(s\delta)$. □

From now on we concentrate on higher levels.

Theorem 11.1.2 *Let $\lambda \in P_+$ and $\langle \lambda, c\rangle > 0$. Then $\mu \in P$ is a weight of $L(\lambda)$ if and only if there exists $w \in W$ such that $w(\mu) \in P_+$ and $w(\mu) \leq \lambda$.*

Proof Assume that μ is a weight of $L(\lambda)$. We know that then $w(\mu)$ is also a weight of $L(\lambda)$ for all $w \in W$. Take w for which the height of $\lambda - w(\mu)$ is minimal. The minimality shows that $\langle w(\mu), \alpha_i^\vee \rangle \geq 0$, i.e. $w(\mu) \in P_+$.

Conversely, assume that $\mu \in P_+$ and $\mu \leq \lambda$. We have to prove that μ is a weight of $L(\lambda)$. Let $\mu = \lambda - \alpha$ where $\alpha = \sum_{i=0}^{\ell} k_i\alpha_i$. We may assume $\alpha \neq 0$.

We first show that every connected component of $\operatorname{supp}\alpha$ contains an i with $\langle \lambda, \alpha_i^\vee \rangle > 0$. Otherwise there exists a connected component S of $\operatorname{supp}\alpha$ with $\langle \lambda, \alpha_i^\vee \rangle = 0$ for all $i \in S$. We have $L(\lambda)_\mu \subset U(\mathfrak{n}_-){-\alpha}v_\lambda$, and by the PBW theorem, $U(\mathfrak{n}_-)_{-\alpha}$ is spanned by the monomials of the form $\prod_{\beta\in\Delta_+} e_{-\beta}^{k_\beta}$ where $\sum k_\beta\beta = \alpha$ and each β involves simple roots which lie in the same connected component of $\operatorname{supp}\alpha$. Now, the $e_{-\beta}$ with simple roots in different connected components commute with each other, so we may bring the $e_{-\beta}$ with simple roots in S to the right of the above product. But for such β we have $e_{-\beta}v_\lambda = 0$. It follows that $U(\mathfrak{n}_-)_{-\alpha}v_\lambda = 0$, giving a contradiction.

Now let Ψ be defined by

$$\Psi = \{\gamma \in Q_+ \mid \gamma \leq \alpha,\ \lambda - \gamma \text{ is a weight of } L(\lambda)\}.$$

The set Ψ is finite. Let $\beta \in \Psi$ be an element of maximal height. Then $\beta \leq \alpha$. We need to show that $\beta = \alpha$. Let $\beta = \sum m_i\alpha_i$. We have $m_i \leq k_i$ for all i. Let $I = \{0, 1, \ldots, \ell$ and $J = \{i \in I \mid k_i = m_i\}$. Again, we need to show that $J = I$. If not, consider the non-empty subset of I given by $\operatorname{supp}\alpha \setminus (\operatorname{supp}\alpha \cap J)$. This set splits into connected components. Let M be one of them and take $i \in M$. Then $\lambda - \beta$ is a weight of $L(\lambda)$ but $\lambda - \beta - \alpha_i$ is not. Thus $\langle \lambda - \beta, \alpha_i^\vee \rangle \geq 0$. Also $lan\mu, \alpha_i^\vee \rangle \geq 0$ since $\mu \in P_+$ and so $\langle \lambda - \alpha, \alpha_i^\vee \rangle \geq 0$. Thus we have

$$\langle \alpha, \alpha_i^\vee \rangle \leq \langle \lambda, \alpha_i^\vee \rangle \leq \langle \beta, \alpha_i^\vee \rangle.$$

Let $\gamma = \sum_{j\in M}(k_j - m_j)\alpha_j$. We have $k_j - m_j > 0$ for all $j \in M$. We also have

$$\langle \gamma, \alpha_i^\vee \rangle = \sum_{j\in M}(k_j - m_j)a_{ij}.$$

However $\langle \gamma, \alpha_i^\vee \rangle = \langle \alpha - \beta, \alpha_i^\vee \rangle$ since $\operatorname{supp}(\alpha - \beta) = \operatorname{supp}\alpha \setminus J$ and M is a connected component of $\operatorname{supp}\alpha \setminus J$. Thus $\langle \gamma, \alpha_i^\vee \rangle \leq 0$ for each $i \in M$.

Let A_M be be the principal minor corresponding to M. Let u be the column vector with entries $k_i - m_i$ for $i \in M$. Then we have $u > 0$ and $Au \leq 0$. It follows that A_M does not have finite type, i.e. $M = I$. Thus $\operatorname{supp}\alpha = I$ and $J = \varnothing$. But then for all $i \in I$, $\lambda - \beta$ is a weight of $L(\lambda)$ but $\lambda - \beta - \alpha_i$ is not. Thus $\langle \lambda - \beta, \alpha_i^\vee \rangle \leq 0$ for all $i \in I$. Hence $\langle \alpha, \alpha_i^\vee \rangle \leq \langle \lambda, \alpha_i^\vee \rangle \leq \langle \beta, \alpha_i^\vee \rangle$ for all $i \in I$. We now have $u > 0$ and $Au \leq 0$. Since A is affine we deduce that $Au = 0$. This shows that $\langle \alpha, \alpha_i^\vee \rangle = \langle \beta, \alpha_i^\vee \rangle$ for all $i \in I$. Hence $\langle \alpha, \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle$ for all i, i.e. $\langle \mu, \alpha_i^\vee \rangle = 0$. But then we have $\langle \mu, c \rangle = 0$, and so $\langle \lambda, c \rangle = 0$, contradiction. □

Corollary 11.1.3 *If μ is a weight of $L(\lambda)$ then $\mu - \delta$ is also a weight.*

Proof Since μ is a weight there exists $w \in W$ such that $w(\mu) \in P_+$. Then $w(\mu - \delta) = w(\mu) - \delta \in P_+$. Since $w(\mu) - \delta \leq \lambda$ it follows from the theorem that $w(\mu) - \delta$ is a weight of $L(\lambda)$. □

It follows from the corollary that $\mu - i\delta$ is a weight for all positive integers i. On the other hand, there exist only finitely many positive integers i such that $\mu + i\delta \leq \lambda$.

Definition 11.1.4 A weight μ of $L(\lambda)$ is called an *maximal weight* if $\mu + \delta$ is not a weight.

Corollary 11.1.5 *For each weight μ of $L(\lambda)$ there are a unique maximal weight ν and a unique non-negative integer i such that $\mu = \nu - i\delta$.*

Proof Consider the sequence $\mu, \mu + i\delta, \mu + 2\delta, \ldots$. There exists i such that $\mu + i\delta$ is a weight of $L(\lambda)$ but $\mu + (i+1)\delta$ is not. Let $\nu = \mu + i\delta$. Then ν is an maximal weight of $L(\lambda)$ and $\mu = \nu - i\delta$. If $\mu = \nu' - i'\delta$ where ν' is an maximal weight and i' is a non-negative integer we show that $\nu = \nu'$ and $i = i'$. Otherwise we may assume that $i < i'$. Then $\nu' = \nu + (i' - i)\delta$ is a weight. Then $\nu + \delta$ is also a weight. Contradiction. □

A *string of weights* of $L(\lambda)$ is a set $\nu, \nu - \delta, \nu - 2\delta, \ldots$ where ν is an maximal weight. Each weight lies in a unique string of weights.

Lemma 11.1.6 *The set of maximal weights of $L(\lambda)$ is invariant under the Weyl group.*

Proof Let $w \in W$. Then μ is a weight if and only if $w(\mu)$ is a weight. Thus if μ is an maximal weight then $w(\mu)$ is a weight but $w(\mu) + \delta = w(\mu + \delta)$ is not a weight. □

Corollary 11.1.7 *Each maximal weight of $L(\lambda)$ has form $w(\mu)$ where $w \in W$ and μ is a dominant maximal weight.*

Recall the fundamental alcove

$$C_{\mathrm{af}} = \{\lambda \in \mathring{\mathfrak{h}}^*_{\mathbb{R}} \mid (\lambda|\alpha_i) \geq 0 \text{ for } 1 \leq i \leq \ell \text{ and } (\lambda|\theta) \leq 1\}.$$

We also recall that

$$\mathfrak{h}^* = \mathring{\mathfrak{h}}^* \oplus (\mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta),$$

and for $\lambda \in \mathfrak{h}^*$ we have

$$\lambda = \bar{\lambda} + \langle \lambda, c\rangle \Lambda_0 + a_0^{-1}\langle \lambda, d\rangle \delta$$

where $\bar{\lambda} \in \mathring{\mathfrak{h}}^*$. Let $\bar{Q}$ be the set of $\bar{\lambda}$ given by λ in the root lattice Q.

Proposition 11.1.8 *Let $\lambda \in P_+$ have level $k > 0$. Then the projection map $\mu \mapsto \bar{\mu}$ gives a bijection between the set of dominant maximal weights of $L(\lambda)$ and $(\bar{\lambda} + \bar{Q}) \cap kC_{\mathrm{af}}$.*

Proof Let μ be a dominant weight of $L(\lambda)$. Then $\mu = \lambda - \sum_i m_i\alpha_i$ for $m_i \in \mathbb{Z}_{\geq 0}$. Hence $\bar{\mu} = \bar{\lambda} - \overline{(\sum_i m_i\alpha_i)}$ and so $\bar{\mu} \in \bar{\lambda} + \bar{Q}$.

Now $\mu = \bar{\mu} + k\Lambda_0 + a_0^{-1}\langle \mu, d\rangle\delta$. Since $\mu \in P_+$ we have $\langle \mu, \alpha_i^\vee\rangle \geq 0$ for $i = 0, \ldots, \ell$. Now $\langle \Lambda_0, \alpha_i\rangle = \langle \delta, \alpha_i^\vee\rangle = 0$ for $i = 1, \ldots, \ell$. So $\langle \bar{\mu}, \alpha_i^\vee\rangle \geq 0$ and hence $(\bar{\mu}|\alpha_i) \geq 0$ for $i = 1, \ldots, \ell$. We also have

$$(\bar{\mu}|\theta) = (\mu|\theta) = (\mu|\delta - a_0\alpha_0) = \langle \mu, c\rangle - \langle \mu, \alpha_0^\vee\rangle = k - \langle \mu, \alpha_0^\vee\rangle.$$

Since $\langle \mu, \alpha_0^\vee\rangle \geq 0$ we have $(\bar{\mu}|\theta) \leq k$. Thus $\bar{\mu} \in kC_{\mathrm{af}}$. Hence the projection maps dominant maximal weights of $L(\lambda)$ into $(\bar{\lambda} + \bar{Q}) \cap kC_{\mathrm{af}}$.

We next show that this map is surjective. Let $\nu \in (\bar{\lambda} + \bar{Q}) \cap kC_{\mathrm{af}}$. Since $\bar{\alpha}_i = \alpha_i$ for $i = 1, \ldots, \ell$ and $\bar{\alpha}_0 = \overline{-a_0^{-1}\theta + a_0^{-1}\delta} = -a_0^{-1}\theta$ we have

$$\nu = \bar{\lambda} + k_1\alpha_1 + \cdots + k_\ell\alpha_\ell - k_0a_0^{-1}\theta \qquad (k_i \in \mathbb{Z}).$$

Since $\theta = a_1\alpha_1 + \cdots + a_\ell\alpha_\ell$ we have

$$\nu = \bar{\lambda} + (m - k_0a_0^{-1})\theta - (ma_1 - k_1)\alpha_1 - \cdots - (ma_\ell - k_\ell)\alpha_\ell.$$

Choose $m \in \mathbb{Z}$ with $m \geq k_i/a_i$ for $i = 0, \ldots, \ell$. Then

$$\nu = \bar{\lambda} + (m_0a_0^{-1})\theta - m_1\alpha_1 - \cdots - m_\ell\alpha_\ell$$

where $m_i = ma_i - k_i$ are non-negative integers for $i = 0, 1, \ldots, \ell$. Let $\mu = \lambda - \sum_{i=0}^{\ell} m_i\alpha_i$. Then

$$\bar{\mu} = \bar{\lambda} + (m_0a_0^{-1})\theta - m_1\alpha_1 - \cdots - m_\ell\alpha_\ell = \nu.$$

We next show that $\mu \in P_+$. For $i = 1, \dots, \ell$ we have

$$\langle \mu, \alpha_i^\vee \rangle = \langle \bar{\mu}, \alpha_i^\vee \rangle = \langle \nu, \alpha_i^\vee \rangle \geq 0.$$

Also

$$\langle \mu, \alpha_0^\vee \rangle = \langle \bar{\mu}, c - a_0\theta^\vee \rangle = k - (\bar{\mu}|\theta) = k - (\nu|\theta) \geq 0.$$

Hence $\mu \in P_+$ and also $\mu \leq \lambda$, so μ is a weight of $L(\lambda)$. Replacing μ by the maximal weight in the chain of weights containing μ we may assume that μ is a dominant maximal weight. Thus our map is surjective.

To show that the map is injective, let μ, μ' be dominant maximal weights of $L(\lambda)$ with $\bar{\mu} = \bar{\mu'}$. Writing $\mu = \bar{\mu} + k\Lambda_0 + a_0^{-1}\langle \mu, d\rangle\delta$ and $\mu' = \bar{\mu'} + k\Lambda_0 + a_0^{-1}\langle \mu', d\rangle\delta$, we get $\mu - \mu' = a_0^{-1}(\langle \mu, d\rangle - \langle \mu', d\rangle)\delta$. Now $\lambda - \mu, \lambda - \mu' \in Q$, hence $\mu - \mu' \in Q$ and $a_0^{-1}(\langle \mu, d\rangle - \langle \mu', d\rangle)\delta \in Q$. This shows that $a_0^{-1}(\langle \mu, d\rangle - \langle \mu', d\rangle) \in \mathbb{Z}$. Thus $\mu = \mu' + r\delta$ for $r \in \mathbb{Z}$. Since μ, μ' are both maximal we must have $r = 0$, i.e. $\mu = \mu'$. □

Corollary 11.1.9 *The set of dominant maximal weights of $L(\lambda)$ is finite.*

Proof $\bar{Q}$ is a lattice in $\mathring{\mathfrak{h}}^*$ and $\bar{\lambda} + \bar{Q}$ is a coset of that lattice. On the other hand the set kC_{af} is bounded. Hence the intersection $(\bar{\lambda}+\bar{Q}) \cap kC_{\mathrm{af}}$ must be finite. □

We now have a procedure for describing all weights of $L(\lambda)$. First determine the finite set $(\bar{\lambda} + \bar{Q}) \cap kC_{\mathrm{af}}$ where $k = \langle \lambda, c\rangle$. For each element ν in this finite set there is a unique dominant weight μ of $L(\lambda)$ with $\bar{\mu} = \nu$. This gives the set of all dominant maximal weights. By applying elements of the Weyl group to these we obtain all maximal weights. Finally, by subtracting positive integral multiples of δ from the maximal weights we obtain all weights of $L(\lambda)$.

We next consider the weights in a string $\mu, \mu - \delta, \mu - 2\delta, \dots$. We wish to show that the multiplicities of these weights form an increasing function as we move down the string. In order to do this we introduce the subalgebra

$$\mathfrak{t} = \bigoplus_{m \in \mathbb{Z}} \mathfrak{g}_{m\delta}.$$

Thus $\mathfrak{t}$ is spanned by $\mathfrak{h}$ and the root spaces of the imaginary roots. This algebra has a triangular decomposition

$$\mathfrak{t} = \mathfrak{t}_- \oplus \mathfrak{h} \oplus \mathfrak{t}_+,$$

where $\mathfrak{t}_\pm = \sum_{\pm i>0} \mathfrak{g}_{i\delta}$. One can define the category $\mathcal{O}$ of $\mathfrak{t}$-modules in the usual manner. One can also define Verma modules for $\mathfrak{t}$:

$$M(\lambda) = U(\mathfrak{t})/(U(\mathfrak{t})\mathfrak{t}_+ + \sum_{x\in\mathfrak{h}} U(\mathfrak{t})(x - \langle\lambda, x\rangle) \qquad (\lambda \in \mathfrak{h}^*).$$

Consider the expression

$$\Omega_0 = 2\sum_{i>0}\sum_j e^{(j)}_{-i\delta} e^{(j)}_{i\delta}$$

where $\{e^{(j)}_{i\delta}\}$ is a basis of $\mathfrak{g}_{i\delta}$ and $\{e^{(j)}_{-i\delta}\}$ is the dual basis of $\mathfrak{g}_{-i\delta}$. Thus

$$(e^{(j)}_{i\delta}|e^{(k)}_{-i\delta}) = \delta_{jk}, \quad [e^{(j)}_{i\delta}, e^{(k)}_{-i\delta}] = \delta_{jk} ic.$$

Although the expression for Ω_0 is an infinite sum the action of Ω_0 on any $\mathfrak{t}$-module in category $\mathcal{O}$ is well defined, since all but a finite number of the terms will act as zero.

Lemma 11.1.10 *Let $\lambda \in \mathfrak{h}^*$ and $M(\lambda)$ be the associated Verma module for $\mathfrak{t}$. Let $u \in U(\mathfrak{t})_{m\delta}$ where $m \in \mathbb{Z}\setminus\{0\}$. Then $\Omega_0 u - u\Omega_0$ acts on $M(\lambda)$ in the same way as $-2m\langle\lambda, c\rangle u$.*

Proof Note that a basis element $e^{(j)}_{r\delta}$ commutes with all $e^{(k)}_{i\delta}, e^{(k)}_{-i\delta}$ except for $e^{(j)}_{-r\delta}$. So

$$\Omega_0 u - u\Omega_0 = 2(e^{(j)}_{-r\delta} e^{(j)}_{r\delta} e^{(j)}_{r\delta} - e^{(j)}_{r\delta} e^{(j)}_{-r\delta} e^{(j)}_{r\delta}) = -2rce^{(j)}_{r\delta} = -2r\langle\lambda, c\rangle e^{(j)}_{r\delta}$$

on $M(\lambda)$. Thus the lemma holds if u is a basis element $e^{(j)}_{m\delta}$. It follows that the lemma also holds if $u \in \mathfrak{g}_{m\delta}$. Next suppose that $u = u_1u_2$ where on $M(\lambda)$ we have

$$\Omega_0 u_i - u_i\Omega_0 = -2r_i\langle\lambda, c\rangle u_i \qquad (i = 1, 2).$$

Then on $M(\lambda)$

$$\begin{aligned} \Omega_0 u - u\Omega_0 &= \Omega_0 u_1 u_2 - u_1 u_2 \Omega_0 \\ &= u_1\Omega_0 u_2 - 2r_1\langle\lambda, c\rangle u - u_1\Omega_0 u_2 - 2r_2\langle\lambda, c\rangle u \\ &= -2(r_1 + r_2)\langle\lambda, c\rangle u. \end{aligned}$$

The required result now follows from the PBW theorem. □

Proposition 11.1.11 *Let $\langle\lambda, c\rangle \neq 0$. Then the Verma module $M(\lambda)$ for $\mathfrak{t}$ is irreducible.*

Proof Suppose if possible that $M(\lambda)$ has a proper submodule K. Let v be a highest weight vector of K. Then $v \in M(\lambda)_{\lambda - m\delta}$ for some positive integer m. Thus $v = uv_\lambda$ for some $u \in U(\mathfrak{t}_-)_{-m\delta}$. By the previous lemma,

$$(\Omega_0 u - u\Omega_0)v_\lambda = -2m\langle\lambda, c\rangle uv_\lambda.$$

Thus $\Omega_0 v - u\Omega_0 v_\lambda = -2m\langle\lambda, c\rangle v$. Now $\Omega_0 v_\lambda = 0$ and $\Omega_0 v = 0$ since v_λ and v are highest weight vectors. By assumption this implies $v = 0$ giving a contradiction. □

We now restrict the $\mathfrak{g}$-module $L(\lambda)$ to $\mathfrak{t}$.

Proposition 11.1.12 *Suppose $\lambda \in P_+$ with $\langle\lambda, c\rangle > 0$. Then the $\mathfrak{t}$-module $L(\lambda)$ is completely reducible. Its irreducible components are Verma modules for $\mathfrak{t}$.*

Proof Let

$$U = \{v \in L(\lambda) \mid \mathfrak{t}_+ v = 0\},$$

and pick a basis B of U consisting of weight vectors. Suppose that $v \in B$ has weight μ. Then we have a surjective homomorphism $M(\mu) \to U(\mathfrak{t})v$. Now $\langle\mu, c\rangle = \langle\lambda, c\rangle > 0$, so $M(\mu)$ is irreducible and the homomorphism $M(\mu) \to U(\mathfrak{t})v$ is an isomorphism. Let $V = \sum_{v \in B} U(\mathfrak{t})v$. This sum of modules is a direct sum. Indeed, consider

$$U(\mathfrak{t})v \cap \sum_{v' \in B, v' \neq v} U(\mathfrak{t})v'.$$

Since $U(\mathfrak{t})v$ is irreducible the intersection is either trivial or $U(\mathfrak{t})v$. In the latter case $v \in \sum_{v' \in B, v' \neq v} U(\mathfrak{t})v'$. This is impossible since

$$U \cap \sum_{v' \in B, v' \neq v} U(\mathfrak{t})v' = \sum_{v' \neq v} \mathbb{C}v'.$$

Thus $V = \oplus_{v \in B} U(\mathfrak{t})v$.

We wish to show that $V = L(\lambda)$. If not consider the $\mathfrak{t}$-module $L(\lambda)/V$. Let μ be a weight of $L(\lambda)/V$ such that $\mu + i\delta$ is not a weight for any $i > 0$. Then $\mathfrak{t}_+ L(\lambda)_\mu \subset V_\mu$. Now consider the map $\Omega_0 : L(\lambda) \to L(\lambda)$. Since the action of Ω_0 preserves weight spaces (it is "of weight 0"), we have $\Omega_0 : L(\lambda) \to L(\lambda)$. So $L(\lambda)_\mu$ decomposes as a direct sum of generalized eigenspaces

$$L(\lambda)_\mu = \oplus_{\zeta \in \mathbb{C}} (L(\lambda)_\mu)_\zeta.$$

Since $L(\lambda)_\mu \not\subset V$, there exists $\zeta \in \mathbb{C}$ such that $(L(\lambda)_\mu)_\zeta \not\subset V$. Choose $v \in (L(\lambda)_\mu)_\zeta$ with $v \notin V$. Then

$$(\Omega_0 - \zeta 1)^k v = 0$$

for k large enough, and $\Omega_0 v \in V$ since $\mathfrak{t}_+ L(\lambda)_\mu \subset V$. If $\zeta \neq 0$ the polynomials $(t-\zeta)^k$ and t are coprime so we can deduce $v \in V$ giving contradiction. Hence $\zeta = 0$.

Now $\mathfrak{t}_+ v \neq 0$ since $v \notin V$ and hence $v \notin U$. So there exists $m > 0$ and $u \in U(\mathfrak{t}_+)_{m\delta}$ with $uv \neq 0$ and $\mathfrak{t}_+ uv = 0$. Let $v' = uv$. Then $v' \neq 0$ and $\Omega_0 v' = 0$. Now all weights ν of $L(\lambda)$ satisfy $\langle \nu, c\rangle = \langle \lambda, c\rangle$ so by Lemma 11.1.10, we have

$$\Omega_0 uv - u\Omega_0 v = -2m\langle \lambda, c\rangle uv,$$

that is

$$(\Omega_0 + 2m\langle \lambda, c\rangle)v' = u\Omega_0 v.$$

It follows that

$$(\Omega_0 + 2m\langle \lambda, c\rangle)^2 v' = (\Omega_0 + 2m\langle \lambda, c\rangle)u\Omega_0 v = u\Omega_0^2 v,$$

and continuing thus we obtain

$$(\Omega_0 + 2m\langle \lambda, c\rangle)^k v' = u\Omega_0^k v = 0.$$

But the polynomials $(t + 2m\langle \lambda, c\rangle)^k$ and t are coprime. Thus $(\Omega_0 + 2m\langle \lambda, c\rangle)^k v' = 0$ and $\Omega_0 v' = 0$ imply $v' = 0$ giving a contradiction. □

Proposition 11.1.13 *Let μ be a weight of $L(\lambda)$ where $\lambda \in P_+$ with $\langle \lambda, c\rangle > 0$. Then* $\dim L(\lambda)_{\mu-\delta} \geq \dim L(\lambda)_\mu$.

Proof Choose a non-zero element $x \in \mathfrak{g}_{-\delta}$ and consider the action of x on the $\mathfrak{t}$-module $L(\lambda)$. Since $L(\lambda)$ is a direct sum of Verma modules, it is free as a module over $U(\mathfrak{t}-)$, so x acts on it injectively. Thus we get an injective map $L(\lambda)_\mu \to L(\lambda)_{\mu-\delta}$. □

11.2 The fundamental modules for $\widehat{\mathfrak{sl}}_2$

By symmetry it suffices to determine the character of $L(\Lambda_0)$. Note that $\bar{\alpha}_0 = -\alpha_1$ and the lattice $\bar{Q} = \mathbb{Z}\alpha_1$, $\theta = \alpha_1, \theta^\vee = \alpha_1^\vee$. The fundamental alcove is given by

$$C_{\text{af}} = \{\lambda \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* \mid \langle \lambda, \alpha_1\rangle \geq 0,\ \langle \lambda, \theta^\vee\rangle \leq 1\} = \{\lambda \in \mathring{\mathfrak{h}}_{\mathbb{R}}^* \mid 0 \leq \langle \lambda, \alpha_1\rangle \leq 1\}.$$

Thus

$$(\bar{\Lambda}_0 + \bar{Q}) \cap C_{\mathrm{af}} = \{m\alpha_1 \mid m \in \mathbb{Z},\ 0 \le 2m \le 1\} = \{0\}.$$

Thus $L(\Lambda_0)$ has only one dominant maximal weight which must be the highest weight Λ_0. The other maximal weights are transforms of Λ_0 under the affine Weyl group W. The stabilizer of Λ_0 in W is $\overset{\circ}{W} = \langle r_1 \rangle$. So the maximal weights have the form

$$t_{m\alpha_1}(\Lambda_0) = \Lambda_0 + m\alpha_1 - m^2\delta \qquad (m \in \mathbb{Z})$$

The set of all weights of $L(\Lambda_0)$ is

$$\{\Lambda_0 + m\alpha_1 - m^2\delta - k\delta \mid m \in \mathbb{Z}, k \in \mathbb{Z}_{\ge 0}\}.$$

The weights $\Lambda_0 + m\alpha_1 - m^2\delta$ have multiplicity 1 and the multiplicity $\Lambda_0 + m\alpha_1 - m^2\delta - k\delta$ is independent of m. To determine the multiplicity of these weights consider Weyl-Kac formula

$$\mathrm{ch}\, L(\Lambda_0) = \frac{\sum_{w \in W} \varepsilon(w) e(w(\Lambda_0 + \rho) - \rho)}{\prod_{\alpha \in \Delta_+} (1 - e(-\alpha))^{\mathrm{mult}\,\alpha}}.$$

Now

$$\sum_{w \in W} \varepsilon(w) e(w(\Lambda_0 + \rho) - \rho) = \sum_{\overset{\circ}{w} \in \overset{\circ}{W}} \sum_{n \in \mathbb{Z}} \varepsilon(\overset{\circ}{w}) e(\overset{\circ}{w}\, t_{n\alpha_1}(\Lambda_0 + \rho) - \rho).$$

Now $\rho = \frac{1}{2}\alpha_1 + 2\Lambda_0$, hence

$$t_{n\alpha_1}(\Lambda_0 + \rho) = 3\Lambda_0 + (3n + \frac{1}{2})\alpha_1 - (3n^2 + n)\delta,$$

so

$$t_{n\alpha_1}(\Lambda_0 + \rho) - \rho = \Lambda_0 + 3n\alpha_1 - (3n^2 + n)\delta.$$

Also

$$r_1 t_{n\alpha_1}(\Lambda_0 + \rho) = 3\Lambda_0 - (3n + \frac{1}{2})\alpha_1 - (3n^2 + n)\delta,$$

so

$$r_1 t_{n\alpha_1}(\Lambda_0 + \rho) - \rho = \Lambda_0 - (3n + 1)\alpha_1 - (3n^2 + n)\delta.$$

Thus

$$\sum_{w \in W} \varepsilon(w) e(w(\Lambda_0 + \rho) - \rho) = e(\Lambda_0) \sum_{n \in \mathbb{Z}} \big(e(3n\alpha_1) - e(-(3n+1)\alpha_1)\big) e(-(3n^2+n)\delta).$$

We write $e(-\alpha_1) = z$ and $e(-\delta) = q^{1/2}$. Then our expression is

$$e(\Lambda_0) \sum_{n\in\mathbb{Z}} \left(z^{-3n} - z^{3n+1}\right) q^{\frac{n(3n+1)}{2}}.$$

Now we can factor this expression using Macdonald's identity for type $A^{(2)}_{2\ell}$. So we get

$$\begin{aligned}
& e(\Lambda_0) \prod_{n>0} (1-q^n)(1-q^n z^{-1})(1-q^{n-1}z)(1-q^{2n-1}z^{-2})(1-q^{2n-1}z^2) \\
= \; & e(\Lambda_0)(1-z) \prod_{n>0} (1-q^n)(1-q^n z^{-1})(1-q^n z)(1-q^{2n-1}z^{-2})(1-q^{2n-1}z^2) \\
= \; & e(\Lambda_0)(1-z) \prod_{n>0} (1-q^n)(1-q^n z^{-1})(1-q^{\frac{2n-1}{2}} z^{-1})(1-q^n z)(1-q^{\frac{2n-1}{2}} z) \\
& \times (1+q^{\frac{2n-1}{2}} z^{-1})(1+q^{\frac{2n-1}{2}} z) \\
= \; & e(\Lambda_0)(1-z) \prod_{k>0} (1-q^{k/2} z^{-1})(1-q^{k/2} z) \prod_{n>0} (1-q^n)(1+q^{\frac{2n-1}{2}} z^{-1})(1+q^{\frac{2n-1}{2}} z).
\end{aligned}$$

We now make use of the Macdonald's identity for type $A^{(1)}_1$:

$$\prod_{n>0} (1-q^n)(1-q^{n-1}z')(1-q^n z'^{-1}) = \sum_{m\in\mathbb{Z}} (-1)^m z'^m q^{\frac{m(m-1)}{2}}.$$

Taking $z' = -z^{-1} q^{\frac{1}{2}}$ we obtain

$$\prod_{n>0} (1-q^n)(1+q^{\frac{2n-1}{2}} z^{-1})(1+q^{\frac{2n-1}{2}} z) = \sum_{m\in\mathbb{Z}} z^{-m} q^{\frac{m^2}{2}}.$$

Hence

$$\begin{aligned}
\operatorname{ch} L(\lambda) &= \frac{e(\Lambda_0)(1-z)\prod_{k>0}(1-q^{k/2}z^{-1})(1-q^{k/2}z)\sum_{m\in\mathbb{Z}} z^{-m} q^{\frac{m^2}{2}}}{(1-z)\prod_{k>0}(1-q^{k/2}z^{-1})(1-q^{k/2}z)(1-q^{k/2})} \\
&= \frac{\sum_{n\in\mathbb{Z}} e(\Lambda_0 + n\alpha_1 - n^2\delta)}{\prod_{k>0}(1-e(-k\delta))} \\
&= \sum_{n\in\mathbb{Z}} e(\Lambda_0 + n\alpha_1 - n^2\delta) \sum_{k\geq 0} p(k) e(-k\delta) \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\geq 0} p(k) e(\Lambda_0 + n\alpha_1 - n^2\delta - k\delta).
\end{aligned}$$

Hence:

Proposition 11.2.1 *The weights of the fundamental module*

Proof

□

Bibliography

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