Lectures on Infinite Dimensional Lie Algebras

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Part one

Kac-Moody Algebras

1
Main Definitions

1.1 Some Examples

1.1.1 Special Linear Lie Algebras

Let $\mathfrak{g} = \mathfrak{s}l_n = \mathfrak{s}l_n(\mathbb{C})$. Choose the subalgebra $\mathfrak{h}$ consisting of all diagonal matrices in $\mathfrak{g}$. Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1}$,

$$\alpha_1^\vee, \dots, \alpha_{n-1}^\vee$$

is a basis of $\mathfrak{h}$. Next define $\varepsilon_1 \dots, \varepsilon_n \in \mathfrak{h}^*$ by

$$\varepsilon_i : \mathrm{diag}(a_1, \dots, a_n) \mapsto a_i.$$

Then, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$,

$$\alpha_1, \dots, \alpha_{n-1}$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the $(n-1) \times (n-1)$ matrix $A := (a_{ij})$ is

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

This matrix is called the *Cartan matrix*. Define

$$X_{\varepsilon_i - \varepsilon_j} := e_{ij},\ X_{-\varepsilon_i + \varepsilon_j} := e_{ji} \qquad (1 \le i < j \le n)$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

and

$$\{\alpha_1^\vee, \dots, \alpha_{n-1}^\vee\} \cup \{X_{\varepsilon_i - \varepsilon_j} \mid 1 \le i \ne j \le n\}$$

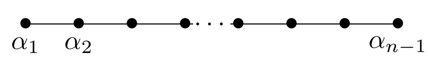
is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i < n$. It is easy to check that

$$e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}, \alpha_1^\vee, \dots \alpha_{n-1}^\vee \tag{1.1}$$

generate $\mathfrak{g}$ and the following relations hold:

$$[e_i, f_j] = \delta_{ij} \alpha_i^\vee, \tag{1.2}$$
$$[\alpha_i^\vee, \alpha_j^\vee] = 0, \tag{1.3}$$
$$[\alpha_i^\vee, e_j] = a_{ij} e_j, \tag{1.4}$$
$$[\alpha_i^\vee, f_j] = -a_{ij} f_j, \tag{1.5}$$
$$(\operatorname{ad} e_i)^{1-a_{ij}}(e_j) = 0 \qquad (i \ne j), \tag{1.6}$$
$$(\operatorname{ad} f_i)^{1-a_{ij}}(f_j) = 0 \qquad (i \ne j). \tag{1.7}$$

A (special case of a) theorem of Serre claims that $\mathfrak{g}$ is actually generated by the elements of (1.1) subject *only* to these relations. What is important for us is the fact that the Cartan matrix contains all the information needed to write down the Serre's presentation of A. Since the Cartan matrix is all the data we need, it makes sense to find a nicer geometric way to picture the same data. Such picture is called the Dynkin diagram, and in our case it is:



Here vertices i and $i+1$ are connected because $a_{i,i+1} = a_{i+1,i} = -1$, others are not connected because $a_{ij} = 0$ for $|i - j| > 1$, and we don't have to record a_{ii} since it is always going to be 2.

1.1.2 Symplectic Lie Algebras

Let V be a $2n$-dimensional vector space and $\varphi : V \times V \to \mathbb{C}$ be a non-degenerate symplectic bilinear form on V. Let

$$\mathfrak{g} = \mathfrak{s}p(V, \varphi) = \{X \in \mathfrak{g}l(V) \mid \varphi(Xv, w) + \varphi(v, Xw) = 0 \text{ for all } v, w \in V\}.$$

An easy check shows that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}l(V)$. It is known from linear algebra that over $\mathbb{C}$ all non-degenerate symplectic forms are equivalent, i.e. if φ' is another such form then $\varphi'(v,w) = \varphi(gv, gw)$ for some fixed $g \in GL(V)$. It follows that

$$\mathfrak{s}p(V, \varphi') = g(\mathfrak{s}p(V, \varphi))g^{-1} \cong \mathfrak{s}p(V, \varphi),$$

thus we can speak of just $\mathfrak{s}p(V)$. To think of $\mathfrak{s}p(V)$ as a Lie algebra of matrices, choose a symplectic basis $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$, that is

$$\varphi(e_i, e_{-i}) = -\varphi(e_{-i}, e_i) = 1,$$

and all other $\varphi(e_i, e_j) = 0$. Then the Gram matrix is

$$G = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix},$$

where

$$s = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ & & \vdots & & \\ 0 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \tag{1.8}$$

It follows that the matrices of $\mathfrak{s}p(V)$ in the basis of e_i's are precisely the matrices from the Lie algebra

$$\mathfrak{s}p_{2n} = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = sB^t s, C = sC^t s, D = -sA^t s\},$$

so $\mathfrak{s}p(V) \cong \mathfrak{s}p_{2n}$. Note that $sX^t s$ is the transpose of X with respect to the second main diagonal.

Choose the subalgebra $\mathfrak{h}$ consisting of all diagonal matrices in $\mathfrak{g}$. Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1} - e_{-i,-i} + e_{-i-1,-i-1}$, for $1 \le i < n$ and $\alpha_n^\vee = e_{nn} - e_{-n,-n}$,

$$\alpha_1^\vee, \dots, \alpha_{n-1}^\vee, \alpha_n^\vee$$

is a basis of $\mathfrak{h}$. Next, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i < n$, and $\alpha_n := 2\varepsilon_n$,

$$\alpha_1, \dots, \alpha_{n-1}, \alpha_n$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the Cartan matrix A is the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

Define

$$\begin{aligned} X_{2\varepsilon_i} &= e_{i,-i}, \quad (1 \le i \le n) \\ X_{-2\varepsilon_i} &= e_{-i,i}, \quad (1 \le i \le n) \\ X_{\varepsilon_i - \varepsilon_j} &= e_{ij} - e_{-j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i + \varepsilon_j} &= e_{ji} - e_{-i,-j} \quad (1 \le i < j \le n) \\ X_{\varepsilon_i + \varepsilon_j} &= e_{i,-j} + e_{j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i - \varepsilon_j} &= e_{-j,i} + e_{-i,j} \quad (1 \le i < j \le n). \end{aligned}$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

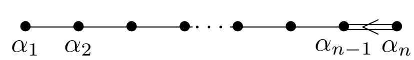
and

$$\{\alpha_1^\vee, \dots, \alpha_n^\vee\} \cup \{X_\alpha\}$$

is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i \le n$. It is easy to check that

$$e_1, \dots, e_n, f_1, \dots, f_n, \alpha_1^\vee, \dots \alpha_n^\vee \tag{1.9}$$

generate $\mathfrak{g}$ and the relations (1.2-1.7) hold. Again, Serre's theorem claims that $\mathfrak{g}$ is actually generated by the elements of (1.11) subject *only* to these relations. The Dynkin diagram in this case is:



The vertices $n-1$ and n are connected the way they are because $a_{n-1,n} = -2$ and $a_{n,n-1} = -1$, and in other places we follow the same rules as in the case $\mathfrak{sl}$.

1.1.3 Orthogonal Lie Algebras

Let V be an N-dimensional vector space and $\varphi : V \times V \to \mathbb{C}$ be a non-degenerate symmetric bilinear form on V. Let

$$\mathfrak{g} = \mathfrak{so}(V, \varphi) = \{X \in \mathfrak{gl}(V) \mid \varphi(Xv, w) + \varphi(v, Xw) = 0 \text{ for all } v, w \in V\}.$$

An easy check shows that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(V)$. It is known from linear algebra that over $\mathbb{C}$ all non-degenerate symmetric bilinear forms are equivalent, i.e. if φ' is another such form then $\varphi'(v, w) = \varphi(gv, gw)$ for some fixed $g \in GL(V)$. It follows that

$$\mathfrak{so}(V, \varphi') = g(\mathfrak{so}(V, \varphi))g^{-1} \cong \mathfrak{so}(V, \varphi),$$

thus we can speak of just $\mathfrak{so}(V)$. To think of $\mathfrak{so}(V)$ as a Lie algebra of matrices, choose a basis $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$ if $N = 2n$ and $e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}$ if $N = 2n + 1$, such that the Gram matrix of φ in this basis is

$$\begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} 0 & 0 & s \\ 0 & 2 & 0 \\ s & 0 & 0 \end{pmatrix},$$

respectively, where s is the $n \times n$ matrix as in (1.8). It follows that the matrices of $\mathfrak{so}(V)$ in the basis of e_i's are precisely the matrices from the Lie algebra

$$\mathfrak{so}_{2n} = \{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid B = -sB^t s, C = -sC^t s, D = -sA^t s\},$$

if $N = 2n$, and

$$\mathfrak{so}_{2n+1} = \{\begin{pmatrix} A & 2sx^t & B \\ y & 0 & x \\ C & 2sy^t & D \end{pmatrix} \mid B = -sB^t s, C = -sC^t s, D = -sA^t s\},$$

if $N = 2n + 1$ (here x, y are arbitrary $1 \times n$ matrices). We have in all cases that $\mathfrak{so}(V) \cong \mathfrak{so}_N$.

Choose the subalgebra $\mathfrak{h}$ consisting of all diagonal matrices in $\mathfrak{g}$. We now consider even and odd cases separately. First, let $N = 2n$.

Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1} - e_{-i,-i} + e_{-i-1,-i-1}$, for $1 \le i < n$ and $\alpha_n^\vee = e_{n-1,n-1} + e_{nn} - e_{-n+1,-n+1} - e_{-n,-n}$,

$$\alpha_1^\vee, \dots, \alpha_{n-1}^\vee, \alpha_n^\vee$$

is a basis of $\mathfrak{h}$. Next, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i < n$, and $\alpha_n :=$

$\varepsilon_{n-1} + \varepsilon_n$,

$$\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the Cartan matrix A is the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & \ldots & -1 & 0 & 2 \end{pmatrix}.$$

Define

$$\begin{aligned} X_{\varepsilon_i - \varepsilon_j} &= e_{ij} - e_{-j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i + \varepsilon_j} &= e_{ji} - e_{-i,-j} \quad (1 \le i < j \le n) \\ X_{\varepsilon_i + \varepsilon_j} &= e_{i,-j} - e_{j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i - \varepsilon_j} &= e_{-j,i} - e_{-i,j} \quad (1 \le i < j \le n). \end{aligned}$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

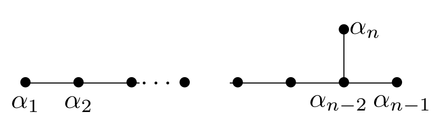
and

$$\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \cup \{X_\alpha\}$$

is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i \le n$. It is easy to check that

$$e_1, \ldots, e_n, f_1, \ldots, f_n, \alpha_1^\vee, \ldots \alpha_n^\vee \tag{1.10}$$

generate $\mathfrak{g}$ and the relations (1.2-1.7) hold. Again, Serre's theorem claims that $\mathfrak{g}$ is generated by the elements of (1.11) subject *only* to these relations. The Dynkin diagram in this case is:



Let $N = 2n+1$. Then, setting $\alpha_i^\vee := e_{ii} - e_{i+1,i+1} - e_{-i,-i} + e_{-i-1,-i-1}$, for $1 \le i < n$ and $\alpha_n^\vee = 2e_{nn} - 2e_{-n,-n}$,

$$\alpha_1^\vee, \ldots, \alpha_{n-1}^\vee, \alpha_n^\vee$$

is a basis of $\mathfrak{h}$. Next, setting $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \le i < n$, and $\alpha_n := \varepsilon_n$,

$$\alpha_1, \ldots, \alpha_{n-1}, \alpha_n$$

is a basis of $\mathfrak{h}^*$. Let

$$a_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle.$$

Then the Cartan matrix A is the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \ldots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & -2 & 2 \end{pmatrix}.$$

(It is transpose to the one in the symplectic case). Define

$$\begin{aligned} X_{\varepsilon_i} &= 2e_{i,0} + e_{0,-i}, \quad (1 \le i \le n) \\ X_{-\varepsilon_i} &= 2e_{-i,0} + e_{0,i}, \quad (1 \le i \le n) \\ X_{\varepsilon_i - \varepsilon_j} &= e_{ij} - e_{-j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i + \varepsilon_j} &= e_{ji} - e_{-i,-j} \quad (1 \le i < j \le n) \\ X_{\varepsilon_i + \varepsilon_j} &= e_{i,-j} - e_{j,-i} \quad (1 \le i < j \le n) \\ X_{-\varepsilon_i - \varepsilon_j} &= e_{-j,i} - e_{-i,j} \quad (1 \le i < j \le n). \end{aligned}$$

Note that

$$[h, X_\alpha] = \alpha(h) X_\alpha \qquad (h \in \mathfrak{h}),$$

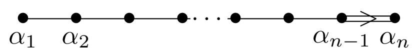
and

$$\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \cup \{X_\alpha\}$$

is a basis of $\mathfrak{g}$. Set $e_i = X_{\alpha_i}$ and $f_i = X_{-\alpha_i}$ for $1 \le i \le n$. It is easy to check that

$$e_1, \ldots, e_n, f_1, \ldots, f_n, \alpha_1^\vee, \ldots \alpha_n^\vee \tag{1.11}$$

generate $\mathfrak{g}$ and the relations (1.2-1.7) hold. Again, Serre's theorem claims that $\mathfrak{g}$ is actually generated by the elements of (1.11) subject *only* to these relations. The Dynkin diagram in this case is:



1.2 Generalized Cartan Matrices

Definition 1.2.1 A matrix $A \in M_n(\mathbb{Z})$ is a *generalized Cartan matrix* (GCM) if

(C1) $a_{ii} = 2$ for all i;
(C2) $a_{ij} \le 0$ for all $i \ne j$;
(C3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Two GCMs A and A' are *equivalent* if they have the same degree n and there is $\sigma \in S_n$ such that $a'_{ij} = a_{\sigma(i),\sigma(j)}$. A GCM is called *indecomposable* if it is not equivalent to a diagonal sum of smaller GCMs.

Throughout we are going to assume that $A = (a_{ij})_{1\le i,j\le n}$ is a generalized Cartan matrix of rank ℓ.

Definition 1.2.2 A *realization* of A is a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ where $\mathfrak{h}$ is a complex vector space, $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathfrak{h}^*$, and $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathfrak{h}$ such that

(i) both Π and $\Pi^\vee$ are linearly independent;
(ii) $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for all i, j;
(iii) $\dim \mathfrak{h} = 2n - \ell$.

Two realizations $(\mathfrak{h}, \Pi, \Pi^\vee)$ and $(\mathfrak{h}', \Pi', (\Pi')^\vee)$ are *isomorphic* if there exists an isomorphism $\varphi : \mathfrak{h} \to \mathfrak{h}'$ of vector spaces such that $\varphi(\alpha_i^\vee) = ((\alpha'_i)^\vee)$ and $\varphi^*(\alpha'_i) = (\alpha_i)$ for $i = 1, 2, \dots, n$.

Example 1.2.3 (i) Let $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. We have $n = \ell = 3$. Let $e_1, \dots, e_4$ be the standard basis of $\mathbb{C}^4$, $\varepsilon_1, \dots, \varepsilon_4$ be the dual basis, and $\mathfrak{h} = \{(a_1, \dots, a_4) \mid a_1 + \dots + a_4 = 0\}$. Finally, take $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \varepsilon_4\}$ and $\Pi^\vee = \{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$.

Another realization comes as follows. Let $\mathfrak{h} = \mathbb{C}^3$, and α_i denote the ith coordinate function. Now take $\alpha_i^\vee$ to be the ith row of A. It is clear that the two realizations are isomorphic.

(ii) Let $A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$. We have $n = 3$, $\ell = 2$. Take $\mathfrak{h} = \mathbb{C}^4$ and let α_i denote the ith coordinate function (we only need the first three). Now take $\alpha_i^\vee$ to be the ith row of the matrix $\begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 2 & -1 & 0 \\ -1 & -1 & 2 & 1 \end{pmatrix}$.

Proposition 1.2.4 *For each A there is a unique up to isomorphism realization. Realizations of matrices A and B are isomorphic if and only if $A = B$.*

Proof Assume for simplicity that A is of the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{11} is a non-singular $\ell \times \ell$ matrix. Let

$$C = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}.$$

Note $\det C = \pm \det A_{11}$, so C is non-singular. Let $\mathfrak{h} = \mathbb{C}^{2n-\ell}$. Define $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ to be the first n coordinate functions, and $\alpha_1^\vee, \dots, \alpha_n^\vee$ to be the first n row vectors of C.

Now, let $(\mathfrak{h}', \Pi', (\Pi')^\vee$ be another realization of A. We complete $(\alpha')_1^\vee, \dots, (\alpha')_n^\vee$ to a basis $(\alpha')_1^\vee, \dots, (\alpha')_{2n-\ell}^\vee$ of $\mathfrak{h}'$. Then the matrix $(\langle (\alpha_i')^\vee, \alpha_j' \rangle)$ has form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ B_1 & B_2 \end{pmatrix}.$$

By linear independence, this matrix has rank n. Thus it has n linearly independent rows. Since the rows rows $\ell + 1, \dots, n$ are linear combinations of rows $1, \dots, \ell$, the matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ B_1 & B_2 \end{pmatrix}$$

is non-singular. We now complete $\alpha_1', \dots, \alpha_n'$ to $\alpha_1', \dots, \alpha_{2n-\ell}'$, so that the matrix $(\langle (\alpha_i')^\vee, \alpha_j' \rangle)$ is

$$\begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ B_1 & B_2 & 0 \end{pmatrix}.$$

This matrix is non-singular, so $\alpha'_1, \dots, \alpha'_{2n-\ell}$ is a basis of $(\mathfrak{h}')^*$. Since A_{11} is non-singular, by adding suitable linear combinations of the first ℓ rows to the last $n-\ell$ rows, we may achieve $B_1 = 0$. Thus it is possible to choose $(\alpha')^\vee_{n+1}, \dots, (\alpha')^\vee_{2n-\ell}$, so that $(\alpha')^\vee_1, \dots, (\alpha')^\vee_{2n-\ell}$ are a basis of $\mathfrak{h}'$ and

$$(\langle (\alpha'_i)^\vee, \alpha'_j \rangle) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ 0 & B'_2 & 0 \end{pmatrix}.$$

The matrix B'_2 must be non-singular since the whole matrix is non-singular. We now make a further change to $(\alpha')^\vee_{n+1}, \dots, (\alpha')^\vee_{2n-\ell}$ equivalent to multiplying the above matrix by

$$\begin{pmatrix} I_\ell & 0 & 0 \\ 0 & I_{n-\ell} & 0 \\ 0 & 0 & (B'_2)^{-1} \end{pmatrix}.$$

Then we obtain

$$(\langle (\alpha'_i)^\vee, \alpha'_j \rangle) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}.$$

This is equal to the matrix C above. Thus the map $\alpha_i^\vee \mapsto (\alpha'_i)^\vee$ gives an isomorphism $\mathfrak{h} \to \mathfrak{h}'$ whose dual is given by $\alpha'_i \mapsto \alpha_i$. This shows that the realizations $(\mathfrak{h}, \Pi, \Pi^\vee$ and $(\mathfrak{h}', \Pi', (\Pi')^\vee$ are isomorphic.

Finally, assume that $\varphi : (\mathfrak{h}, \Pi, \Pi^\vee) \to (\mathfrak{h}', \Pi', (\Pi')^\vee)$ is an isomorphism of realizations of A and B respectively. Then

$$b_{ij} = \langle (\alpha'_i)^\vee, \alpha'_j \rangle = \langle \varphi(\alpha_i^\vee), \alpha'_j \rangle = \langle \alpha_i^\vee, \varphi^*(\alpha'_j) \rangle = \langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}.$$

□

Throughout we assume that $(\mathfrak{h}, \Pi, \Pi^\vee)$ is a realization of A.

We refer to the elements of Π as *simple roots* and the elements of $\Pi^\vee$ as *simple coroots*, to Π and $\Pi^\vee$ as *root basis* and coroot basis, respectively. Also set

$$Q = \oplus_{i=1}^n \mathbb{Z}\alpha_i, \quad Q_+ = \oplus_{i=1}^n \mathbb{Z}_+\alpha_i.$$

We call Q *root lattice*. *Dominance ordering* is a partial order $\geq$ on $\mathfrak{h}^*$ defined as follows: $\lambda \geq \mu$ if and only if $\lambda - \mu \in Q_+$. For $\alpha = \sum_{i=1}^n k_i\alpha_i \in Q$, the number

$$\text{ht } \alpha := \sum_{i=1}^n k_i$$

is called the *height* of α.

1.3 The Lie algebra $\tilde{\mathfrak{g}}(A)$

Definition 1.3.1 *The Lie algebra $\tilde{\mathfrak{g}}(A)$ is defined as the algebra with generators e_i, f_i $(i = 1, \dots, n)$ and $\mathfrak{h}$ and relations*

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee, \tag{1.12}$$

$$[h, h'] = 0 \qquad (h, h' \in \mathfrak{h}), \tag{1.13}$$

$$[h, e_i] = \langle \alpha_i, h\rangle e_i \qquad (h \in \mathfrak{h}), \tag{1.14}$$

$$[h, f_i] = -\langle \alpha_i, h\rangle f_i \qquad (h \in \mathfrak{h}). \tag{1.15}$$

It follows from the uniqueness of realizations that $\tilde{\mathfrak{g}}(A)$ depends only on A (this boils down to the following calculation: $\langle \alpha_i', \varphi(h)\rangle = \langle \varphi^*(\alpha_i'), h\rangle = \langle \alpha_i, h\rangle$).

Denote by $\tilde{n}_+$ (resp. $\tilde{n}_-$) the subalgebra of $\tilde{\mathfrak{g}}(A)$ generated by all e_i (resp. f_i).

Lemma 1.3.2 *Let V be an $\mathfrak{h}$-module such that $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ where the* weight space *V_λ is defined as $\{v \in V \mid hv = \langle\lambda, h\rangle v$ for all $h \in \mathfrak{h}\}$. Let U be a submodule of V. Then $U = \bigoplus_{\lambda\in\mathfrak{h}^*}(U \cap V_\lambda)$.*

Proof Any elementr $v \in V$ can be written in the form $v = v_1 + \dots + v_m$ where $v_j \in V_{\lambda_j}$, and theere is $h \in \mathfrak{h}$ such that $\lambda_j(h)$ are all distinct. For $v \in U$, we have

$$h^k(v) = \sum_{j=1}^{m} \lambda_j(h)^k v_j \in U \qquad (k = 0, 1, \dots, m-1).$$

We got a system of linear equations with non-singular matrix. It follows that all $v_j \in U$. □

Theorem 1.3.3 *Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$. Then*

(i) $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+$.
(ii) $\tilde{\mathfrak{n}}_+$ *(resp. $\tilde{n}_-$) is freely generated by the e_i's (resp. f_i's).*
(iii) *The map $e_i \mapsto f_i$, $f_i \mapsto e_i$, $h \mapsto -h$ $(h \in \mathfrak{h})$ extends uniquely to an involution $\tilde{\omega}$ of $\tilde{\mathfrak{g}}$.*
(iv) *One has the root space decomposition with respect to $\mathfrak{h}$:*

$$\tilde{\mathfrak{g}} = \Big(\bigoplus_{\alpha\in Q_+, \alpha\neq 0} \tilde{\mathfrak{g}}_{-\alpha}\Big) \oplus \mathfrak{h} \oplus \Big(\bigoplus_{\alpha\in Q_+, \alpha\neq 0} \tilde{\mathfrak{g}}_{\alpha}\Big),$$

where $\tilde{\mathfrak{g}}_\alpha = \{x \in \tilde{\mathfrak{g}} \mid [h,x] = \alpha(h)x$ *for all* $h \in \mathfrak{h}\}$. *Moreover, each* $\tilde{\mathfrak{g}}_\alpha$ *is finite dimensional, and* $\tilde{\mathfrak{g}}_{\pm\alpha} \subset \tilde{\mathfrak{n}}_\pm$ *for* $\pm\alpha \in Q_+$, $\alpha \neq 0$.

(v) *Among the ideals of* $\tilde{\mathfrak{g}}$ *which have trivial intersection with* $\mathfrak{h}$, *there is unique maximal ideal* $\mathfrak{r}$. *Moreover,*

$$\mathfrak{r} = (\mathfrak{r} \cap \tilde{\mathfrak{n}}_-) \oplus (\mathfrak{r} \cap \tilde{\mathfrak{n}}_+) \qquad (\textit{direct sum of ideals}).$$

Proof Let V be a complex vector space with basis $v_1, \dots, v_n$ and let $\lambda \in \mathfrak{h}^*$. Define the action of the generators on the tensor algebra $T(V)$ as follows:

(a) $f_i(a) = v_i \otimes a$ for $a \in T(V)$.

(b) $h(1) = \langle \lambda, h \rangle$ and then inductively on s,

$$h(v_j \otimes a) = -\langle \alpha_j, h \rangle v_j \otimes a + v_j \otimes h(a)$$

for $a \in T^{s-1}(V)$.

(c) $e_i(1) = 0$ and then inductively on s,

$$e_i(v_j \otimes a) = \delta_{ij} \alpha_i^\vee(a) + v_j \otimes e_i(a)$$

for $a \in T^{s-1}(V)$.

To see that these formulas define a representation of $\tilde{\mathfrak{g}}$, let us check the relations. For the first relation:

$$\begin{aligned}(e_i f_j - f_j e_i)(a) &= e_i(v_j \otimes a) - v_j \otimes e_i(a) \\ &= \delta_{ij}\alpha_i^\vee(a) + v_j \otimes e_i(a) - v_j \otimes e_i(a) \\ &= \delta_{ij}\alpha_i^\vee(a).\end{aligned}$$

The second relation is obvious since $\mathfrak{h}$ acts diagonally. For the third relation, apply induction on s, the relation being obvious for $s = 0$. For $s > 0$, take $a = v_k \otimes a_1$ where $a_1 \in T^{s-1}(V)$. Then using induction we have

$$\begin{aligned}(he_j - e_jh)(v_k \otimes a_1) &= h(\delta_{jk}\alpha_j^\vee(a_1)) + h(v_k \otimes e_j(a_1)) \\ &\quad - e_j(-\langle\alpha_k, h\rangle v_k \otimes a_1) - e_j(v_k \otimes h(a_1)) \\ &= \delta_{jk}\alpha_j^\vee(h(a_1)) - \langle\alpha_k, h\rangle v_k \otimes e_j(a_1) \\ &\quad + v_k \otimes h(e_j(a_1)) + \langle\alpha_k, h\rangle\delta_{jk}\alpha_j^\vee(a_1) \\ &\quad + \langle\alpha_k, h\rangle v_k \otimes e_j(a_1) - \delta_{jk}\alpha_j^\vee(h(a_1)) \\ &\quad - v_k \otimes e_jh(a_1) \\ &= v_k \otimes [h, e_j](a_1) + \langle\alpha_j, h\rangle\delta_{jk}\alpha_j^\vee(a_1) \\ &= v_k \otimes \langle\alpha_j, h\rangle e_j(a_1) + \langle\alpha_j, h\rangle\delta_{jk}\alpha_j^\vee(a_1) \\ &= \langle\alpha_j, h\rangle(v_k \otimes e_j(a_1) + \delta_{jk}\alpha_j^\vee(a_1)) \\ &= \langle\alpha_j, h\rangle e_j(v_k \otimes a_1).\end{aligned}$$

Finally, for the fourth relation:

$$\begin{aligned}(hf_j - f_jh)(a) &= h(v_j \otimes a) - v_j \otimes h(a) \\ &= -\langle\alpha_j, h\rangle v_j \otimes a + v_j \otimes a - v_j \otimes h(a) \\ &= -\langle\alpha_j, h\rangle v_j \otimes a.\end{aligned}$$

Now we prove (i)-(v).

(iii) is easy to check using the defining relations.

(ii) Consider the map $\varphi : \tilde{\mathfrak{n}}_- \to T(V),\ u \mapsto u(1)$. We have $\varphi(f_i) = v_i$, and for any Lie word $w(f_1, \dots, f_n)$ we have

$$\varphi(w(f_1, \dots, f_n)) = w(v_1, \dots, v_n).$$

Now, for two words w and w', we have

$$\begin{aligned}\varphi([w(f_1, \dots f_n), w'(f_1, \dots f_n)]) &= [w(v_1, \dots v_n), w'(v_1, \dots v_n)] \\ &= [\varphi(w(f_1, \dots f_n)), \varphi(w'(f_1, \dots f_n))],\end{aligned}$$

so φ is a Lie algebra homomorphism. Now $T(V) = F(v_1, \dots, v_n)$, the free associative algebra on $v_1, \dots, v_n$. Moreover, the free Lie algebra $FL(v_1, \dots, v_n)$ lies in $T(V)$ and is spanned by all Lie words in $v_1, \dots, v_n$. Thus $FL(v_1, \dots, v_n)$ is the image of φ. But there is a Lie algebra homomorphism $\varphi' : FL(v_1, \dots, v_n) \to \tilde{\mathfrak{n}}_-,\ v_i \mapsto f_i$, which is inverse to φ, so φ is an isomorphism. It follows that the f_i generate $\tilde{\mathfrak{n}}_-$ freely. The similar result for $\tilde{\mathfrak{n}}_+$ follows by applying the automorphism $\tilde{\omega}$.

(i) It is clear from relations that $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}_- + \mathfrak{h} + \tilde{\mathfrak{n}}_+$. Let $u = n_- + h + n_+ = 0$. Then in T(V) we have $0 = u(1) = n_-(1) + \langle\lambda, h\rangle$. It follows that

$\langle \lambda, h \rangle = 0$ for all λ, whence $h = 0$. Now $0 = n_-(1) = \varphi(n_-)$, whence $n_- = 0$.

(iv) It follows from the last two defining relations that

$$\tilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in Q_+,\ \alpha \neq 0} \tilde{\mathfrak{g}}_{\pm\alpha}.$$

Moreover, $\dim \tilde{\mathfrak{g}}_\alpha \leq n^{|\,\mathrm{ht}\ \alpha|}$.

(v) By Lemma 1.3.2, for any ideal $\mathfrak{i}$ of $\tilde{\mathfrak{g}}$, we have $\mathfrak{i} = \bigoplus_{\alpha \in \mathfrak{h}^*} (\tilde{\mathfrak{g}}_\alpha \cap \mathfrak{i})$. Since $\mathfrak{h} = \tilde{\mathfrak{g}}_0$, the sum of the ideals which have trivial intersection with $\mathfrak{h}$ is the unique maximal ideal with this property. It is also clear that the sum in (v) is direct. Finally, $[f_i, \mathfrak{r} \cap \tilde{\mathfrak{n}}_+] \subset \tilde{n}_+$. Hence $[\tilde{\mathfrak{g}}, \mathfrak{r} \cap \tilde{\mathfrak{n}}_+] \subset \mathfrak{r} \cap \tilde{\mathfrak{n}}_+$. Similarly for $\mathfrak{r} \cap \tilde{\mathfrak{n}}_-$. □

Remark 1.3.4 Note that the formula (b) in the proof of the theorem implies that the natural homomorphism $\mathfrak{h} \to \tilde{\mathfrak{g}}$ is an injection. This justifies out notation.

1.4 The Lie algebra $\mathfrak{g}(A)$

Definition 1.4.1 We define the *Kac-Moody algebra* $\mathfrak{g} = \mathfrak{g}(A)$ to be the quotient $\tilde{\mathfrak{g}}(A)/\mathfrak{r}$ where $\mathfrak{r}$ is the ideal from Theorem 1.3.3(v).

We refer to A as the *Cartan matrix* of $\mathfrak{g}$, and to n as the *rank* of $\mathfrak{g}$.

In view of Remark 1.3.4, we have a natural embedding $\mathfrak{h} \to \mathfrak{g}(A)$. The image of this embedding is also denoted $\mathfrak{h}$ and is called a *Cartan subalgebra* of $\mathfrak{g}$.

We keep the same notation for the images of the elements $e_i, f_i, \mathfrak{h}$ in $\mathfrak{g}$. The elements e_i and f_i are called *Chevalley generators*.

We have the following root decomposition with respect to $\mathfrak{h}$:

$$\mathfrak{g} = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

with $\mathfrak{g}_0 = \mathfrak{h}$. The number

$$\mathrm{mult}\ \alpha := \dim \mathfrak{g}_\alpha$$

is called the *multiplicity* of α. The element $\alpha \in Q$ is called a *root* if $\alpha \neq 0$ and $\mathrm{mult}\ \alpha \neq 0$. A root $\alpha > 0$ is called *positive*, a root $\alpha < 0$ is called *negative*. Every root is either positive or negative. We denote by $\Delta, \Delta_+, \Delta_-$ the sets of the roots, positive roots, and negative roots, respectively.

The subalgebra of $\mathfrak{g}$ generated by the e_i's (resp. f_i's) is denoted by $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$). From Theorem 1.3.3, we have

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

It follows that $\mathfrak{g}_\alpha \subset \mathfrak{n}_+$ if $\alpha > 0$ and $\mathfrak{g}_\alpha \subset \mathfrak{n}_-$ if $\alpha < 0$. So for $\alpha > 0$, $\mathfrak{g}_\alpha$ is a span of the elements of the form $[\dots[[e_{i_1}, e_{i_2}], e_{i_3}]\dots e_{i_s}]$ such that $\alpha_{i_1} + \dots + \alpha_{i_s} = \alpha$. Similarly for $\alpha < 0$. It follows that

$$\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i, \quad \mathfrak{g}_{-\alpha_i} = \mathbb{C}f_i, \quad \mathfrak{g}_{s\alpha_i} = 0 \qquad (s \neq \pm 1).$$

Since every root is either positive or negative, we deduce

Lemma 1.4.2 *If $\beta \in \Delta_+ \setminus \{\alpha_i\}$, then $(\beta + \mathbb{Z}\alpha_i) \cap \Delta \subset \Delta_+$.*

From Theorem 1.3.3(v), $\mathfrak{r}$ is $\tilde{\omega}$-invariant, so we get the *Chevalley involution*

$$\omega : \mathfrak{g} \to \mathfrak{g},\ e_i \mapsto -f_i,\ f_i \mapsto -e_i,\ h \mapsto h \qquad (h \in \mathfrak{h}).$$

It is clear that $\omega(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$, so $\operatorname{mult}\alpha = \operatorname{mult}(-\alpha)$ and $\Delta_- = -\Delta_+$.

Proposition 1.4.3 *Let A_1 be an $n \times n$ GCM, A_2 be an $m \times m$ GCM, and $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ be the direct sum matrix. Let $(\mathfrak{h}_i, \Pi_i, \Pi_i^\vee)$ be a realization of A_i. Then $(\mathfrak{h}_1 \oplus \mathfrak{h}_2, \Pi_1 \sqcup \Pi_2, \Pi_1^\vee \sqcup \Pi_2^\vee)$ is a realization of A, and $\mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2) \cong \mathfrak{g}(A)$, the isomorphism sending $(h_1, h_2) \mapsto (h_1, h_2)$, $(e_i, 0) \mapsto e_i$, $(0, e_j) \mapsto e_{n+j}$, $(f_i, 0) \mapsto f_i$, $(0, f_j) \mapsto f_{n+j}$.*

Proof The first statement is obvious. For the second one, observe that generators $(h_1, h_2), (e_i, 0), (0, e_j), (f_i, 0), (0, f_j)$ of $\mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$ satisfy the defining relations of $\tilde{\mathfrak{g}}(A)$. So there exists a surjective homomorphism

$$\tilde{\pi} : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$$

which acts on the generators as the inverse of the isomorphism promised in the proposition. Moreover, since $\mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$ has no ideals with intersect $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ trivially, it follows that $\tilde{\pi}$ factors through the surjective homomorphism

$$\pi : \mathfrak{g}(A) \to \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2).$$

It suffices to show that π is injective. If not, its kernel must be an ideal whose intersection with $\mathfrak{h}$ is non-trivial. But then $\dim \pi(\mathfrak{h}) < \dim \mathfrak{h}$ giving a contradiction with the fact that $\pi(h) = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ has dimension $\dim \mathfrak{h}_1 + \dim \mathfrak{h}_2 = \dim \mathfrak{h}$. $\square$

Denote by

$$\mathfrak{g}' = \mathfrak{g}'(A)$$

the subalgebra of $\mathfrak{g}(A)$ generated by all Chevalley generators e_i and f_j.

Proposition 1.4.4 *Let $\mathfrak{h}' \subset \mathfrak{h}$ be the span of $\alpha_1^\vee, \dots, \alpha_n^\vee$.*

(i) $\mathfrak{g}' = \mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+$.
(ii) $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$.

Proof (i) It is clear that $\mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+ \subset \mathfrak{g}'$. Conversely, if a Lie word in the Chevalley generators is not equal to zero and belongs to $\mathfrak{h}$, it follows from the relations that it belongs to $\mathfrak{h}'$.

(ii) It is clear that $\mathfrak{g}'$ is an ideal in $\mathfrak{g}$, and it follows from (i) that $\mathfrak{g}/\mathfrak{g}' \cong \mathfrak{h}/\mathfrak{h}'$ is abelian, so $[\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{g}'$. Conversely, $\alpha_i^\vee = [e_i, f_i]$, $e_i = [\frac{1}{2}\alpha_i^\vee, e_i]$, and $f_i = [f_i, \frac{1}{2}\alpha_i^\vee]$, so $\mathfrak{g}' \subset [\mathfrak{g}, \mathfrak{g}]$. □

Let $s = (s_1, \dots, s_n) \in \mathbb{Z}^n$. The *s-grading*

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j(s)$$

of $\mathfrak{g}$ is obtained by setting

$$\mathfrak{g}_j(s) = \bigoplus \mathfrak{g}_\alpha$$

where the sum is over all $\alpha = \sum_i k_i \alpha_i \in Q$ such that $\sum_i s_i k_i = j$. Note that

$$\deg e_i = -\deg f_i = -s_i, \quad \deg \mathfrak{h} = 0.$$

The case $s = (1, \dots, 1)$ gives the *principal grading* of $\mathfrak{g}$.

Lemma 1.4.5 *If an element a of $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) commutes with all f_i (resp. all e_i), then $a = 0$.*

Proof Note that in the principal grading $\mathfrak{g}_{-1} = \operatorname{span}(f_1, \dots, f_n)$ and $\mathfrak{g}_1 = \operatorname{span}(e_1, \dots, e_n)$. So $[a, \mathfrak{g}_{-1}] = 0$. Then

$$\sum_{i,j \geq 0} (\operatorname{ad} \mathfrak{g}_1)^i (\operatorname{ad} \mathfrak{h})^j a$$

is an ideal of $\mathfrak{g}$ contained in $\mathfrak{n}_+$. This ideal must be zero, whence $a = 0$. □

Proposition 1.4.6 *The center of $\mathfrak{g}$ and $\mathfrak{g}'$ is*

$$\mathfrak{c} = \{h \in \mathfrak{h} \mid \langle \alpha_i, h\rangle = 0 \text{ for all } i = 1, \dots, n\}. \tag{1.16}$$

Moreover, $\dim \mathfrak{c} = n - \ell$.

Proof Let $c \in \mathfrak{g}$ be central and $c = \sum_i c_i$ be decomposition with respect to the principal grading. Then $[c, \mathfrak{g}_{-1}] = 0$ implies $[c_i, \mathfrak{g}_{-1}] = 0$, whence $c_i = 0$ for $i > 0$ and similarly $c_i = 0$ for $i < 0$. So $c \in \mathfrak{h}$, and then $0 = [c, e_i] = \langle \alpha_i, c\rangle e_i$ implies $c \in \mathfrak{c}$. Converse is clear. Finally, $\mathfrak{c} \subset \mathfrak{h}'$, since otherwise $\dim \mathfrak{c} > n - \ell$. □

Lemma 1.4.7 *Let I_1, I_2 be disjoint subsets of $\{1, \dots, n\}$ such that $a_{ij} = 0 = a_{ji}$ for all $i \in I_1, j \in I_2$. Let $\beta_s = \sum_{i \in I_s} k_i^{(s)} \alpha_i$ $(s = 1, 2)$. If $\alpha = \beta_1 + \beta_2$ is a root of $\mathfrak{g}$, then either β_1 or β_2 is zero.*

Proof Let $i \in I_1, j \in I_2$. Then $[\alpha_i^\vee, e_j] = 0, [\alpha_j^\vee, e_i] = 0, [e_i, f_j] = 0, [e_j, f_i] = 0$. Using Leibnitz formula and Lemma 1.4.5, we conclude that $[e_i, e_j] = [f_i, f_j] = 0$. Denote by $\mathfrak{g}^{(s)}$ be the subalgebra generated by all e_i, f_i for $i \in I_s$. We have shown that $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ commute. Now, since $\mathfrak{g}_\alpha$ is contained in the subalgebra generated by $\mathfrak{g}^{(1)}$ and $\mathfrak{g}^{(2)}$ it follows that it is contained in one of them. □

Proposition 1.4.8

(i) $\mathfrak{g}$ *is a simple Lie algebra if and only if* $\det A \neq 0$ *and for each pair of indices i, j the following condition holds:*

$$\text{there are indices } i_1, \dots, i_s \text{ such that } a_{ii_1} a_{i_1 i_2} \dots a_{i_s j} \neq 0. \tag{1.17}$$

(ii) *If the condition (1.17) holds then every ideal of $\mathfrak{g}$ either contains $\mathfrak{g}'$ or is contained in the center.*

Proof (i) If $\det A = 0$, then the center of $\mathfrak{g}$ is non-trivial by Proposition 1.4.6. If (1.17) is violated, then we can split $\{1, \dots, n\}$ into two non-trivial sunsets I_1 and I_2 such that $a_{ij} = a_{ji} = 0$ whenever $i \in I_1, j \in I_2$. Then $\mathfrak{g}$ is a direct sum of two ideals by Proposition 1.4.3. Conversely, let $\det A \neq 0$ and (1.17) hold. If $\mathfrak{i} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{i}$ contains a non-zero element $h \in \mathfrak{h}$. By Proposition 1.4.6, $\mathfrak{c} = 0$, and hence $[h, e_j] = ae_j \neq 0$ for some j. Hence $e_j \in \mathfrak{i}$, and $\alpha_j^\vee = [e_j, f_j] \in \mathfrak{i}$. Now from (1.17) it follows that $e_j, f_j, \alpha_j^\vee \in \mathfrak{i}$ for all $\mathfrak{i}$. Since $\det A \neq 0$, $\mathfrak{h}$ is a span of the $\alpha_j^\vee$'s, and $\mathfrak{i} = \mathfrak{g}$.

(ii) is proved similarly—exercise. □

We finish with some terminology concerning duality. Note that A^t is also GCM, and $(\mathfrak{h}^*, \Pi^\vee, \Pi)$ is its realization. The algebras $\mathfrak{g}(A)$ and $\mathfrak{g}(A^t)$ are called *dual* to each other. Then the *dual root lattice*

$$Q^\vee := \sum_{i=1}^{n} \mathbb{Z}\alpha_i^\vee$$

corresponding to $\mathfrak{g}(A)$ is the root lattice corresponding to $\mathfrak{g}(A^t)$. Also, denote by

$$\Delta^\vee \subset Q^\vee$$

the root system $\Delta(A^t)$ and refer to it as the *dual root system of* $\mathfrak{g}$.

1.5 Examples

The following clumsy but easy result will be useful for dealing with examples:

Proposition 1.5.1 *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{C}$ *and* $\mathfrak{h}$ *be a finite dimensional abelian subalgebra of* $\mathfrak{g}$ *with* $\dim \mathfrak{h} = 2n - \ell$. *Suppose* $\Pi = \{\alpha_1, \dots, \alpha_n\}$ *is a linearly independent system of* $\mathfrak{h}^*$ *and* $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\}$ *a linearly independent system of* $\mathfrak{h}$ *satisfying* $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$. *Suppose also that* $e_1, \dots, e_n, f_1, \dots, f_n$ *are elements of* $\mathfrak{g}$ *satisfying relations (1.12)-(1.15). Suppose* $e_1, \dots, e_n, f_1, \dots, f_n$ *and* $\mathfrak{h}$ *generate* $\mathfrak{g}$ *and that* $\mathfrak{g}$ *has no non-zero ideals* $\mathfrak{i}$ *with* $\mathfrak{i} \cap \mathfrak{h} = 0$. *Then* $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}(A)$.

Proof There is surjective homomorphism $\theta : \tilde{\mathfrak{g}}(A) \to \mathfrak{g}$. The restriction of θ to $\mathfrak{h} \subset \tilde{\mathfrak{g}}(A)$ is an isomorphism onto $\mathfrak{h} \subset \mathfrak{g}$, cf. Remark 1.3.4. So $\ker \theta \cap \mathfrak{h} = 0$. It follows that $\ker \theta \subset \mathfrak{r}$. In fact, $\ker \theta = \mathfrak{r}$, since $\mathfrak{g}$ has no nonzero ideal $\mathfrak{i}$ with $\mathfrak{i} \cap \mathfrak{h} = 0$. $\square$

Example 1.5.2 Let

$$A = A_n := \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}.$$

We claim that $\mathfrak{g}(A) \cong \mathfrak{sl}_{n+1}$. We take $\mathfrak{h} \subset \mathfrak{sl}_{n+1}$ to be diagonal matrices of trace 0. Let $\varepsilon_i \in \mathfrak{h}^*$ be the ith coordinate function, i.e.

$$\varepsilon_i(\mathrm{diag}(a_1, \dots, a_n)) = a_i \qquad (1 \le i \le n).$$

Now take

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad \alpha_i^\vee = e_{ii} - e_{i+1,i+1} \qquad (1 \le i \le n),$$

and

$$e_i = e_{i,i+1}, \quad f_i = e_{i+1,i} \qquad\qquad (1 \le i \le n).$$

It is easy to see that all assumptions of Proposition 1.5.1 are satisfied. For example, to see that $\mathfrak{sl}_{n+1}$ does not contain nonzero ideals $\mathfrak{i}$ with $\mathfrak{i} \cap \mathfrak{h} = 0$, note that any such ideal would have to be a direct sum of the root subspaces, and it is easy to see that no such is an ideal. In fact, an argument along these lines shows that $\mathfrak{sl}_{n+1}$ is a simple Lie algebra, i.e. it has no non-trivial ideals. Note that the roots of $\mathfrak{sl}_{n+1}$ are precisely

$$\varepsilon_i - \varepsilon_j \qquad (1 \le i \ne j \le n+1),$$

with the corresponding root spaces $\mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{C}e_{ij}$.

Moreover, a similar argument shows that if $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra with Cartan matrix A, then $\mathfrak{g} \cong \mathfrak{g}(A)$.

Before doing the next example we explain several general constructions.

Let $\mathfrak{g}$ be an arbitrary Lie algebra. A 2-*cocycle* on $\mathfrak{g}$ is a bilinear map

$$\psi : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

satisfying

$$\psi(y, x) = -\psi(x, y) \qquad (x, y \in \mathfrak{g}), \tag{1.18}$$

$$\psi([x, y], z) + \psi([y, z], x) + \psi([z, x], y) = 0 \qquad (x, y, z \in \mathfrak{g}). \tag{1.19}$$

If ψ is a 2-cocyle and

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}c$$

for some formal element c, then $\tilde{\mathfrak{g}}$ is a Lie algebra with respect to

$$[x + \lambda c, y + \mu c] = [x, y] + \psi(x, y)c.$$

We refer to $\tilde{\mathfrak{g}}$ as the *central extension* of $\mathfrak{g}$ with respect to the cocycle ψ.

Let $D : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra derivation, i.e. D is a linear map and

$$D([x, y]) = [D(x), y] + [x, D(y)] \qquad (x, y \in \mathfrak{g}).$$

Let

$$\hat{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}d$$

for some formal element d. Then $\hat{\mathfrak{g}}$ is a Lie algebra with respect to

$$[x + \lambda d, y + \mu d] = [x, y] + \lambda d(y) - \mu d(x).$$

We refer to $\hat{g}$ as the Lie algebra obtained from $\mathfrak{g}$ by *adjoining* the derivation D. Sometimes we use the same letter d for both d and D.

A typical example of derivation comes as follows. Let $\mathfrak{g} = \oplus_j \in \mathbb{Z}\mathfrak{g}_j$ be a Lie algebra grading on $\mathfrak{g}$. Then the map $\mathfrak{g}$ sending x to jx for any $x \in \mathfrak{g}$ is a derivation.

Let

$$\mathcal{L} = \mathbb{C}[t, t^{-1}],$$

and for any Lie algebra $\mathfrak{g}$ define the corresponding *loop algebra*

$$\mathcal{L}(\mathfrak{g}) := \mathcal{L} \otimes \mathfrak{g}.$$

This is an infinite dimensional Lie algebra with bracket

$$[P \otimes x, Q \otimes y] = PQ \otimes [x, y] \qquad (P, Q \in \mathcal{L},\ x, y \in \mathfrak{g}).$$

If $(\cdot|\cdot)$ is a bilinear form on $\mathfrak{g}$, it can be extended to a $\mathcal{L}$-valued bilinear form

$$(\cdot|\cdot)_t : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathcal{L}$$

by setting

$$(P \otimes x | Q \otimes y)_t = PQ(x|y).$$

We define the *residue* function

$$\text{Res } : \mathcal{L} \to \mathbb{C},\ \sum_{i \in \mathbb{Z}} c_i t^i \mapsto c_{-1}.$$

Lemma 1.5.3 *Let $(\cdot|\cdot)$ be a symmetric invariant bilinear form on $\mathfrak{g}$. The function $\psi : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathbb{C}$ defined by*

$$\psi(a, b) = \text{Res}\,(\frac{da}{dt}|b)_t$$

is a 2-cocycle on $\mathcal{L}(\mathfrak{g})$. Moreover, $\psi(t^i \otimes x, t^j \otimes y) = i\delta_{i,-j}(x|y)$.

Proof Note that

$$\begin{aligned}
\psi(t^i \otimes x, t^j \otimes y) &= \operatorname{Res}\,(it^{i-1} \otimes x | t^j \otimes y)_t \\
&= \operatorname{Res}\, it^{i+j-1}(x|y) \\
&= \begin{cases} i(x|y) & \text{if } i+j=0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

from which (1.18) follows. Moreover, we have

$$\begin{aligned}
\psi([t^i \otimes x, t^j \otimes y], t^k \otimes z) &= \psi(t^{i+j}[x,y], t^k \otimes z) \\
&= \begin{cases} (i+j)([x,y]|z) & \text{if } i+j+k=0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Now, if $i+j+k \neq 0$, (1.19) is clear. If $i+j+k=0$, the required sum is

$$\begin{aligned}
& -k([x,y]|z) - i([y,z]|x) - j([z,x]|y) \\
= & -k([x,y]|z) - i([x,y]|z) - j([x,y]|z) = 0
\end{aligned}$$

since the form is symmetric and invariant. □

If $\mathfrak{g}$ is a simple finite dimensional Lie algebra it possesses unique up to a scalar non-degenerate symmetric invariant form $(\cdot|\cdot)$, so Lemma 1.5.3 allows us to define a 2-cocycle ψ on $\mathcal{L}(\mathfrak{g})$, and the previous discussion then allows us to consider the corresponding central extension

$$\bar{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c.$$

Moreover, $\bar{\mathcal{L}}(\mathfrak{g})$ is graded with $\deg t^j \otimes x = j$, $\deg c = 0$. We then have the corresponding derivation

$$d : \bar{\mathcal{L}}(\mathfrak{g}) \to \bar{\mathcal{L}}(\mathfrak{g}),\ t^j \otimes x \mapsto jt^j \otimes x,\ c \mapsto 0.$$

Finally, by adjoining d to $\bar{\mathcal{L}}(\mathfrak{g})$ we get the Lie algebra

$$\hat{\mathcal{L}}(\mathfrak{g}) := \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d,$$

with operation

$$\begin{aligned}
&[t^m \otimes x + \lambda c + \mu d, t^n \otimes y + \lambda_1 c + \mu_2 d] \\
&= (t^{m+n} \otimes [x,y] + \mu n t^n \otimes y - \mu_1 m t^m \otimes x) + m\delta_{m,-n}(x|y)c.
\end{aligned}$$

Example 1.5.4 Let $A = A_1^{(1)} := \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We claim that

$$\mathfrak{g}(A) \cong \hat{\mathcal{L}}(\mathfrak{sl}_2),$$

sometimes also denoted $\widehat{\mathfrak{sl}_2}$. First of all recall that the non-degenerate symmetric invariant form on $\mathfrak{sl}_2$ is just the trace form

$$(x|y) = \operatorname{tr}(xy) \qquad (x, y \in \mathfrak{sl}_2).$$

Then

$$(e, f) = 1,\ (h, h) = 2,\ (e, e) = (e, h) = (f, h) = (f, f) = 0.$$

Now set

$$\mathfrak{h} = \mathbb{C}h \oplus \mathbb{C}c \oplus \mathbb{C}d$$

and note that $\dim \mathfrak{h} = 2n - \ell$. Next define

$$\alpha_0^\vee = c - 1 \otimes h, \quad \alpha_1^\vee = 1 \otimes h$$

and $\alpha_0, \alpha_1 \in \mathfrak{h}^*$ via

$$\langle \alpha_i, \alpha_i^\vee \rangle = 2, \quad \langle \alpha_i, \alpha_j^\vee \rangle = -2 \qquad (0 \le i \ne j \le 1)$$

and

$$\langle \alpha_0, c \rangle = 0,\ \langle \alpha_0, d \rangle = 1,\ \langle \alpha_1, c \rangle = 0,\ \langle \alpha_1, d \rangle = 0.$$

It is clear that we have defined a realization of A. Next set

$$e_0 = t \otimes f,\ e_1 = 1 \otimes e,\ f_0 = t^{-1} \otimes e,\ f_1 = 1 \otimes f.$$

It is now easy to check the remaining conditions of Proposition 1.5.1. Indeed,

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee,\ [h, e_i] = \alpha_i(h)e_i,\ [h, f_i] = -\alpha_i(h)f_i \qquad (h \in \mathfrak{h})$$

follow from definitions. Next, $\widehat{\mathfrak{sl}_2}$ is generated by $\mathfrak{h}, e_0, e_1, f_0, f_1$: if $\mathfrak{m}$ is the subalgebra generated by them, then clearly $1 \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$. Set $\mathfrak{i} := \{x \in \mathfrak{sl}_2 \mid t \otimes x \in \mathfrak{m}$. We have $f \in \mathfrak{i}$, so $\mathfrak{i} \ne 0$. Also, if $x \in \mathfrak{i}, y \in \mathfrak{sl}_2$, then $[x, y] \in \mathfrak{i}$, thus $\mathfrak{i}$ is an ideal of $\mathfrak{sl}_2$, whence $\mathfrak{i} = \mathfrak{sl}_2$, and $t \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$. We may now use the relation

$$[t \otimes x, t^{k-1} \otimes y] = t^k \otimes [x, y] \qquad (k > 0)$$

to deduce by induction on k that $t^k \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$ for all $k > 0$. Analogously $t^k \otimes \mathfrak{sl}_2 \subset \mathfrak{m}$ for all $k < 0$.

It remains to show that $\widehat{\mathfrak{sl}_2}$ has no non-zero ideals $\mathfrak{i}$ having trivial intersection with $\mathfrak{h}$. For this we study root space decomposition of $\widehat{\mathfrak{sl}_2}$. Let $\delta \in \mathfrak{h}^*$ be defined from

$$\delta(\alpha_1^\vee) = \delta(\alpha_2^\vee) = 0, \quad \delta(d) = 1.$$

We claim that the roots are precisely

$$\{\pm\alpha_1 + k\delta \mid k \in \mathbb{Z}\} \cup \{k\delta \mid k \in \mathbb{Z} \setminus \{0\}\}.$$

Indeed,

$$\mathfrak{g}_{\alpha_1+k\delta} = \mathbb{C}(t^k \otimes e),\ \mathfrak{g}_{-\alpha_1+k\delta} = \mathbb{C}(t^k \otimes f), \qquad (k \in \mathbb{Z})$$

and

$$\mathfrak{g}_{k\delta} = \mathbb{C}(t^k \otimes h) \qquad (k \in \mathbb{Z} \setminus \{0\}).$$

Since $\delta = \alpha_1 + \alpha_2$, positive roots are of the form $\{(k+1)\alpha_1 + k\alpha_1, k\alpha_1 + (k+1)\alpha_2, (k+1)\alpha_1 + (k+1)\alpha_2\}$ for $k \in \mathbb{Z}_{\geq 0}$.

Let $\mathfrak{i}$ be a non-zero ideal of $\widehat{\mathfrak{s}l_2}$ which has trivial intersection with $\mathfrak{h}$. It follows from Lemma 1.3.2 that some $t^i \otimes x \in \mathfrak{i}$ where $x = e, f$ or h. Take y to be f, e or h, respectively. Then $(x|y) \neq 0$, and

$$[t^i \otimes x, t^{-i} \otimes y] = [x, y] + i(x|y)c \in \mathfrak{i} \cap \mathfrak{h},$$

and hence

$$[x, y] + i(x|y)c = 0.$$

since $[x, y]$ is a multiple of $1 \otimes h$, we must have $i = 0$, whence $[x, y] = 0$. But since $i = 0$ we cannot have $x = h$, and then $[x, y] = 0$ is a contradiction.

In conclusion we introduce the element $\Lambda_0 \in \mathfrak{h}^*$ which is defined from

$$\Lambda_0 : \alpha_0^\vee \mapsto 1,\ \alpha_1^\vee \mapsto 0,\ d \mapsto 0.$$

Then $\{\alpha_0, \alpha_1, \Lambda_0\}$ and $\{\alpha_1, \delta, \Lambda_0\}$ are bases of $\mathfrak{h}^*$.

2

Invariant bilinear form and generalized Casimir operator

2.1 Symmetrizable GCMs

A GCM $A = (a_{ij})$ is called *symmetrizable* if there exists a non-singular diagonal matrix $D = \text{diag}(\varepsilon_1, \dots, \varepsilon_n)$ and a symmetric matrix B such that

$$A = DB. \tag{2.1}$$

If A is symmetrizable, we also call $\mathfrak{g} = \mathfrak{g}(A)$ symmetrizable.

Lemma 2.1.1 *Let A be a GCM. Then A is symmetrizable if and only if*

$$a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} = a_{i_2 i_1} a_{i_3 i_2} \dots a_{i_1 i_k}$$

for all $i_1, i_2, \dots, i_k \in \{1, \dots, n\}$.

Proof If A is symmetrisable then $a_{ij} = \varepsilon_i b_{ij}$, hence

$$\begin{aligned} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1} &= d_{i_1} \dots d_{i_k} a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_k i_1}, \\ a_{i_2 i_1} a_{i_3 i_2} \dots a_{i_1 i_k} &= d_{i_1} \dots d_{i_k} b_{i_2 i_1} b_{i_3 i_2} \dots b_{i_1 i_k}, \end{aligned}$$

and these are equal since B is symmetric.

For the converse, we may assume that A is indecomposable. Thus for each $i \in \{1, \dots, n\}$ there exists a sequence $1 = j_1, \dots, j_t = i$ with

$$a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{t-1} j_t} \neq 0.$$

We choose a number $\varepsilon_1 \neq 0$ in $\mathbb{R}$ and define

$$\varepsilon_i = \frac{a_{j_t j_{t-1}} \dots a_{j_2 j_1}}{a_{j_1 j_2} a_{j_2 j_3} \dots a_{j_{t-1} j_t}} \varepsilon_1. \tag{2.2}$$

To see that this definition depends only on i, not on the sequence chosen from 1 to i, let $1 = k_1, \ldots, k_u = i$ be another such sequence. Then

$$\frac{a_{j_t j_{t-1}} \ldots a_{j_2 j_1}}{a_{j_1 j_2} a_{j_2 j_3} \ldots a_{j_{t-1} j_t}} = \frac{a_{k_u k_{u-1}} \ldots a_{k_2 k_1}}{a_{k_1 k_2} a_{k_2 k_3} \ldots a_{k_{u-1} k_u}},$$

since it is equivalent to

$$a_{1k_2} a_{k_2 k_3} \ldots a_{k_{t-1} i} a_{i j_{t-1}} \ldots a_{j_2 1} = a_{k_2 1} a_{k_3 k_2} \ldots a_{i k_{u-1}} a_{j_{t-1} j_t} \ldots a_{1 j_2},$$

which is one of the given conditions on the matrix A. Thus $\varepsilon_i \in \mathbb{R}$ is well defined and $\varepsilon_i \neq 0$.

Let $b_{ij} = a_{ij}/\varepsilon_i$. It remains to show that $b_{ij} = b_{ji}$ or $a_{ij}/\varepsilon_i = a_{ji}/\varepsilon_j$. If $a_{ij} = 0$ this is clear since then $a_{ji} = 0$. If $a_{ij} \neq 0$, let $1 = j_1, \ldots, j_t = i$ be a sequence from 1 to i of the type described above. Then $1 = j_1, \ldots, j_t, j$ is another such sequence from 1 to j. These sequences may be used to obtain ε_i and ε_j respectively, and we have

$$\varepsilon_j = \frac{a_{ji}}{a_{ij}} \varepsilon_i,$$

as required. □

Lemma 2.1.2 *Let A be a symmetrizable indecomposable GCM. Then A can be expressed in the form $A = DB$ where $D = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$, B is symmetric, with $\varepsilon_1, \ldots, \varepsilon_n$ positive integers and $b_{ij} \in \mathbb{Q}$. Also D is determined by these conditions up to a scalar multiple.*

Proof We choose ε_1 to be any positive rational number. Then (2.2) shows that we can choose all ε_i to be positive rational numbers. Multiplying by a positive scalar we can make all ε_i positive integers. Also $b_{ij} = a_{ij}/\varepsilon_i \in \mathbb{Q}$. The proof of Lemma 2.1.1 also shows that D is unique up to a scalar multiple. □

Remark 2.1.3 If A is symmetrizable, in view of the above lemma, we may and **always will assume** that $\varepsilon_1, \ldots, \varepsilon_n$ are positive integers and B is a rational matrix.

2.2 Invariant bilinear form on $\mathfrak{g}$

Let A be a symmetrizable GCM as above. Fix a linear complement $\mathfrak{h}''$ to $\mathfrak{h}'$ in $\mathfrak{h}$:

$$\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''.$$

Define a symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ by the following two conditions:

$$(\alpha_i^\vee|h) = \langle \alpha_i, h\rangle \varepsilon_i \qquad (h \in \mathfrak{h}); \tag{2.3}$$

$$(h'|h'') = 0 \qquad (h', h'' \in \mathfrak{h}''). \tag{2.4}$$

Note that

$$(\alpha_i^\vee|\alpha_j^\vee) = b_{ij}\varepsilon_i\varepsilon_j. \tag{2.5}$$

Lemma 2.2.1

(i) *The kernel of the restriction* $(\cdot|\cdot)|_{\mathfrak{h}'}$ *is* $\mathfrak{c}$.

(ii) $(\cdot|\cdot)$ *is non-degenerate on* $\mathfrak{h}$.

Proof (i) is clear from (1.16).

(ii) We have rank B = rank $A = \ell$. Symmetric matrix of rank ℓ has a non-singular $\ell \times \ell$ principal minor. (Indeed, the case $\ell = n$ is trivial. If $\ell < n$, for some i the ith row of B is a linear combination of the remaining rows. Let B' be the matrix obtained from B by removing the ith row. Then rank $B' = \ell$. Let B'' be the matrix obtained from B' by removing its ith column. Then rank $B'' = \ell$. Thus by induction B'' has a non-singular $\ell \times \ell$ principal minor.)

It follows that A has a non-singular $\ell \times \ell$ principal minor. Assume for simplicity that A has the form $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ where A_{11} is a non-singular $\ell \times \ell$ matrix. By the proof of Proposition 1.2.4, we may extend to bases $\alpha_1^\vee, \dots, \alpha_{2n-\ell}^\vee$ of $\mathfrak{h}$ and $\alpha_1, \dots, \alpha_{2n-\ell}$ of $\mathfrak{h}^*$ such that

$$(\langle \alpha_i^\vee, \alpha_j \rangle) = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}.$$

Write D in the form $D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}$ where D_1 is $\ell \times \ell$. Then

$$(\langle \alpha_i^\vee, \alpha_j \rangle) = \begin{pmatrix} D_1B_{11} & D_1B_{12} & 0 \\ D_2B_{21} & D_2B_{22} & I_{n-\ell} \\ 0 & I_{n-\ell} & 0 \end{pmatrix}.$$

Note next from definitions that

$$(\alpha_i^\vee|\alpha_j^\vee) = \begin{pmatrix} D_1B_{11}D_1 & D_1B_{12}D_2 & 0 \\ D_2B_{21}D_1 & D_2B_{22}D_2 & D_2 \\ 0 & D_2 & 0 \end{pmatrix}.$$

This matrix is non-singular, since its determinant is

$$\pm(\det D_1)^2(\det D_2)^2\det B_{11}.$$

□

Since $(\cdot|\cdot)$ is non-degenerate we have an isomorphism $\nu:\mathfrak{h}\to\mathfrak{h}^*$ such that

$$\langle\nu(h_1),h_2\rangle=(h_1|h_2)\qquad(h_1,h_2\in\mathfrak{h}),$$

and the induced bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$. Note from (2.3) that

$$\nu(\alpha_i^\vee)=\varepsilon_i\alpha_i. \tag{2.6}$$

So, by (2.5),

$$(\alpha_i|\alpha_j)=b_{ij}=a_{ij}\varepsilon_i^{-1}. \tag{2.7}$$

Since all $\varepsilon_i>0$ (Remark 2.1.3), it follows that

$$(\alpha_i|\alpha_i) > 0 \qquad (1\le i\le n). \tag{2.8}$$

$$(\alpha_i|\alpha_j) \le 0 \qquad (i\ne j). \tag{2.9}$$

$$\alpha_i^\vee = \frac{2}{(\alpha_i|\alpha_i)}\nu^{-1}(\alpha_i). \tag{2.10}$$

So we get the usual expression for Cartan matrix:

$$a_{ij}=\frac{2(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)}.$$

Example 2.2.2 (i) If A is as in Example 1.5.2, then the Gram matrix of $(\cdot|\cdot)$ in the basis $\alpha_1^\vee,\dots,\alpha_n^\vee$ is A itself. In fact, we may take $(\cdot|\cdot)$ to be the trace form restricted to $\mathfrak{h}$.

(ii) If A is as in Example 1.5.4, choose $\mathfrak{h}'':=\mathbb{C}d$. Then the Gram matrix of $(\cdot|\cdot)$ in the basis $\alpha_0^\vee,\alpha_1^\vee,d$ and the transported form in the basis $\alpha_0,\alpha_1,\Lambda_0$ is

$$\begin{pmatrix}2 & -2 & 1\\ -2 & 2 & 0\\ 1 & 0 & 0\end{pmatrix},$$

while the Gram matrix of the same forms in the bases $\alpha_1^\vee,c,d$ and $\alpha_1,\delta,\Lambda_0$ is

$$\begin{pmatrix}2 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}.$$

Theorem 2.2.3 *Let $\mathfrak{g}$ be symmetrizable. Fix decomposition (2.1) for A. Then there exist a non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ such that*

(i) *$(\cdot|\cdot)$ is invariant, i.e. for all $x, y, z \in \mathfrak{g}$ we have*

$$([x, y]|z) = (x|[y, z]). \tag{2.11}$$

(ii) *$(\cdot|\cdot)|_{\mathfrak{h}}$ is as above.*

(iii) *$(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ of $\alpha + \beta \neq 0$.*

(iv) *$(\cdot|\cdot)|_{\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}}$ is non-degenerate for $\alpha \neq 0$.*

(v) *$[x, y] = (x|y)\nu^{-1}(\alpha)$ for $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}, \alpha \in \Delta$.*

Proof Set $\mathfrak{g}(N) := \oplus_{j=-N}^{N} \mathfrak{g}_j$, $N = 0, 1, \dots$, where $\mathfrak{g} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is the principal grading. Start with the form $(\cdot|\cdot)$ on $\mathfrak{g}(0) = \mathfrak{h}$ defined above and extend it to $\mathfrak{g}(1)$ as follows:

$$(e_i|f_j) = \delta_{ij}\varepsilon_i, \ (\mathfrak{g}_0|\mathfrak{g}_{\pm 1}) = 0, \ (\mathfrak{g}_{\pm 1}|\mathfrak{g}_{\pm 1}) = 0.$$

An explicit check shows that the form $(\cdot|\cdot)$ on $\mathfrak{g}(1)$ satisfies (2.11) if both $[x, y]$ and $[y, z]$ belong to $\mathfrak{g}(1)$. Now we proceed by induction to extend the form to an arbitrary $\mathfrak{g}(N)$, $N \geq 2$. By induction we assume that the form has been extended to $\mathfrak{g}(N-1)$ so that it satisfies $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ for $|i|, |j| \leq N - 1$ with $i + j \neq 0$, and (2.11) for all $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j, z \in \mathfrak{g}_k$ with $|i + j|, |j + k| \leq N - 1$. We show that the form can be extended to $\mathfrak{g}(N)$ with analogous properties. First we require that $(\mathfrak{g}_i|\mathfrak{g}_j) = 0$ for all $|i|, |j| \leq N$ with $i + j \neq 0$. It remains to define $(x|y) = (y|x)$ for $x \in \mathfrak{g}_N, y \in \mathfrak{g}_{-N}$. Note that y is a linear combination of Lie monomials in $f_1, \dots, f_n$ of degree N. Since $N \geq 2$, each Lie monomial is a bracket of Lie monomials of degrees s and t with $s + t = N$. It follows that y can be written in the form

$$y = \sum_i [u_i, v_i] \qquad (u_i \in \mathfrak{g}_{-a_i}, \ v_i \in \mathfrak{g}_{-b_i}) \tag{2.12}$$

where $a_i, b_i > 0$ and $a_i + b_i = N$. The expression of y in this form need not be unique. Now define

$$(x|y) := \sum_i ([x, u_i]|v_i). \tag{2.13}$$

The RHS is known since $[x, u_i]$ and v_i lie in $\mathfrak{g}(N-1)$. We must therefore show that RHS remains the same if a different expression (2.12) for y is chosen. In a similar way we can write x in the form

$$x = \sum_j [w_j, z_j] \qquad (w_j \in \mathfrak{g}_{-c_j}, \ z_j \in \mathfrak{g}_{-d_j})$$

where $c_j, d_j > 0$ and $c_j + d_j = N$. We will show that

$$\sum_j (w_j|[z_j, y]) = \sum_i ([x, u_i]|v_i).$$

This will imply that the RHS of (2.13) is independent of the given expression for y. In fact it is sufficient to show that

$$(w_j|[z_j, [u_i, v_i]]) = ([[w_j, z_j], u_i]|v_i).$$

Now

$$\begin{aligned}
([[w_j, z_j], u_i]|v_i) &= ([[w_j, u_i], z_j]|v_i) + ([w_j, [z_j, u_i]]|v_i) \\
&= ([w_j, u_i]|[z_j, v_i]) - ([z_j, u_i]|[w_j, v_i]) \\
&= ([w_j, u_i]|[z_j, v_i]) - ([w_j, v_i]|[z_j, u_i]) \\
&= (w_j|[u_i, [z_j, v_i]]) - (w_j|[v_i, [z_j, u_i]]) \\
&= (w_j|[z_j, [u_i, v_i]]).
\end{aligned}$$

We must now check (2.11) for all $x \in \mathfrak{g}_i, y \in \mathfrak{g}_j, z \in \mathfrak{g}_k$ with $|i + j|, |j + k| \leq N$. We may assume that $i + j + k = 0$ and at least one of $|i|, |j|, |k|$ is N. We suppose first that just one of $|i|, |j|, |k|$ is N. Then the other two are non-zero. If $|i| = N$ then (2.11) holds by definition of the form on $\mathfrak{g}(N)$. Similarly for $|k| = N$. So suppose $|j| = N$. We may assume that y has form $y = [u, v]$ where $u \in \mathfrak{g}_a, v \in \mathfrak{g}_b$, $a + b = N$, and $0 < |a| < |j|, 0 < |b| < |j|$. Then

$$\begin{aligned}
([x, y]|z) &= ([x, [u, v]]|z) \\
&= ([[v, x], u]|z) + ([[x, u], v]|z) \\
&= ([v, x]|[u, z]) + ([x, u]|[v, z]) \\
&= ([x, v]|[z, u]) + ([x, u]|[v, z]) \\
&= (x|[v, [z, u]]) + (x|[u, [v, z]]) \\
&= (x|[[u, v], z]) \\
&= (x|[y, z]).
\end{aligned}$$

Now suppose two of $|i|, |j|, |k|$ are equal to N. Then i, j, k are $N, -N, 0$ in some order. Thus one of x, y, z lies in $\mathfrak{h}$. Suppose $x \in \mathfrak{h}$. We may again assume that $y = [u, v]$. Then

$$\begin{aligned}([x,y]|z) &= ([x,[u,v]]|z)\\ &= ([[x,u],v]|z) - ([[x,v],u]|z)\\ &= ([x,u]|[v,z]) - ([x,v]|[u,z]) \quad \text{(by definition of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N))\\ &= (x|[u,[v,z]]) - (x|[v,[u,z]]) \quad \text{(by invariance of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N-1))\\ &= (x|[[u,v],z])\\ &= (x|[y,z]).\end{aligned}$$

If $z \in \mathfrak{h}$ the result follows by symmetry. Finally, let $y \in \mathfrak{h}$. Then we may assume that $z = [u, v]$ where $u \in \mathfrak{g}_a, v \in \mathfrak{g}_b$, $a + b = k$, and $0 < |a| < |k|, 0 < |b| < |k|$. Then

$$\begin{aligned}(x|[y,z]) &= (x|[y,[u,v]])\\ &= (x|[u,[y,v]]) + (x|[[y,u],v])\\ &= ([x,u]|[y,v]) + ([x,[y,u]]|v) \quad \text{(by definition of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N))\\ &= ([[x,u],y]|v) + ([x,[y,u]]|v) \quad \text{(by invariance of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N-1))\\ &= ([[x,y],u]|v)\\ &= ([x,y]|[u,v]) \quad \text{(by definition of } (\cdot|\cdot) \text{ on } \mathfrak{g}(N))\\ &= ([x,y]|z).\end{aligned}$$

By induction, we have defined a symmetric bilinear form on $\mathfrak{g}$ which satisfies (i) and (ii). Let $\mathfrak{i}$ be the radical of $(\cdot|\cdot)$. Then $\mathfrak{i}$ is an ideal in $\mathfrak{g}$. If $\mathfrak{i} \neq 0$ then $\mathfrak{i} \cap \mathfrak{h} \neq 0$, which contradicts Lemma 2.2.1(ii). Thus $(\cdot|\cdot)$ is non-degenerate.

The form also satisfies (iii), since for all $h \in \mathfrak{h}, x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$, using invariance, we have

$$0 = ([h,x]|y) + (x|[h,y]) = (\langle \alpha, h\rangle + \langle \alpha, h\rangle)(x|y).$$

Now (iv) also follows from the non-degeneracy of the form.

Finally, let $\alpha \in \Delta$, $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta, h \in \mathfrak{h}$. Then

$$([x,y] - (x|y)\nu^{-1}(\alpha)|h) = (x|[y,h]) - (x|y)\langle \alpha, h\rangle = 0,$$

which implies (v). □

The form $(\cdot|\cdot)$ constructed in the theorem above is called the *standard invariant form* on $\mathfrak{g}$. It is uniquely determined by the conditions (i) and (ii) of the theorem (indeed, if $(\cdot|\cdot)_1$ is another such form then $(\cdot|\cdot) - (\cdot|\cdot)_1$ is too, but its radical is non-trivial ideal containing $\mathfrak{h}$, which is a contradiction).

Throughout $(\cdot|\cdot)$ denotes the standard invariant form on symmetrizable $\mathfrak{g}$.

Example 2.2.4 (i) The standard invariant form is just the trace form on $\mathfrak{sl}_{n+1}$ is the trace form.

(ii) The standard invariant form on $\widehat{\mathfrak{sl}}_2$ is given by

$$\begin{aligned}(t^m \otimes x | t^n \otimes y) &= \delta_{m,-n} \mathrm{tr}\,(xy),\\ (\mathbb{C}c + \mathbb{C}d | \mathcal{L}(\mathfrak{sl}_2)) &= 0,\\ (c|c) = (d|d) &= 0,\\ (c|d) &= 0.\end{aligned}$$

2.3 Generalized Casimir operator

Let $\mathfrak{g}$ be symmetrizable. By Theorem 2.2.3(iii),(iv), we can choose dual bases $\{e_\alpha^{(i)}\}$ and $\{e_{-\alpha}^{(i)}\}$ in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$. Then

$$(x|y) = \sum_s (x|e_{-\alpha}^{(s)})(y|e_\alpha^{(s)}) \qquad (x \in \mathfrak{g}_\alpha,\ y \in \mathfrak{g}_{-\alpha}). \tag{2.14}$$

Lemma 2.3.1 *If $\alpha, \beta \in \Delta$ and $z \in \mathfrak{g}_{\beta-\alpha}$, then in $\mathfrak{g} \otimes \mathfrak{g}$ we have*

$$\sum_s e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}] = \sum_t [e_{-\beta}^{(t)}, z] \otimes e_\beta^{(t)}. \tag{2.15}$$

Proof Define a bilinear form $(\cdot|\cdot)$ on $\mathfrak{g} \otimes \mathfrak{g}$ via

$$(x \otimes y | x_1 \otimes y_1) := (x|x_1)(y|y_1).$$

Taje $e \in \mathfrak{g}_\alpha, f \in \mathfrak{g}_{-\beta}$. It suffices to prove that pairing of both sides of (2.15) with $e \otimes f$ gives the same result. We have, using (2.14),

$$\begin{aligned}\sum_s (e_{-\alpha}^{(s)} \otimes [z, e_\alpha^{(s)}] | e \otimes f) &= \sum_s (e_{-\alpha}^{(s)}|e)([z, e_\alpha^{(s)}]|f)\\ &= \sum_s (e_{-\alpha}^{(s)}|e)(e_\alpha^{(s)}|[f, z])\\ &= (e|[f, z]).\end{aligned}$$

Similarly,

$$\sum_t ([e_{-\beta}^{(t)}, z] \otimes e_\beta^{(t)} | e \otimes f) = ([z, e]|f),$$

as required. □

Corollary 2.3.2 *In the notation of Lemma 2.3.1, we have*

$$\sum_s [e_{-\alpha}^{(s)}, [z, e_\alpha^{(s)}]] = -\sum_t [[z, e_{-\beta}^{(t)}], e_\beta^{(t)}] \qquad (\textit{in } \mathfrak{g}), \tag{2.16}$$

$$\sum_s e_{-\alpha}^{(s)}[z, e_\alpha^{(s)}] = -\sum_t [z, e_{-\beta}^{(t)}] e_\beta^{(t)} \qquad (\textit{in } U(\mathfrak{g})). \tag{2.17}$$

Definition 2.3.3 A $\mathfrak{g}$-module V us called *restricted* if for every $v \in V$ we have $\mathfrak{g}_\alpha v = 0$ for all but finitely many positive roots α.

Let $\rho \in \mathfrak{h}^*$ be any functional satisfying

$$\langle \rho, \alpha_i^\vee \rangle = 1 \qquad (1 \le i \le n).$$

Then, by (2.10),

$$(\rho|\alpha_i) = (\alpha_i|\alpha_i)/2 \qquad (1 \le i \le n). \tag{2.18}$$

For a restricted $\mathfrak{g}$-module V we define a linear operator Ω_0 on V as follows:

$$\Omega_0 = 2 \sum_{\alpha \in \Delta_+} \sum_i e_{-\alpha}^{(i)} e_\alpha^{(i)}.$$

One can check that this definition is independent on choice of dual bases. Let $u_1, u_2, \ldots$ and $u^1, u^2, \ldots$ be dual bases of $\mathfrak{h}$. Note that

$$(\lambda|\mu) = \sum_i \langle \lambda, u^i \rangle \langle \mu, u_i \rangle \qquad (\lambda, \mu \in \mathfrak{h}^*). \tag{2.19}$$

Indeed,

$$\begin{aligned} (\lambda|\mu) &= (\nu^{-1}(\lambda)|\nu^{-1}(\mu)) \\ &= \sum_i (\nu^{-1}(\lambda)|u^i)(\nu^{-1}(\mu)|u_i) \\ &= \sum_i \langle \lambda, u^i \rangle \langle \mu, u_i \rangle. \end{aligned}$$

Also,

$$[\sum_i u^i u_i, x] = x((\alpha|\alpha) + 2\nu^{-1}(\alpha)) \qquad (x \in \mathfrak{g}_\alpha). \tag{2.20}$$

Indeed,

$$\begin{aligned} [\sum_i u^i u_i, x] &= \sum_i \langle \alpha, u^i \rangle x u_i + \sum_i u^i \langle \alpha, u_i \rangle x \\ &= \sum_i \langle \alpha, u^i \rangle \langle \alpha, u_i \rangle x + x \left(\sum_i u^i \langle \alpha, u_i \rangle + u_i \langle \alpha, u^i \rangle \right). \end{aligned}$$

Define the generalized *Casimir operator* to be the following linear operator Ω on V:

$$\Omega := 2\nu^{-1}(\rho) + \sum_i u^i u_i + \Omega_0.$$

Example 2.3.4 (i) Let $\mathfrak{g} = \mathfrak{s}l_2$. Then we have

$$\Omega = h + h(1/2)h + 2fe = ef + fe + h(1/2)h,$$

i.e. $\Omega = \sum v^i v_i$ for a pair $\{v^i\}$ and $\{v_i\}$ of dual bases of $\mathfrak{s}l_2$. This is a general fact for a finite dimensional simple Lie algebra.

(ii) Let $\mathfrak{g} = \widehat{\mathfrak{s}l}_2$. We can take a pair of dual bases u^i and u_i of $\mathfrak{h}$ as follows

$$\{\alpha_1^\vee, c, d\} \quad \text{and} \quad \{(1/2)\alpha_1^\vee, d, c\},$$

and

$$2\nu^{-1}(\rho) = \alpha_1^\vee + 4d.$$

Finally,

$$\Omega_0 = \sum_{k=1}^{+\infty}(t^{-k}h)(t^k h) + 2\sum_{k=0}^{+\infty}(t^{-k}f)(t^k e) + 2\sum_{k=1}^{+\infty}(t^{-k}e)(t^k f).$$

For the purposes of he following theorem consider root space decomposition of $U(\mathfrak{g})$:

$$U(\mathfrak{g}) = \bigoplus_{\beta\in Q} U_\beta,$$

where

$$U_\beta = \{u \in U(\mathfrak{g}) | [h, u] = \langle\beta, h\rangle u \text{ for all } h \in \mathfrak{h}\}.$$

Set

$$U'_\beta = U_\beta \cap U(\mathfrak{g}'),$$

so that $U(\mathfrak{g}') = \bigoplus_{\beta\in Q} U'_\beta$.

Theorem 2.3.5 *Let $\mathfrak{g}$ be symmetrizable.*

(i) *If V be a restricted $\mathfrak{g}'$-module and $u \in U'_\alpha$ then*

$$[\Omega_0, u] = -u\big(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)\big). \tag{2.21}$$

(ii) *If V is a restricted $\mathfrak{g}$-module then Ω commutes with the action of $\mathfrak{g}$ on V.*

Proof Note that elements of $\mathfrak{h}$ commute with Ω since Ω is of weight 0. Now (ii) follows from (i) and (2.20). Next, note that if (i) holds for $u \in U'_\alpha$ and $u_1 \in U'_\beta$, then it also holds for $uu_1 \in U'_{\alpha+\beta}$:

$$\begin{aligned}[\Omega_0, uu_1] &= [\Omega_0, u]u_1 + u[\Omega_0, u_1] \\ &= -u\big(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha)\big)u_1 \\ &\quad -uu_1\big(2(\rho|\beta) + (\beta|\beta) + 2\nu^{-1}(\beta)\big) \\ &= -uu_1\big(2(\rho|\alpha) + (\alpha|\alpha) + 2\nu^{-1}(\alpha) \\ &\quad +2(\alpha|\beta) + 2(\rho|\beta) + (\beta|\beta) + 2\nu^{-1}(\beta)\big) \\ &= -uu_1\big(2(\rho|\alpha+\beta) + (\alpha+\beta|\alpha+\beta) + 2\nu^{-1}(\alpha+\beta)\big).\end{aligned}$$

Since the e_{α_i}'s and $e_{-\alpha_i}$'s generate $\mathfrak{g}'$, it suffices to check (2.21) for $u = e_{\alpha_i}$ and $e_{-\alpha_i}$. We explain the calculation for e_{α_i}, the case of $e_{-\alpha_i}$ being similar. We have

$$\begin{aligned}[\Omega_0, e_{\alpha_i}] &= 2\sum_{\alpha\in\Delta_+}\sum_s([e^{(s)}_{-\alpha}, e_{\alpha_i}]e^{(s)}_\alpha + e^{(s)}_{-\alpha}[e^{(s)}_\alpha, e_{\alpha_i}]) \\ &= 2[e_{-\alpha_i}, e_{\alpha_i}]e_{\alpha_i} + 2\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_s([e^{(s)}_{-\alpha}, e_{\alpha_i}]e^{(s)}_\alpha + e^{(s)}_{-\alpha}[e^{(s)}_\alpha, e_{\alpha_i}]).\end{aligned}$$

Note using Theorem 2.2.3(v) that

$$2[e_{-\alpha_i}, e_{\alpha_i}]e_{\alpha_i} = -2\nu^{-1}(\alpha_i)e_{\alpha_i} = -2(\alpha_i|\alpha_i)e_{\alpha_i} - 2e_{\alpha_i}\nu^{-1}(\alpha_i),$$

which is the RHS of (2.21) for $u = e_{\alpha_i}$. So it remains to prove that

$$\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_s([e^{(s)}_{-\alpha}, e_{\alpha_i}]e^{(s)}_\alpha + e^{(s)}_{-\alpha}[e^{(s)}_\alpha, e_{\alpha_i}]) = 0. \tag{2.22}$$

Applying (2.17) to $z = e_{\alpha_i}$, we get

$$\begin{aligned}&\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_s([e^{(s)}_{-\alpha}, e_{\alpha_i}]e^{(s)}_\alpha + e^{(s)}_{-\alpha}[e^{(s)}_\alpha, e_{\alpha_i}]) \\ = &\sum_{\alpha\in\Delta_+\backslash\{\alpha_i\}}\sum_s([e^{(s)}_{-\alpha}, e_{\alpha_i}]e^{(s)}_\alpha - \sum_{\alpha\in\Delta\backslash\{\alpha_i\}}\sum_t[e^{(t)}_{-\alpha-\alpha_i}, e_{\alpha_i}]e^{(t)}_{\alpha+\alpha_i}.\end{aligned}$$

If $\alpha + \alpha_i \notin \Delta$, the last term is interpreted as zero. If $\alpha - \alpha_i \notin \Delta$, then $[e^{(s)}_{-\alpha}, e_{\alpha_i}] = 0$. Thus we may assume $\alpha = \beta + \alpha_i$ in the first term with $\beta \in \Delta_+$ in view of Lemma 1.4.2, which makes that term equal to $\sum_{\beta\in\Delta\backslash\{\alpha_i\}}\sum_t[e^{(t)}_{-\beta-\alpha_i}, e_{\alpha_i}]e^{(t)}_{\beta+\alpha_i}$, which completes the proof of (2.22). □

Corollary 2.3.6 *If in the assumptions of Theorem 2.3.5, $v \in V$ is a high weight vector of weight Λ then*

$$\Omega(v) = (\Lambda + 2\rho|\Lambda)v.$$

If, additionally, v generates V, then

$$\Omega = (\Lambda + 2\rho|\Lambda)I_V.$$

Proof The second statement follows from the first and the theorem. The first statement is a consequence of the definition of Ω and (2.19). □

3

Integrable representations of $\mathfrak{g}$ and the Weyl group

3.1 Integrable modules

Let

$$\mathfrak{g}_{(i)} = \mathbb{C}e_i + \mathbb{C}\alpha_i^\vee + \mathbb{C}f_i.$$

It is clear that $\mathfrak{g}_{(i)}$ is isomorphic to $\mathfrak{sl}_2$ with standard basis.

Lemma 3.1.1 (Serre Relations) *If $i \neq j$ then*

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0. \tag{3.1}$$

Proof We prove the second equality, the first then follows by application of ω. Let $v = f_j$, $\theta_{ij} = (\operatorname{ad} f_i)^{1-a_{ij}} f_j$. We consider $\mathfrak{g}$ as a $\mathfrak{g}_{(i)}$-module via adjoint action. We have $e_i v = 0$ and $\alpha_i^\vee v = -a_{ij} v$. So, by representation theory of $\mathfrak{sl}_2$,

$$e_i\theta_{ij} = (1 - a_{ij})(-a_{ij} - (1 - a_{ij}) + 1)(\operatorname{ad} f_i)^{-a_{ij}} f_j = 0 \qquad (i \neq j).$$

It is also clear from relations that $e_k\theta_{ij} = 0$ if $k \neq i, j$ or if $k = j$ and $a_{ij} \neq 0$. Finally, if $k = j$ and $a_{ij} = 0$, then

$$e_j\theta_{ij} = [e_i, [f_i, f_j]] = [f_i, \alpha_j^\vee] = a_{ji} f_i = 0.$$

It remains to apply Lemma 1.4.5. □

Let V be a $\mathfrak{g}$-module and $x \in \mathfrak{g}$. Then x is *locally nilpotent* on V if for every $v \in V$ there is N such that $x^N v = 0$.

Lemma 3.1.2 *Let $\mathfrak{g}$ be a Lie algebra, V be a $\mathfrak{g}$-module, and $x \in \mathfrak{g}$.*

(i) *If $y_1, y_2, \dots$ generate $\mathfrak{g}$ and $(\operatorname{ad} x)^{N_i} y_i = 0$, $i = 1, 2, \dots,$ then $\operatorname{ad} x$ is locally nilpotent on $\mathfrak{g}$.*

(ii) *If* $v_1, v_2, \dots$ *generate* V *as* $\mathfrak{g}$*-module,* $\operatorname{ad} x$ *is locally nilpotent on* $\mathfrak{g}$*, and* $x^{N_i} v_i = 0,\ i = 1, 2, \dots,$ *then* x *is locally nilpotent on* V*.*

Proof Since $\operatorname{ad} x$ is a derivation, we have

$$(\operatorname{ad} x)^k [y, z] = \sum_{i=0}^{k} \binom{k}{i} [(\operatorname{ad} x)^i y, (\operatorname{ad} x)^{k-i} z].$$

This yields (i) by induction on the length of commutators in the y_i's.

(ii) follows from the formula

$$x^k a = \sum_{i=0}^{k} \binom{k}{i} ((\operatorname{ad} x)^i a) x^{k-i}, \tag{3.2}$$

which holds in any associative algebra. □

Lemma 3.1.3 *Operators* $\operatorname{ad} e_i$ *and* $\operatorname{ad} f_i$ *are locally nilpotent on* $\mathfrak{g}$*.*

Proof Follows from the defining relations, Serre relations, and Lemma 3.1.2(i). □

A $\mathfrak{g}$-module V is called $\mathfrak{h}$-*diagonalizable* if

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where the *weight space* V_λ is defined to be

$$V_\lambda = \{v \in V | hv = \lambda(h) v \text{ for all } h \in \mathfrak{h}\}.$$

If $V_\lambda \neq 0$ we call λ a *weight* of V, and $\dim V_\lambda$ the *multiplicity of the weight* λ denoted $\operatorname{mult}_V \lambda$. $\mathfrak{h}'$-diagonalizable $\mathfrak{g}'$-modules are defined similarly.

A $\mathfrak{g}$ (resp. $\mathfrak{g}'$)-module V is called *integrable* if it is $\mathfrak{h}$ (resp. $\mathfrak{h}'$)-diagonalizable and all e_i, f_i act locally nilpotently on V. For example the $\mathfrak{g}$-module is integrable.

Proposition 3.1.4 *Let* V *be an integrable* $\mathfrak{g}$*-module. As a* $\mathfrak{g}_{(i)}$*-module,* V *decomposes into a direct sum of finite dimensional irreducible* $\mathfrak{h}$*-invariant modules.*

Proof For $v \in V_\lambda$ we have

$$e_i f_i^k v = k(1 - k + \langle \lambda, \alpha_i^\vee \rangle) f_i^{k-1} v + f_i^k e_i v.$$

It follows that the subspace

$$U := \sum_{k,m \geq 0} \mathbb{C} f_i^k e_i^m v$$

is $(\mathfrak{g}_{(i)} + \mathfrak{h})$-invariant. Since e_i and f_i are locally nilpotent on V, $\dim U < \infty$. By Weyl's Complete Reducibility Theorem, U is a direct sum of irreducible $\mathfrak{h}$-invariant $\mathfrak{g}_{(i)}$-submodules (for $\mathfrak{h}$-invariance use the fact that $f_i^k e_i^m v$ and $f_i^{k'} e_i^{m'} v$ are of the same $\alpha_i^\vee$-weight if and only if they are of the same $\mathfrak{h}$-weight). It follows that each $v \in V$ lies in a direct sum of finite dimensional $\mathfrak{h}$-invariant irreducible $\mathfrak{g}_{(i)}$-modules, which implies the proposition. □

Proposition 3.1.5 *Let V be an integrable $\mathfrak{g}$-module, $\lambda \in \mathfrak{h}^*$ be a weight of V, and α_i a simple root of $\mathfrak{g}$. Denote by M the set of all $t \in \mathbb{Z}$ such that $\lambda + t\alpha_i$ is a weight of V, and let $m_t := \operatorname{mult}_V(\lambda + t\alpha_i)$. Then:*

(i) *M is a closed interval $[-p, q]$ of integers, where both p and q are either non-negative integers or ∞; $p - q = \langle \lambda, \alpha_i^\vee \rangle$ when both p and q are finite; if $\operatorname{mult}_V \lambda < \infty$ then p and q are finite.*

(ii) *The map $e_i : V_{\lambda + t\alpha_i} \to V_{\lambda + (t+1)\alpha_i}$ is an embedding for $t \in [-p, -\langle \lambda, \alpha_i^\vee \rangle / 2)$; in particular, the function $t \mapsto m_t$ is increasing on this interval.*

(iii) *The function $t \mapsto m_t$ is symmetric with respect to $t = -\langle \lambda, \alpha_i^\vee \rangle / 2$.*

(iv) *If λ and $\lambda + \alpha_i$ are weights then $e_i(V_\lambda) \neq 0$.*

(v) *If $\lambda + \alpha_i$ (resp. $\lambda - \alpha_i$) is not a weight, then $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ (resp. $\langle \lambda, \alpha_i^\vee \rangle \leq 0$).*

(vi) *$\lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i$ is also a weight of V and*

$$\operatorname{mult}_V(\lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i) = \operatorname{mult}_V \lambda.$$

Proof Set $U := \sum_{k \in \mathbb{Z}} V_{\lambda + k\alpha_i}$. This is a $(\mathfrak{g}_{(i)} + \mathfrak{h})$-module, which in view of Proposition 3.1.4 is a direct sum of finite dimensional $\mathfrak{h}$-invariant irreducible $\mathfrak{g}_{(i)}$-modules. Let $p := -\inf M$ and $q := \sup M$. Then $p, q \in \mathbb{Z}_+$ since $0 \in M$. Now everything follows from representation theory of $\mathfrak{sl}_2$ using the fact $\langle \lambda + t\alpha_i, \alpha_i^\vee \rangle = 0$ for $t = -\langle \lambda, \alpha_i^\vee \rangle / 2$. □

3.2 Weyl group

For each $i = 1, \dots, n$ define the *fundamental reflection* r_i of $\mathfrak{h}^*$ by the formula

$$r_i(\lambda) = \lambda - \langle \lambda, \alpha_i^\vee \rangle \alpha_i \qquad (\lambda \in \mathfrak{h}^*).$$

It is clear that r_i is a reflection with respect to the hyperplane

$$T_i = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle = 0\}.$$

The subgroup $W = W(A)$ of $GL(\mathfrak{h}^*)$ generated by all fundamental reflections is called the *Weyl group* of $\mathfrak{g}$. The action r_i on $\mathfrak{h}^*$ induces the dual fundamental reflection $r_i^\vee$ on $\mathfrak{h}$. Hence the Weyl groups of dual Kac-Moody algebras are contragredient linear groups which allows us to identify them. We will always do this and write r_i for $r_i^\vee$.

Proposition 3.2.1

(i) *Let V be an integrable $\mathfrak{g}$-module. Then* $\text{mult}\,_V\lambda = \text{mult}\,_V w(\lambda)$ *for any $\lambda \in \mathfrak{h}^*$ and $w \in W$. In particular, the set of weights of V is W-invariant.*

(ii) *The root system Δ is W-invariant and* $\text{mult}\,\alpha = \text{mult}\,w(\alpha)$ *for all $\alpha \in \Delta, w \in W$.*

Proof Follows from Proposition 3.1.5. □

Lemma 3.2.2 *If $\alpha \in \Delta_+$ and $r_i(\alpha) < 0$ then $\alpha = \alpha_i$. In particular, $\Delta_+ \setminus \{\alpha_i\}$ is invariant with respect to r_i.*

Proof Follows from Lemma 1.4.2. □

If a is a locally nilpotent operator on a vector space V, and b is another operator on V such that $(\operatorname{ad} a)^n b = 0$ for some N, then

$$(\exp a)b(\exp -a) = (\exp(\operatorname{ad} a))(b). \tag{3.3}$$

Indeed, using (3.2), we get

$$\begin{aligned}(\sum_{i\geq 0}\frac{a^i}{i!})b(\sum_{j\geq 0}(-1)^j\frac{a^j}{j!}) &= \sum_{k\geq 0}\frac{1}{k!}\sum_{i+j=k}\frac{k!}{i!j!}(a^iba^j)\\ &= \sum_{k\geq 0}\frac{1}{k!}(\operatorname{ad} a)^k(b).\end{aligned}$$

Lemma 3.2.3 *Let π be an integrable representation of $\mathfrak{g}$ in V. For $i = 1, \dots, n$ set*

$$r_i^\pi := (\exp \pi(f_i))(\exp \pi(-e_i))(\exp \pi(f_i)).$$

Then

(i) $r_i^\pi(V_\lambda) = V_{r_i(\lambda)}$;

(ii) $r_i^{\mathrm{ad}} \in \mathrm{Aut}\,\mathfrak{g}$;

(iii) $r_i^{\mathrm{ad}}|_{\mathfrak{h}} = r_i$.

Proof Let $v \in V_\lambda$. Then

$$h(r_i^\pi(v)) = r_i^\pi(h(v)) = \langle \lambda, h\rangle r_i^\pi(v) \quad \text{if} \quad \langle \alpha_i, h\rangle = 0. \tag{3.4}$$

Next we prove that

$$\alpha_i^\vee(r_i^\pi(v)) = -\langle \lambda, \alpha_i^\vee\rangle r_i^\pi(v). \tag{3.5}$$

This follows from

$$(r_i^\pi)^{-1}\alpha_i^\vee r_i^\pi = -\alpha_i^\vee, \tag{3.6}$$

and, in view of (3.3), it is enough to check (3.6) holds for the adjoint representation of $\mathfrak{sl}_2$. Applying (3.3) one more time, we see that it is enough to check (3.6) for the natural 2-dimensional representation of $\mathfrak{sl}_2$. But in that representation we have

$$\exp f_i = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \exp(-e_i) = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad r_i^\pi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

which implies (3.6) easily.

Now, any $h \in \mathfrak{h}$ can be written in the form $h = h' + c\alpha_i^\vee$, where c is a constant and $\langle \alpha_i, h'\rangle = 0$. Then using (3.4) and (3.5), we have

$$h(r_i^\pi(v)) = (\langle \lambda, h'\rangle - \langle \lambda, c\alpha_i^\vee\rangle) r_i^\pi(v) = \langle \lambda, r_i(h)\rangle r_i^\pi(v) = \langle r_i(\lambda), h\rangle r_i^\pi(v),$$

which proves (i).

For (iii), take $h \in \mathfrak{h}$ and write it again in the form $h = h' + c\alpha_i^\vee$ as above. Then it is clear that $r_i^{\mathrm{ad}} h' = h'$, and we just have to prove that $r_i^{\mathrm{ad}}(\alpha_i^\vee) = -\alpha_i^\vee$. We have

$$\begin{aligned}
(\exp \operatorname{ad} f_i)(\alpha_i^\vee) &= \alpha_i^\vee + 2f_i; \\
(\exp \operatorname{ad}(-e_i))(\alpha_i^\vee + 2f_i) &= \alpha_i^\vee + 2e_i + 2f_i - 2\alpha_i^\vee - 2e_i \\
&= -\alpha_i^\vee + 2f_i; \\
(\exp \operatorname{ad} f_i)(-\alpha_i^\vee + 2f_i) &= -\alpha_i^\vee - 2f_i + 2f_i \\
&= -\alpha_i^\vee.
\end{aligned}$$

(ii) follows from (3.3) applied to the adjoint representation:

$$\begin{aligned} r_i^{\mathrm{ad}}[x,y] &= (\exp \operatorname{ad} f_i)(\exp \operatorname{ad}(-e_i))(\exp \operatorname{ad}(f_i))(\operatorname{ad} x)(y) \\ &= (\exp \operatorname{ad} f_i)(\exp \operatorname{ad}(-e_i))(\exp \operatorname{ad}(f_i))(\operatorname{ad} x)(\exp \operatorname{ad}(-f_i))(\exp \operatorname{ad} e_i)(\exp \operatorname{ad}(-f_i)) \\ &\quad \times(\exp \operatorname{ad} f_i)(\exp \operatorname{ad}(-e_i))(\exp \operatorname{ad}(f_i))(y) \\ &= r_i^{\mathrm{ad}}(x)(r_i^{\pi}(y)) \\ &= [r_i^{\mathrm{ad}}(x), r_i^{\pi}(y)]. \end{aligned}$$

□

Proposition 3.2.4 *The bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ is W-invariant.*

Proof Note that $|r_i(\alpha_i)|^2 = |-\alpha_i|^2 = |\alpha_i|^2$. Now let $\Lambda, \Phi \in \mathfrak{h}^*$ and write $\Lambda = c\alpha_i + \lambda, \Phi = d\alpha_i + \varphi$ where $(\lambda|\alpha_i) = (\varphi|\alpha_i) = 0$, and c, d are constants. Then $r_i(\Lambda) = \lambda - c\alpha_i, r_i(\Phi) = \varphi - d\alpha_i$, so

$$(r_i(\Lambda)|r_i(\Phi)) = (\lambda - c\alpha_i|\varphi - d\alpha_i) = (\lambda, \varphi) + (c\alpha_i|d\alpha_i) = (\Lambda|\Phi).$$

□

3.3 Weyl group as a Coxeter group

Lemma 3.3.1 *If α_i is a simple root and $r_{i_1} \dots r_{i_t}(\alpha_i) < 0$ then there exists s such that $1 \le s \le t$ and*

$$r_{i_1} \dots r_{i_t} r_i = r_{i_1} \dots \widehat{r_{i_s}} \dots r_{i_t}.$$

Proof Set $\beta_k = r_{i_{k+1}} \dots r_{i_t}(\alpha_i)$ for $k < t$ and $\beta_t = \alpha_i$. Then $\beta_t > 0$ and $\beta_0 < 0$. Hence for some s we have $\beta_{s-1} < 0$ and $\beta_s > 0$. But $\beta_{s-1} = r_{i_s}\beta_s$, so by Lemma 3.2.2, $\beta_s = \alpha_{i_s}$, and we get

$$\alpha_{i_s} = w(\alpha_i), \quad \text{where} \quad w = r_{i_{s+1}} \dots r_{i_t}. \tag{3.7}$$

By Lemma 3.2.3, $w = \tilde{w}|_{\mathfrak{h}}$ for some $\tilde{w}$ from the subgroup of $\operatorname{Aut}\mathfrak{g}$ generated by the r_i^{ad}. Applying $\tilde{w}$ to both sides of the equation $[\mathfrak{g}_{\alpha_i}, \mathfrak{g}_{-\alpha_i}] = \mathbb{C}\alpha_i^\vee$, we see that $\mathbb{C}w(\alpha_i^\vee) = \mathbb{C}\alpha_{i_s}^\vee$. Since $\langle w(\alpha_i), w(\alpha_i^\vee)\rangle = \langle \alpha_i, \alpha_i^\vee\rangle = 2$, we now conclude that

$$w(\alpha_i^\vee) = \alpha_{i_s}^\vee \tag{3.8}$$

It now follows that $r_{i_s} = w r_i w^{-1}$:

$$w r_i w^{-1}(\lambda) = w(w^{-1}(\lambda) - \langle w^{-1}(\lambda), \alpha_i^\vee\rangle\alpha_i) = \lambda - \langle \lambda, \alpha_{i_s}^\vee\rangle\alpha_{i_s} = r_{i_s}(\lambda).$$

It remains to multiply both sides of $r_{i_s} = wr_iw^{-1}$ by $r_{i_1} \dots r_{i_{s-1}}$ on the left and by $r_{i_{s+1}} \dots r_{i_t} r_i$ on the right. $\square$

Decomposition $w = r_{i_1} \dots r_{i_s}$ is called *reduced* if s is minimal among all presentations of w as a product of simple reflections r_i. Then s is called the *length* of w and is denoted $\ell(w)$. Note that $\det r_i = -1$, so

$$\det w = (-1)^{\ell(w)} \qquad (w \in W). \tag{3.9}$$

Lemma 3.3.2 *Let $w = r_{i_1} \dots r_{i_t} \in W$ be a reduced decomposition and α_i be a simple root. Then*

(i) $\ell(wr_i) < \ell(w)$ *if and only if* $w(\alpha_i) < 0$*;*

(ii) (Exchange Condition) *If $\ell(wr_i) < \ell(w)$ then there exists s such that $1 \le s \le t$ and*

$$r_{i_s} r_{i_{s+1}} \dots r_{i_t} = r_{i_{s+1}} \dots r_{i_t} r_i$$

Proof By Lemma 3.3.1, $w(\alpha_i) < 0$ implies $\ell(wr_i) < \ell(w)$. Now, if $w(\alpha_i) > 0$, then $wr_i(\alpha_i) < 0$ and it follows that $\ell(w) = \ell(wr_ir_i) < \ell(wr_i)$, completing the proof of (i).

(ii) If $\ell(wr_i) < \ell(w)$ then (i) implies $w(\alpha_i) < 0$, and we deduce the required Exchange Condition from Lemma 3.3.1 by multiplying it with $r_{i_{s-1}} \dots r_{i_1}$ on the left and r_i on the right. $\square$

Lemma 3.3.3 $\ell(w)$ *equals the number of roots $\alpha > 0$ such that $w(\alpha) < 0$.*

Proof Denote

$$n(w) := |\{\alpha \in \Delta_+ \mid w(\alpha) < 0\}.$$

It follows from Lemma 3.2.2 that $n(wr_i) = n(w) \pm 1$, whence $n(w) \le \ell(w)$.

We now apply induction on $\ell(w)$ to prove that $\ell(w) = n(w)$. If $\ell(w) = 0$ then $w = 1$ (by convention), and clearly $n(w) = 0$. Assume that $\ell(w) = t > 0$, and $w = r_{i_1} \dots r_{i_{t-1}} r_{i_t}$. Denote $w' = r_{i_1} \dots r_{i_{t-1}}$. By induction, $n(w') = t - 1$. Let $\beta_1, \dots, \beta_{t-1}$ be the positive roots which are sent to negative roots by w'. By Lemma 3.3.2(i), $w'(\alpha_{i_t}) > 0$, whence $w(\alpha_{i_t}) < 0$. It follows from Lemma 3.2.2 that $r_{i_t}(\beta_1), \dots, r_{i_t}(\beta_{t-1}), \alpha_{i_t}$ are distinct positive roots which are mapped to negative roots by w, so $\mathfrak{n}(w) \ge \ell(w)$. $\square$

Lemma 3.3.4 (Deletion Condition) *Let* $w = r_{i_1} \dots r_{i_s}$. *Suppose* $\ell(w) < s$. *Then there exist* $1 \le j < k \le s$ *such that*

$$w = r_{i_1} \dots \widehat{r_{i_j}} \dots \widehat{r_{i_k}} \dots r_{i_s}.$$

Proof Since $\ell(w) < s$ there exists $2 \le k \le s$ such that

$$\ell(r_{i_1} \dots r_{i_k}) < \ell(r_{i_1} \dots r_{i_{k-1}}) = k - 1$$

. Then by Lemmas 3.3.2(i) and 3.3.1,

$$r_{i_1} \dots r_{i_k} = r_{i_1} \dots \widehat{r_{i_j}} \dots r_{i_{k-1}}$$

for some $1 \le j < k$. □

Now for $1 \le i \ne j \le n$ define

$$m_{ij} := \begin{cases} 2 & \text{if } a_{ij}a_{ji} = 0, \\ 3 & \text{if } a_{ij}a_{ji} = 1, \\ 4 & \text{if } a_{ij}a_{ji} = 2, \\ 5 & \text{if } a_{ij}a_{ji} = 3, \\ \infty & \text{if } a_{ij}a_{ji} \ge 4. \end{cases}$$

Lemma 3.3.5 *Let* $1 \le i \ne j \le n$. *Then the order of* $(r_i r_j)$ *is* m_{ij}.

Proof The subspace $\mathbb{R}\alpha_i + \mathbb{R}\alpha_j$ is invariant with respect to r_i and r_j, and we can make all calculations in this 2-dimensional space. The matrices of r_i and r_j in the basis α_i, α_j are $\begin{pmatrix} -1 & -a_{ij} \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -a_{ji} & -1 \end{pmatrix}$, respectively. So the matrix of $r_i r_j$ is $\begin{pmatrix} -1 + a_{ij}a_{ji} & a_{ij} \\ -a_{ji} & -1 \end{pmatrix}$. The characteristic polynomial of this matrix is $\lambda^2 + (2 - a_{ij}a_{ji})\lambda + 1$, and now the result is an elementary calculation. □

Proposition 3.3.6 *W is generated by $r_1, \dots, r_n$ subject only to the Coxeter relations*

$$r_i^2 = 1 \qquad (1 \le i \le n), \tag{3.10}$$

$$(r_i r_j)^{m_{ij}} = 1 \qquad (1 \le i \ne j \le n), \tag{3.11}$$

where w^∞ *is interpreted as* 1. *So* W *is a Coxeter group.*

Proof This is a general fact. All we need is Deletion Condition. We need to show that every relation

$$r_1 \dots r_{i_s} = 1$$

in W is a consequence of (3.10) and (3.11). We have $\det r_i = -1$ for all i, so $s = 2q$. We apply induction on q. If $q = 1$ the relation looks like $s_{i_1} s_{i_2} = 1$. Hence $s_{i_2} = s_{i_1}^{-1} = s_{i_1}$. So our relation is $s_{i_1}^2 = 1$, which is one of (3.10).

For inductive step, rewrite the given relation as follows:

$$r_{i_1} \dots r_{i_q} r_{i_{q+1}} = r_{i_{2q}} \dots r_{i_{q+2}}. \tag{3.12}$$

Then $\ell(r_{i_1} \dots r_{i_q} r_{i_{q+1}}) < q + 1$, so by the Deletion Condition,

$$r_{i_1} \dots r_{i_q} r_{i_{q+1}} = r_{i_1} \dots \widehat{r_{i_j}} \dots \widehat{r_{i_k}} \dots r_{i_{q+1}} \tag{3.13}$$

for some $1 \le j < k \le q + 1$. Now, unless $j = 1$ and $k = q + 1$, this is a consequence of a relation with fewer than $2q$ terms—for example, if $j > 1$, (3.13) is equivalent to

$$r_{i_2} \dots r_{i_q} r_{i_{q+1}} = r_{i_2} \dots \widehat{r_{i_j}} \dots \widehat{r_{i_k}} \dots r_{i_{q+1}}.$$

So, by induction, (3.13) can be deduced from the defining relations. The relation

$$r_{i_1} \dots \widehat{r_{i_j}} \dots \widehat{r_{i_k}} \dots r_{i_{q+1}} = r_{i_{2q}} \dots r_{i_{q+2}}$$

has $2q - 2$ terms, so is also a consequence of the defining relations. Therefore (3.12) is a consequence of the defining relations, unless $j = 1$ and $k = q + 1$.

In the exceptional case (3.13) is

$$r_{i_1} \dots r_{i_q} r_{i_{q+1}} = r_{i_2} \dots r_{i_q},$$

or

$$r_{i_1} \dots r_{i_q} = r_{i_2} \dots r_{i_{q+1}}. \tag{3.14}$$

Now we write (3.12) in the alternative form

$$r_{i_2} \dots r_{i_{2q}} r_{i_1} = 1. \tag{3.15}$$

In exactly the same way this relation will be a consequence of the defining relations unless

$$r_{i_2} \dots r_{i_{q+1}} = r_{i_3} \dots r_{i_{q+2}}. \tag{3.16}$$

If this relation is a consequence of the defining relations then (3.12) is also a consequence of the defining relations by the above argument, and we are done. Now, (3.12) is equivalent to

$$r_{i_3} r_{i_2} r_{i_3} \dots r_{i_q} r_{i_{q+1}} r_{i_{q+2}} r_{i_{q+1}} \dots r_{i_4} = 1, \tag{3.17}$$

and this will be a consequence of the defining relations unless

$$r_{i_3} r_{i_2} r_{i_3} \dots r_{i_q} = r_{i_2} r_{i_3} \dots r_{i_q} r_{i_{q+1}},$$

We may therefore assume that this is true. But we must also have (3.17). So $r_{i_1} = r_{i_3}$. Hence the given relation will be a consequence of the defining relations unless $r_{i_1} = r_{i_3}$. However, an equivalent forms of the given relation are also $r_{i_2} \dots r_{i_{2q}} r_{i_1} = 1$, $r_{i_3} \dots r_{i_{2q}} r_{i_1} r_{i_2} = 1$, etc. Thus this relation will be a consequence of the defining relations unless $r_{i_1} = r_{i_3} = \dots = r_{i_{2q-1}}$ and $r_{i_2} = r_{i_4} = \dots = r_{i_{2q}}$. Thus we may assume that the given relation has form $(r_{i_1} r_{i_2})^q = 1$. Then $m_{i_1 i_2}$ divides q, and the relation is a consequence of the Coxeter relation (3.11). □

3.4 Geometric properties of Weyl groups

Let $(\mathfrak{h}_\mathbb{R}, \Pi, \Pi^\vee)$ be a realization of A over $\mathbb{R}$, so that

$$(\mathfrak{h}, \Pi, \Pi^\vee) = (\mathbb{C} \otimes_\mathbb{R} \mathfrak{h}_\mathbb{R}, \Pi, \Pi^\vee).$$

Note that $\mathfrak{h}_\mathbb{R}$ is W-invariant since $\mathbb{Q}^\vee \subset \mathfrak{h}_\mathbb{R}$. The set

$$C = \{h \in \mathfrak{h}_\mathbb{R} \mid \langle \alpha_i, h \rangle \geq 0 \text{ for } i = 1, \dots, n\}$$

is called the *fundamental chamber*, the sets of the form $w(C)$ are called *chambers*, and their union

$$X := \bigcup_{w \in W} w(C)$$

is called the *Tits cone*. There are corresponding dual objects $C^\vee, X^\vee$, etc. in $\mathfrak{h}_\mathbb{R}^*$.

Proposition 3.4.1

(i) *For $h \in C$, the group $W_h := \{w \in W \mid w(h) = h\}$ is generated by the fundamental reflections contained in it.*

(ii) *The fundamental chamber is the fundamental domain for the action of W on X, i.e. every W-orbit intersects C in exactly one point. In particular, W acts regularly on the set of chambers.*

(iii) *$X = \{h \in \mathfrak{h}_\mathbb{R} \mid \langle \alpha, h \rangle < 0$ for a finite number of $\alpha \in \Delta_+\}$. In particular X is a convex cone.*

(iv) $C = \{h \in \mathfrak{h}_{\mathbb{R}} \mid h - w(h) = \sum_i c_i \alpha_i^\vee,$ *where* $c_i \geq 0,$ *for any* $w \in W\}$.

(v) *The following conditions are equivalent:*

(a) $|W| < \infty$;
(b) $X = \mathfrak{h}_{\mathbb{R}}$;
(c) $|\Delta| < \infty$;
(d) $|\Delta^\vee| < \infty$;

Proof Take $w \in W$ and let $w = r_{i_1} \dots r_{i_s}$ be a reduced decomposition. Take $h \in C$ and assume that $h' = w(h) \in C$. We have $\langle \alpha_{i_s}, h \rangle \geq 0$, hence $\langle w(\alpha_{i_s}), w(h) \rangle = \langle w(\alpha_{i_s}), h' \rangle \geq 0$. It follows from Lemma 3.3.2(i) that $w(\alpha_{i_s}) < 0$, hence $\langle w(\alpha_{i_s}), h' \rangle \leq 0$, and $\langle w(\alpha_{i_s}), h' \rangle = 0$, whence $\langle \alpha_{i_s}, h \rangle = 0$. Hence $r_{i_s}(h) = h$. Now for the proof of (i) and (ii) it suffices to apply induction on $\ell(w)$.

(iii) Set $X' := \{h \in \mathfrak{h}_{\mathbb{R}} \mid \langle \alpha, h \rangle < 0$ for a finite number of $\alpha \in \Delta_+\}$. Let $h \in X'$ and $w \in W$. Then $\langle \alpha, w(h) \rangle = \langle w^{-1}\alpha, h \rangle$. Only finitely many positive α's are sent to negatives by w^{-1}, see Lemma 3.3.3. So X' is W-invariant, and clearly $C \subset X'$. Therefore $X \subset X'$. To prove the converese embedding, take $h \in X'$ and set $M_h := \{\alpha \in \Delta_+ \mid \langle \alpha, h \rangle < 0\}$. By definition M_h is finite. If $M_h \neq \varnothing$, then some simple root $\alpha_i \in M_h$. But then it follows from Lemma 3.2.2 that $|M_{r_i(h)}| < |M_h|$. Now induction on $|M_h|$ completes the proof of (iii).

(iv) $\supset$ is clear. The converse embedding is proved by induction on $s = \ell(w)$. For $s = 0$ the result is clear and for $s = 1$ it is equivalent to the definition of C. Let $s > 1$ and $w = r_{i_1} \dots r_{i_s}$. We have

$$h - w(h) = (h - r_{i_1} \dots r_{i_{s-1}}(h)) + r_{i_1} \dots r_{i_{s-1}}(h - r_{i_s}(h)).$$

It follows from (the dual version of) Lemma 3.3.2(i) that $r_{i_1} \dots r_{i_{s-1}}(\alpha_{i_s}^\vee) \in Q_+^\vee$, which implies that the second summand is in $Q_+^\vee$. The first summand is there too by inductive assumption.

(v) (a) $\Rightarrow$ (b). Let $h \in \mathfrak{h}_{\mathbb{R}}$, and choose an element h' from the (finite) orbit $W \cdot h$ for which $\operatorname{ht}(h' - h)$ is maximal. Then $h' \in C$, whence $h \in X$.

(b) $\Rightarrow$ (c) Take h in the interior of C. Then $\langle \alpha, -h \rangle < 0$ for all $\alpha \in \Delta_+$, and it remains to apply (iii).

(c) $\Rightarrow$ (a) It suffices to prove that the action of W on the roots is faithful. Assume that $w(\alpha) = \alpha$ for all $\alpha \in \Delta$, and $w = r_{i_1} \dots r_{i_s}$ be a reduced decomposition. But then $w(\alpha_{i_s}) < 0$ by Lemma 3.3.2(i).

(d) $\Leftrightarrow$ (a) is similar to (c) $\Leftrightarrow$ (a), but using dual root system. $\square$

Example 3.4.2 (i) Let $\mathfrak{g} = \mathfrak{sl}_{n+1}$. Then r_i acts on $\varepsilon_1, \ldots, \varepsilon_{n+1}$ by swapping ε_i and ε_{i+1}, from which it follows that $W \cong S_{n+1}$. Introduce $\Lambda_1, \ldots, \Lambda_n \in \mathfrak{h}^*$ as the dual basis to $\alpha_1^\vee, \ldots, \alpha_n^\vee$:

$$\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{ij} \qquad (1 \leq i, j \leq n).$$

Then

$$C^\vee = \mathbb{R}_{\geq 0}\Lambda_1 \oplus \cdots \oplus \mathbb{R}_{\geq 0}\Lambda_n.$$

and $X^\vee = \mathfrak{h}_{\mathbb{R}}^*$.

(ii) Let $\mathfrak{g} = \widehat{\mathfrak{sl}}_2$. Then $\mathfrak{h}_{\mathbb{R}}^* = \mathbb{R}\alpha_1 \oplus \mathbb{R}\delta \oplus \mathbb{R}\Lambda_0$ and the sum $(\mathbb{R}\alpha_1) \oplus (\mathbb{R}\delta \oplus \mathbb{R}\Lambda_0)$ is orthogonal. Moreover,

$$\begin{aligned} r_0 : \alpha_1 &\mapsto -\alpha_1 + 2\delta, \quad \delta \mapsto \delta, \quad \Lambda_0 \mapsto \alpha_1 - \delta + \Lambda_0; \\ r_1 : \alpha_1 &\mapsto -\alpha_1, \qquad\quad \delta \mapsto \delta, \qquad\quad \Lambda_0 \mapsto \Lambda_0, \end{aligned}$$

whence

$$r_0 r_1(\lambda\alpha_1 + \mu\delta + \nu\Lambda_0) = (\lambda + \nu)\alpha_1 + (\mu - 2\lambda - \nu)\delta + \nu\Lambda_0. \tag{3.18}$$

Consider the affine subspace

$$\mathfrak{h}_1^* = \{\lambda \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle \lambda, c \rangle = 1\} \subset \mathfrak{h}_{\mathbb{R}}^*,$$

invariant with respect to the action of W. So W acts on $\mathfrak{h}_1^*$ with affine transformations. Elements of $\mathfrak{h}_1^*$ are of the form

$$\lambda\alpha_1 + \mu\delta + \Lambda_0 \qquad (\lambda, \mu \in \mathbb{R}).$$

Moreover, it is clear that r_0 and r_1 act trivially on δ. So the action of W on $\mathfrak{h}_1^*$ factors through to give an action of W on $\mathfrak{h}_1^*/\mathbb{R}\delta$ which can be identified with $\mathbb{R}\alpha_1$. We will denote the induced affine action of $w \in W$ on $\mathbb{R}\alpha_1$ via $\bar{w}$. An easy calculation gives:

$$\bar{r}_1 : \lambda\alpha_1 \mapsto -\lambda\alpha_1, \quad \bar{r}_0 : \lambda\alpha_1 \mapsto -\lambda\alpha_1 + \alpha_1,$$

whence

$$\bar{r}_0\bar{r}_1(\lambda\alpha_1) = \lambda\alpha_1 + \alpha_1$$

is a 'shift' by α_1. It follows that the image $\bar{W}$ of W is a semidirect product

$$\bar{W} = \mathbb{Z} \rtimes S_2.$$

In fact the map $w \mapsto \bar{w}$ is injective. This follows from the fact that every element of W can be written uniquely in the form $r_1^\varepsilon(r_0r_1)^k$ where $k \in \mathbb{Z}$ and $\varepsilon = 0$ or 1. Thus

$$W = \mathbb{Z} \rtimes S_2.$$

Next,

$$C = \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid 0 \leq \lambda \leq \frac{1}{2}\nu\}.$$

It follows from (3.18) that

$$\begin{aligned}(r_0r_1)^kC &= \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid \nu \geq 0,\ k\nu \leq \lambda \leq (k+\frac{1}{2})\nu\} \\ r_1(r_0r_1)^kC &= \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid \nu \geq 0,\ -(k+\frac{1}{2})\nu \leq \lambda \leq -k\nu\},\end{aligned}$$

whence

$$X = \{\lambda\alpha_1 + \mu\delta + \nu\Lambda_0 \mid \nu \geq 0\}.$$

In terms of the affine action, C gets identified with the *fundamental alcove*

$$C_{\mathrm{af}} = \{\lambda\alpha_1 \mid 0 \leq \lambda \leq \frac{1}{2}\},$$

which is the fundamental domain for the affine action of W on $\mathbb{R}\alpha_1$.

4

The Classification of Generalized Cartan Matrices

4.1 A trichotomy for indecomposable GCMs

Let $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$. We write

$$v \geq 0 \qquad \text{if all } v_i \geq 0$$

and

$$v > 0 \qquad \text{if all } v_i > 0.$$

Definition 4.1.1 A GCM A has *finite type* if the following three conditions hold:

(i) $\det A \neq 0$;
(ii) there exists $u > 0$ with $Au > 0$;
(iii) $Au \geq 0$ implies $u > 0$ or $u = 0$.

A GCM A has *affine type* if the following three conditions hold:

(i) corank $A = 1$ (i.e. rank $A = n - 1$);
(ii) there exists $u > 0$ with $Au = 0$;
(iii) $Au \geq 0$ implies $Au = 0$.

A GCM A has *indefinite type* if the following two conditions hold:

(i) there exists $u > 0$ with $Au < 0$;
(ii) $Au \geq 0$ and $u \geq 0$ imply $u = 0$.

Remark 4.1.2 Let $\gamma = u_1\alpha_1 + \cdots + u_n\alpha_n$ then the jth coordinate of Au equals $\langle \gamma, \alpha_j^\vee \rangle$.

Example 4.1.3 Let a, b be positive integers, and $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$. Then A is of finite (resp. affine, resp. indefinite) type if and only if $ab \leq 3$ (resp. $ab = 4$, resp. $ab > 4$).

We will prove that a GCM has exactly one of the three types above.

Lemma 4.1.4 *Let* $v^i = (v_{i1}, \ldots, v_{in}) \in \mathbb{R}^n$ *for* $i = 1, \ldots, m$. *Then there exist* $x_1, \ldots, x_n \in \mathbb{R}$ *with*

$$\sum_{j=1}^{n} v_{ij} x_j > 0 \qquad (i = 1, \ldots, m)$$

if and only if

$$\lambda_1 v^1 + \cdots + \lambda_m v^m = 0, \quad \lambda_1, \ldots, \lambda_m \geq 0$$

implies $\lambda_1 = \cdots = \lambda_m = 0$.

Proof Suppose there exists a column vector $x = (x_1, \ldots, x_n)^t$ such that $v^i x > 0$ for all i. Suppose $\lambda_1 v^1 + \cdots + \lambda_m v^m = 0$ with all $\lambda_i \geq 0$. Then

$$\lambda_1 v^1 x + \cdots + \lambda_m v^m x = 0.$$

This implies $\lambda_i = 0$ for all i.

Conversely, suppose $\lambda_1 v^1 + \cdots + \lambda_m v^m = 0$, $\lambda_i \geq 0$ imply $\lambda_i = 0$ for all i. Let

$$S := \left\{ \sum_{i=1}^{m} \lambda_i v^i \mid \lambda_i \geq 0,\ \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Define $f : S \to \mathbb{R}$ by $f(y) = ||y|| := \sqrt{y_1^2 + \cdots + y_n^2}$. Then S is a compact subset of $\mathbb{R}^n$ and f is a continuous function. Thus $f(S)$ is a compact subset of $\mathbb{R}$. Hence there exists $x \in S$ with $||x|| \leq ||x'||$ for all $x' \in S$. Clearly $x \neq 0$ since $0 \notin S$ by assumption. We will show $v_i x > 0$ for all i as required. In fact we will show more, namely, that $(y, x) > 0$ for all $y \in S$, where $(y, x) = y_1 x_1 + \ldots y_n x_n$.

Now S is a convex subset of $\mathbb{R}^n$. So for $y \neq x$ we have $ty + (1-t)x \in S$ for all $0 \leq t \leq 1$. By the choice of x,

$$(ty + (1-t)x, ty + (1-t)x) \geq (x, x)$$

or

$$t(y - x, y - x) + 2(y - x, x) \geq 0.$$

As t can be made arbitrarily small, this implies $(y - x, x) \geq 0$ or $(y, x) \geq (x, x) > 0$. □

Proposition 4.1.5 *Let* C *be an* $m \times n$ *matrix over* $\mathbb{R}$. *Suppose* $u \geq 0$ *and* $C^t u \geq 0$ *imply* $u = 0$. *Then there exists* $v > 0$ *with* $Cv < 0$.

Proof Let $C = (c_{ij})$ and consider the following system of inequalities:

$$\begin{aligned} -\sum_{j=1}^{n} c_{ij}x_j &> 0 \qquad (i = 1, \ldots, m), \\ x_j &> 0 \qquad (j = 1, \ldots, n). \end{aligned}$$

We want to use Lemma 4.1.4 to show that this system has a solution. Thus we consider an equation of the form

$$\sum_{i=1}^{m} \lambda_i(-c_{i1}, \ldots, -c_{in}) + \sum_{j=1}^{n} \mu_j \varepsilon_j = 0,$$

where $\lambda_i, \mu_j \geq 0$ and ε_j is the jth coordinate vector in $\mathbb{R}^n$. Then

$$\sum_{i=1}^{m} \lambda_i c_{ij} = \mu_j \qquad (j = 1, \ldots, n).$$

Let $u = (\lambda_1, \ldots, \lambda_m)^t$. Then $C^t u = (\mu_1, \ldots, \mu_n)^t$. Thus we have $u \geq 0$ and $c^t u \geq 0$. this implies $u = 0$. We also have $C^t u = 0$. Thus all la_i and μ_j are zero. Hence Lemma 4.1.4 shows that the above inequalities have a solution. Thus there exists $v > 0$ with $Cv < 0$. □

We now consider three classes of GCM A. Let

$$\begin{aligned} S_F &= \{A \mid A \text{ has finite type}\} \\ S_A &= \{A \mid A \text{ has affine type}\} \\ S_I &= \{A \mid A \text{ has indeterminate type}\} \end{aligned}$$

It is easy to see that no GCM can lie in more than one of these classes. We want to show that each indecomposable GCM lies in one of the three classes.

Lemma 4.1.6 *Let A be an indecomposable GCM. Then $u \geq 0$ and $Au \geq 0$ imply that $u > 0$ or $u = 0$.*

Proof Suppose $u \geq 0$, $u \neq 0$ and $u \not> 0$. Then we can reorder $1, \ldots, n$ so that $u_1 = \cdots = u_s = 0$ and $u_{s+1}, \ldots, u_n > 0$. Let $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ where P is $s \times s$ and S is $(n-s) \times (n-s)$. Now all entries of the block Q are ≤ 0 since A is GCM, and if Q has a negative entry, then Au has a negative coefficient, giving a contradiction. Thus $Q = 0$, whence $R = 0$ by definition of GCM. Now A is decomposable, a contradiction. □

Now let A be an indecomposable GCM and define

$$K_A = \{u \mid Au \geq 0\}.$$

K_A is a convex cone. We consider its intersection with the convex cone $\{u \mid u \geq 0\}$. We will distinguish between two cases:

$$\begin{aligned} \{u \mid u \geq 0,\ Au \geq 0\} &\neq \{0\}, \\ \{u \mid u \geq 0,\ Au \geq 0\} &= \{0\}. \end{aligned}$$

The first of these cases splits into two subcases, as is shown by the next lemma.

Lemma 4.1.7 *Suppose* $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$. *Then just one of the following cases occurs:*

$$\begin{aligned} K_A &\subset \{u \mid u > 0\} \cup \{0\}, \\ K_A &= \{u \mid Au = 0\} \text{ and } K_A \text{ is a 1-dimensional subspace of } \mathbb{R}^n. \end{aligned}$$

Proof We know there exists $u \neq 0$ with $u \geq 0$ and $Au \geq 0$. By Lemma 4.1.6, $u > 0$. Suppose the first case does not hold. Then there is $v \neq 0$ with $Av \geq 0$ such that some coordinate of v is ≤ 0. If $v \geq 0$ then $v > 0$ by Lemma 4.1.6, so some coordinate of v is negative.

We have $Au \geq 0$ and $Av \geq 0$, hence $A(tu + (1-t)v) \geq 0$ for all $0 \leq t \leq 1$. Since all coordinates of u are positive and some coordinate of v is negative, there exists $0 < t < 1$ with $tu + (1-t)v \geq 0$ and some coordinate of $tu + (1-t)v$ is zero. But then $tu + (1-t)v = 0$ by Lemma 4.1.6. Thus v is a scalar multiple of u. We also have

$$0 = A(tu + (1-t)v) = tAu + (1-t)Av.$$

Since $Au \geq 0$ and $Av \geq 0$ this implies $Av = Au = 0$.

Now let $w \in K_A$. Then $Aw \geq 0$. Either $w \geq 0$ or some coordinate of w is negative. If $w \geq 0$ then $w > 0$ or $w = 0$ by Lemma 4.1.6. Suppose $w > 0$. Then by the above argument with u replaced by w, v is a scalar multiple of w, hence w is a scalar multiple of u. Now suppose some coordinate of w is negative. Then by the above argument with v replaced by w, w is a scalar multiple of u. Thus in all cases w is a scalar multiple of u. Hence $K_A = \mathbb{R}u = \{u \mid Au = 0\}$.

Finally, both cases cannot hold simultaneously since in the first case K_A cannot contain a 1-dimensional subspace. □

We can now identify the first case in the lemma above with the case of matrices of finite type.

Proposition 4.1.8 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A has finite type;*
(ii) $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$ *and* $K_A \subset \{u \mid u > 0\} \cup \{0\}$.

Proof (i) $\Rightarrow$ (ii) Suppose A is of finite type. Then there exists $u > 0$ with $Au > 0$. Hence $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$. Also, $\det A \neq 0$. Thus $\{u \mid Au = 0\}$ is not a 1-dimensional subspace. Hence (ii) holds by Lemma 4.1.7.

(ii) $\Rightarrow$ (i) There cannot exist $u \neq 0$ with $Au = 0$ for this would give a 1-dimensional subspace in K_A. Thus $\det A \neq 0$. Now there exists $u \neq 0$ with $u \geq 0$ and $Au \geq 0$. By Lemma 4.1.6, $u > 0$. If $Au > 0$, A has finite type. So suppose to the contrary that some coordinate of Au is zero. Choose the numbering of $1, \ldots, n$ so that the first s coordinates of Au are 0 and the last $n - s$ are positive. Let $A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ where P is $s \times s$ and S is $(n-s) \times (n-s)$. The block $Q \neq 0$, since A is indecomposable. We choose numbering so that the first row of Q is not the zero vector. Then

$$Au = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} Pu^1 + Qu^2 \\ Ru^1 + Su^2 \end{pmatrix},$$

and $Pu^1 + Qu^2 = 0$ and $Ru^1 + Su^2 > 0$. We also have $u^1, u^2 > 0$. Thus $Qu^2 \leq 0$ since the entries of Q are non-positive, and the first coordinate of Qu^2 is negative. Hence $Pu^1 \geq 0$ and the first coordinate of Pu^1 is positive. Since $Ru^1 + Su^2 > 0$ we can chose $\varepsilon > 0$ such that $R(1+\varepsilon)u^1 + Su^2 > 0$.

We now consider instead of our original vector $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}$, the vector $\begin{pmatrix} (1+\varepsilon)u^1 \\ u^2 \end{pmatrix} > 0$. We have

$$A \begin{pmatrix} (1+\varepsilon)u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} Pu^1 + Qu^2 + \varepsilon Pu^1 \\ Ru^1 + Su^2 + \varepsilon Ru^1 \end{pmatrix} = \begin{pmatrix} \varepsilon Pu^1 \\ R(1+\varepsilon)u^1 + Su^2 \end{pmatrix}.$$

The first coordinate and the last $n - s$ coordinates of this vector are positive and the remaining coordinates are ≥ 0. Thus $A \begin{pmatrix} (1+\varepsilon)u^1 \\ u^2 \end{pmatrix} \geq$

0 and the number of non-zero coordinates in this vector is greater than that in Au. We may now iterate this process, obtaining at each stage at least one more non-zero coordinate than we had before. We eventually obtain a vector $v > 0$ such that $Av > 0$. □

We next identify the second case in Lemma 4.1.7 with that of an affine GCM.

Proposition 4.1.9 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A has affine type;*
(ii) *$\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$, $K_A = \{u \mid Au = 0\}$, and K_A is a 1-dimensional subspace of $\mathbb{R}^n$.*

Proof (i) $\Rightarrow$ (ii) Suppose A is of affine type. Then there exists $u > 0$ with $Au = 0$. It follows that $\{u \mid u \geq 0,\ Au \geq 0\} \neq \{0\}$. Also $\lambda u \in K_A$ for all $\lambda \in \mathbb{R}$. It follows from Lemma 4.1.7 that we are in the second case of that lemma.

(ii) $\Rightarrow$ (i) Note first that corank $A = 1$. Also there exists $u \neq 0$ with $u \geq 0$ and $Au \geq 0$. By Lemma 4.1.6, $u > 0$. So there exists $u > 0$ with $Au \geq 0$. But $K_A = \{u \mid Au = 0\}$, so $Au = 0$. Finally, $Au \geq 0$ implies $Au = 0$. □

Proposition 4.1.10 *Let A be an indecomposable GCM. Then*

(i) *A has finite type if and only if A^t has finite type;*
(ii) *A has affine type if and only if A^t has affine type.*

Proof Let A be of finite type. There does not exist $v > 0$ with $Av < 0$ ($Av < 0 \Rightarrow A(-v) > 0 \Rightarrow (-v) > 0 \Rightarrow v < 0$). So by Proposition 4.1.5, there exists $u \neq 0$ with $u \geq 0$ and $A^t u \geq 0$. So

$$\{u \mid u \geq 0,\ A^t u \geq 0\} \neq \{0\}.$$

By Lemma 4.1.7, either

$$K_{A^t} \subset \{u \mid u > 0\} \cup \{0\}$$

or $K_{A^t} = \{u \mid A^t u = 0\}$ and this is a 1-dimensional subspace. Now $\det A \neq 0$, so $\det A^t \neq 0$. Thus the latter case cannot occur. The former case must therefore occur, so by Proposition 4.1.8, A^t is of finite type.

Let A be of affine type. Again, there does not exist $v > 0$ with $Av < 0$

($Av < 0 \;\Rightarrow\; A(-v) > 0$, which is impossible in the affine case). So by Proposition 4.1.5, there exists $u \neq 0$ with $u \geq 0$ and $A^t u \geq 0$. So

$$\{u \mid u \geq 0,\ A^t u \geq 0\} \neq \{0\}.$$

By Lemma 4.1.7, either

$$K_{A^t} \subset \{u \mid u > 0\} \cup \{0\}$$

or $K_{A^t} = \{u \mid A^t u = 0\}$ and this is a 1-dimensional subspace. Now corank $A = 1$ so corank $A^t = 1$. This shows that we cannot have the first possibility. Thus the second possibility holds, and by Proposition 4.1.9, we see that A^t has affine type. □

We may now identify the case not appearing in Lemma 4.1.7.

Proposition 4.1.11 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A has indefinite type;*
(ii) *$\{u \mid u \geq 0,\ Au \geq 0\} = \{0\}$.*

Proof If A has indefinite type. Then $u \geq 0$ and $Au \geq 0$ imply $u = 0$.

Conversely, suppose $\{u \mid u \geq 0,\ Au \geq 0\} = \{0\}$. Then the same condition holds for A^t, i.e. $\{u \mid u \geq 0,\ Au \geq 0\} = \{0\}$. Indeed this follows from Lemma 4.1.7 and Propositions 4.1.8, 4.1.9, 4.1.10. But then Proposition 4.1.5 implies that there exists $v > 0$ with $Av < 0$. Thus A has indefinite type. □

Theorem 4.1.12 (Trichotomy Theorem) *Let A be an indecomposable GCM. Then exactly one of the following three possibilities holds: A has finite type, A has affine type, or A has indefinite type. Moreover, the type of A is the same as the type of A^t. Finally,*

(i) *A has finite type if and only if there exists $u > 0$ with $Au > 0$.*
(ii) *A has affine type if and only if there exists $u > 0$ with $Au = 0$. This u is unique up to a (positive) scalar.*
(iii) *A has indefinite type if and only if there exists $u > 0$ with $Au < 0$.*

Proof The first two statements have already been proved. We prove the third statement. Let $u > 0$.

(i) Assume that $Au > 0$. A cannot have affine type as then $Au \geq 0$ would imply $Au = 0$. A cannot have indefinite type as then $u \geq 0$ and

$Au \geq 0$ would imply $u = 0$. Thus A has finite type. The converse is clear.

(ii) Assume that $Au = 0$. A cannot have finite type as then $\det A = 0$. A cannot have indefinite type as then $u \geq 0$ and $Au \geq 0$ would imply $u = 0$. Thus A has affine type. The converse is clear, and the remaining statement follows from Proposition 4.1.9.

(iii) (i) Assume that $Au < 0$. Then $A(-u) > 0$. A cannot have finite type as this would imply $-u > 0$ or $-u = 0$. A cannot have affine type as and $A(-u) > 0$ would then imply $-u = 0$. Thus A has indefinite type. The converse is clear. □

Lemma 4.1.13 *Let A be an indecomposable GCM.*

(i) *If A is of finite type then every principal minor A_J is also of finite type.*

(ii) *If A is of affine type then every proper principal minor A_J is of finite type.*

Proof By passing to an equivalent GCM we may assume that $J = \{1, \ldots, m\}$ for some $m \leq n$. Let $K = \{m+1, \ldots, n\}$. Write

$$A = \begin{pmatrix} A_J & Q \\ R & S \end{pmatrix}.$$

(i) We have $Au > 0$ for some $u = \begin{pmatrix} u_J \\ u_K \end{pmatrix} > 0$. We have

$$Au = \begin{pmatrix} A_J u_J + Q u_K \\ R u_J + S u_K \end{pmatrix}.$$

We have $A_J u_J + Q u_K > 0$. But $Q u_K \leq 0$, so $A_J u_J > 0$.

(ii) As in (i) we get $A_J u_J + Q u_K = 0$, and $Q u_K \leq 0$ implies $A_J u_J \geq 0$. Suppose if possible $A_J u_J = 0$. Then $Q u_K = 0$, and since $u_K > 0$ this implies that $Q = 0$, which contradicts the assumption that A is indecomposable. Hence we have $u_J > 0$, $A_J u_J \geq 0$, $A_J u_J \neq 0$. This implies that A_J cannot have affine or indefinite type. □

Remark 4.1.14 In proving results of this section we have never used the full force of the assumption that A is a GCM. Namely we nowhere needed that $a_{ii} = 2$ and $a_{ij} \in \mathbb{Z}$.

4.2 Indecomposable symmetrizable GCMs

Proposition 4.2.1 *Suppose A is a symmetric indecomposable GCM. Then:*

(i) *A has finite type if and only if A is positive definite.*
(ii) *A has affine type if and only if A is positive semidefinite of corank 1.*
(iii) *A has indefinite type otherwise.*

Proof (i) Let A be of finite type. Then there exists $u > 0$ with $Au > 0$. Hence for all $\lambda > 0$ we have $(A + \lambda I)u > 0$. Thus $A + \lambda I$ has finite type by Trichotomy Theorem. (Note that $A + \lambda I$ need not be GCM, but see Remark 4.1.14.) Thus $\det(A + \lambda I) \neq 0$ when $\lambda \geq 0$, that is $\det(A - \lambda I) \neq 0$ when $\lambda \leq 0$. Now the eigenvalues of the real symmetric matrix A are all real. Thus all the eigenvalues of A must be positive.

Conversely, suppose A is positive definite. Then $\det A \neq o$, so A has finite or indefinite type. If A has indefinite type there exists $u > 0$ with $Au < 0$. But then $u^t Au < 0$, contradicting the fact that A is positive definite. Thus A must have finite type.

(ii) Let A have affine type. Then there is $u > 0$ with $Au = 0$. The same argument as in (i) shows that all eigenvalues of A are non-negative. But A has corank 1, so 0 appears with multiplicity 1.

Conversely, suppose A is positive semidefinite of corank 1. Then $\det A = 0$ so A cannot have finite type. Suppose A has indefinite type. Then there exists $u > 0$ with $Au < 0$. Thus $u_t Au < 0$, which contradicts the fact that A is positive semidefinite.

(iii) follows from (i) and (ii). □

Lemma 4.2.2 *Let A an indecomposable GCM of finite or affine type. Suppose that $a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k} a_{i_k i_1} \neq 0$ for some integers $i_1, \dots, i_k$ with $k \geq 3$ such that $i_1 \neq i_2, i_2 \neq i_3, \dots, i_{k-1} \neq i_k, i_k \neq i_1$. Then A is of the form*

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}. \tag{4.1}$$

Proof Choose integers $i_1, \ldots, i_k$ as in the assumption with minimal possible k. We thus have

$$a_{i_r i_s} \neq 0 \text{ is } (r,s) \in \{(1,2),(2,3),\ldots,(k,1),(2,1),(3,2),\ldots(1,k)\}.$$

The minimality of k implies that $a_{i_r,i_s} = 0$ if (r,s) does not lie in the above set.

Let $J = \{i_1, \ldots, i_k\}$. Then the principal minor A_J of A has form

$$A_J = \begin{pmatrix} 2 & -r_1 & 0 & 0 & \ldots & 0 & 0 & -s_k \\ -s_1 & 2 & -r_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -s_2 & 2 & -r_3 & \ldots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & -s_{k-2} & 2 & -r_{k-1} \\ -r_k & 0 & 0 & 0 & \ldots & 0 & -s_{k-1} & 2 \end{pmatrix} \quad (4.2)$$

with positive integers r_i, s_i. In particular we see that A_J is indecomposable. Now A_J must be finite or affine type by Lemma 4.1.13. Thus there exists $u = (u_1, \ldots, u_k) > 0$ with $A_J u \geq 0$. We define the $k \times k$ matrix

$$M := \operatorname{diag}(u_1^{-1}, \ldots, u_k^{-1}) A_J \operatorname{diag}(u_1, \ldots, u_k).$$

Then $m_{ij} = u_i^{-1} a_{ij} u_j$. Thus

$$\sum_j m_{ij} = u_i^{-1} \sum_j (A_J)_{ij} u_j \geq 0.$$

In particular, $\sum_{ij} m_{ij} \geq 0$. Now we have

$$A_J = \begin{pmatrix} 2 & -r'_1 & 0 & 0 & \ldots & 0 & 0 & -s'_k \\ -s'_1 & 2 & -r'_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & -s'_2 & 2 & -r'_3 & \ldots & 0 & 0 & 0 \\ & & & & \vdots & & & \\ 0 & 0 & 0 & 0 & \ldots & -s'_{k-2} & 2 & -r'_{k-1} \\ -r'_k & 0 & 0 & 0 & \ldots & 0 & -s'_{k-1} & 2 \end{pmatrix},$$

where $r'_i = u_i^{-1} r_i u_{i+1}, s'_i = u_{i+1}^{-1} s_i u_i$ and u_{k+1} is interpreted as u_1. We note that $r'_i, s'_i > 0$ and $r'_i s'_i = r_i s_i \in \mathbb{Z}$. We also have

$$\sum_{ij} m_{ij} = 2k - (r'_1 + s'_1) - \cdots - (r'_k + s'_k).$$

Now $\frac{r'_i + s'_i}{2} \geq \sqrt{r'_i s'_i} = \sqrt{r_i s_i} \geq 1$, hence $r'_i + s'_i \geq 2$. Since $\sum_{ij} m_{ij} \geq 0$, we deduce that $r'_i + s'_i = 2$ and $r'_i s'_i = 1$. Hence $r_i s_i = 1$, and since r_i, s_i

are positive integers, we deduce that $r_i = s_i = 1$, i.e. A_J is of the form (4.1).

Let $v = (1, \ldots, 1)$. Then $v > 0$ and $A_J v = 0$. Thus A_J is affine type by Theorem 4.1.12. Lemma 2.1.1 shows that this can only happen when $A_J = A$. □

Theorem 4.2.3 *Indecomposable GCM of finite or affine type is symmetrisable.*

Proof If there is a set of integers $i_1, \ldots, i_k$ as in Lemma 4.2.2, then we know that A is of the form (4.1), in particular it is symmetric. Otherwise A is symmetrizable by Lemma 2.1.1. □

Theorem 4.2.4 *Let A be an indecomposable GCM. Then:*

(i) *A has finite type if and only if all its principal minors have positive determinant.*

(ii) *A has affine type if and only if* $\det A = 0$ *and all proper principal minors have positive determinant.*

(iii) *A has indefinite type if and only if neither of the above conditions holds.*

Proof (i) Suppose A has finite type. Then A is symmetrizable by Theorem 4.2.3, hence $A = DB$ where $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ and B symmetric, see Lemma 2.1.2. Theorem 4.1.12 shows that A and B have the same type. By Lemma 4.1.13 all principal minors of B have finite type, hence by Proposition 4.2.1 they all have positive determinant. Then the same is true for A.

Conversely, let all principal minors of A have positive determinant. Suppose there is a set of integers $i_1, \ldots, i_k$ with $k \geq 3$ such that $i_1 \neq i_2, i_2 \neq i_3, \ldots, i_{k-1} \neq i_k, i_k \neq i_1$ and $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{k-1} i_k} a_{i_k i_1} \neq 0$. As in the proof of the previous theorem, A_J has form (4.2). Analyzing 2×2 and 3×3 principal subminors we conclude that A_J is of the form (4.1). But then $\det A_J = 0$, giving a contradiction. Thus there is no such sequence $i_1, \ldots, i_k$ and so A is symmetrizable by Lemma 2.1.1. Hence $A = DB$ where $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_i > 0$ and B symmetric of the same type as A. Now, it follows from the assumption that all principal minors of B have positive determinant, so B is of finite type.

(ii) If A has affine type, then $\det A = 0$ and all proper principal minors have finite type so have positive determinants by (i).

Conversely, supppose $\det A = 0$ and all proper principal minors have positive determinants. As above, we have two cases:

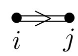
(a) there is a principal minor of the form (4.1). Since $\det A_J = 0$ we must have $A = A_J$, which is affine type.

(b) A is symmetrizable, in which case we reduce to the symmetric case as above. □

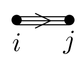
4.3 The classification of finite and affine GCMs

To every GCM A we associate the graph $S(A)$, called the *Dynkin diagram* of A, as follows. The vertices of the Dynkin diagram are labelled by $1, \ldots, n$ (or the corresponding simple roots $\alpha_1, \ldots, \alpha_n$). Let i, j be distinct vertices of $S(A)$. The rules are as follows:

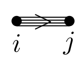
(a) If $a_{ij}a_{ji} = 0$, vertices i, j are not joined.
(b) If $a_{ij} = a_{ji} = -1$, vertices i, j are joined by a single edge.
(c) If $a_{ij} = -1$, $a_{ji} = -2$, vertices i, j are joined as follows



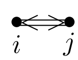
(d) If $a_{ij} = -1$, $a_{ji} = -3$, vertices i, j are joined as follows



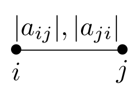
(e) If $a_{ij} = -1$, $a_{ji} = -4$, vertices i, j are joined as follows



(f) If $a_{ij} = -2$, $a_{ji} = -2$, vertices i, j are joined as follows



(g) If $a_{ij}a_{ji} =\geq 5$, vertices i, j are joined as follows



It is clear that the GCM is determined by its Dynkin diagram. Moreover, A is indecomposable if and only if $S(A)$ is connected.

Theorem 4.3.1 *Let A be an indecomposable GCM. Then:*

(i) *A is of finite type if and only if its Dynkin diagram belongs to Figure 4.1. Numbers on the right give* $\det A$.

(ii) *A is of affine type if and only if its Dynkin diagram belongs to Figures 4.2 and 4.3. All diagrams there have $\ell+1$ vertices. Numeric marks are the coordinates of the unique vector $\delta = (a_0, a_1, \dots, a_\ell)$ such that $A\delta = 0$ and the a_i are positive mutually prime integers. Each diagram $X_\ell^{(1)}$ in Figure 4.2 is obtained from the diagram X_ℓ in Figure 4.1 by adding a vertex labeled α_0 and preserving the labeling of other vertices.*

Proof We first prove that the numeric marks in the diagrams from Figures 4.2 and 4.3 are the coordinates of the unique vector $\delta = (a_0, a_1, \dots, a_\ell)$ such that $A\delta = 0$ and the a_i are positive mutually prime integers. Note that $A\delta = 0$ is equivalent to

$$2a_i = \sum_j m_j a_j \qquad \text{for all } i$$

where the sum is over all j which are linked with i; moreover if the number of edges between i and j is equal to $s > 1$ and the arrow points to i then then $m_j = s$, otherwise $m_j = 1$. Now check that the marks work in all cases. Now from Theorem 4.1.12 we conclude that all diagrams from Figures 4.2 and 4.3 are affine and δ is unique.

Since all diagrams from Figure 4.1 are proper subdiagrams of diagrams from Figures 4.2 and 4.3, Theorem 4.2.4 implies that they are of finite type. It remains to show that if A is of finite (resp. affine) type then $S(A)$ appears in Figure 4.1 (resp. Figures 4.2 and 4.3). We establish this by induction on n. The case $n = 1$ is clear. Also, using the condition $\det A \geq 0$ and Theorem 4.2.4, we obtain:

$$\text{finite diagrams of rank 2 are} A_2, C_2, G_2; \qquad (4.3)$$

$$\text{affine diagrams of rank 2 are } A_1^{(1)}, A_2^{(2)}; \qquad (4.4)$$

$$\text{finite diagrams of rank 3 are } A_3, B_3, C_3; \qquad (4.5)$$

$$\text{affine diagrams of rank 3 are } A_2^{(1)}, C_2^{(1)}, G_2^{(1)}, D_3^{(2)}, A_4^{(2)}, D_4^{(3)}. \qquad (4.6)$$

Next, from Lemma 4.2.2, we have

$$\text{if } S(A) \text{ contains a cycle, then } S(A) = A_\ell^{(1)}. \qquad (4.7)$$

Moreover, by induction and Lemma 4.1.13,

$$\text{Any proper subdiagram of } S(A) \text{ appears in Figure 4.1.} \qquad (4.8)$$

Now let $S(A)$ be a finite diagram. Then it does not have graphs appearing in Figures 4.2 and 4.3 as subgraphs and does not have cycles.

This implies that every branch vertex has type D_4 since otherwise we would get an affine subdiagram or a contradiction with (4.8). Using (4.8) again we see that there is at most one branch vertex, in which case it also follows that $S(A)$ is D_ℓ, E_6, E_7, or E_8. Similarly one checks that if $S(A)$ has multiple edges then it must be B_ℓ, C_ℓ, F_4, or G_2. Finally, a graph without branch vertices, cycles and multiple edges must be A_ℓ.

Let $S(A)$ be affine. In view of (4.7) we may assume that $S(A)$ has no cycles. In view of (4.7), $S(A)$ is obtained from a diagram in Figure 4.1 by adjoining one vertex in such a way that every subdiagram is again in Figure 4.1. It is easy to see that in this way we can only get diagrams from Figures 4.2 and 4.3. □

Proposition 4.3.2 *Let A be an indecomposable GCM. Then the following conditions are equivalent:*

(i) *A is of finite type.*
(ii) *A is symmetrizable and the $(\cdot|\cdot)$ on $\mathfrak{h}_{\mathbb{R}}$ is positive definite.*
(iii) $|W| < \infty$.
(iv) $|\Delta| < \infty$.
(v) *$\mathfrak{g}(A)$ is a finite dimensional simple Lie algebra.*
(vi) *There exists $\alpha \in \Delta_+$ such that $\alpha + \alpha_i \notin \Delta$ for all $i = 1, \dots, n$.*

Proof (i) $\Rightarrow$ (ii) follows from Theorems 4.2.3 and 4.2.4.

(ii) $\Rightarrow$ (iii). In view of Proposition 3.2.4, W is a subgroup of the orthogonal group $O((\cdot|\cdot))$ so W is compact. Moreover, W preserves the lattice Q, so W is discrete. It follows that W is finite.

(iii) $\Rightarrow$ (iv) follows from Proposition 3.4.1(v).

(iv) $\Rightarrow$ (vi) is obvious.

(vi) $\Rightarrow$ (i). Let $\alpha \in \Delta_+$ be such that $\alpha + \alpha_i \notin \Delta$ for all i. By Proposition 3.1.5(v), $\langle \alpha, \alpha_i^\vee \rangle \geq 0$ for all i. Write $\alpha = u_1\alpha_1 + \dots + u_n\alpha_n$ with non-negative coefficients u_i. Then $u = (u_1, \dots, u_n) \geq 0$, $u \neq 0$, and $Au \geq 0$. By Trichotomy Theorem, A is finite or affine type, and in the former case we have $\langle \alpha, \alpha_i^\vee \rangle = 0$ for all i. But then $\alpha - \alpha_i \in \Delta_+$ for some i by Lemma 1.4.5, and so $\alpha - \alpha_i + 2\alpha_i = \alpha + \alpha_i \in \Delta_+$ in view of Proposition 3.1.5(vi), giving a contradiction.

Finally, (i) $\Rightarrow$ (v) follows from Proposition 1.4.8 and (v) $\Rightarrow$ (iv) is obvious. □

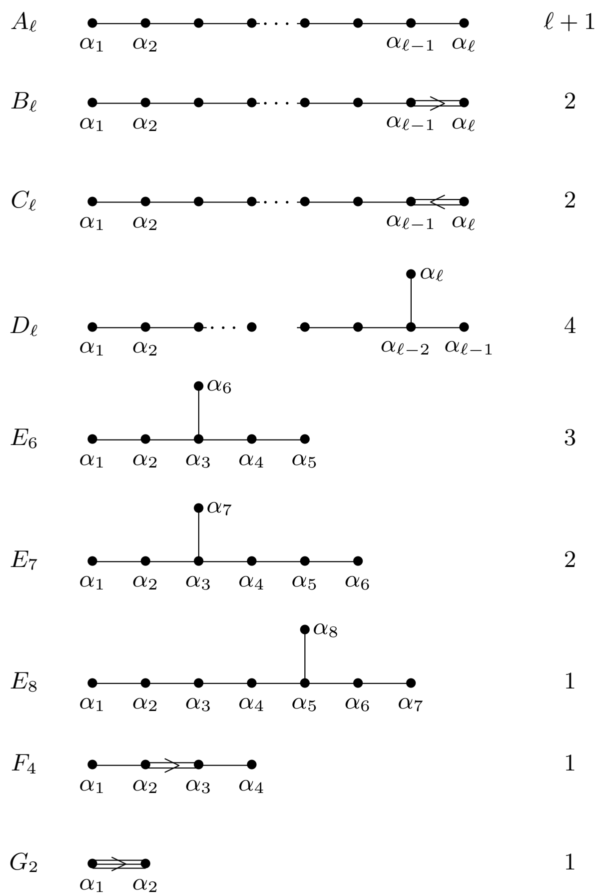


Fig. 4.1. Dynkin diagrams of finite GCMs

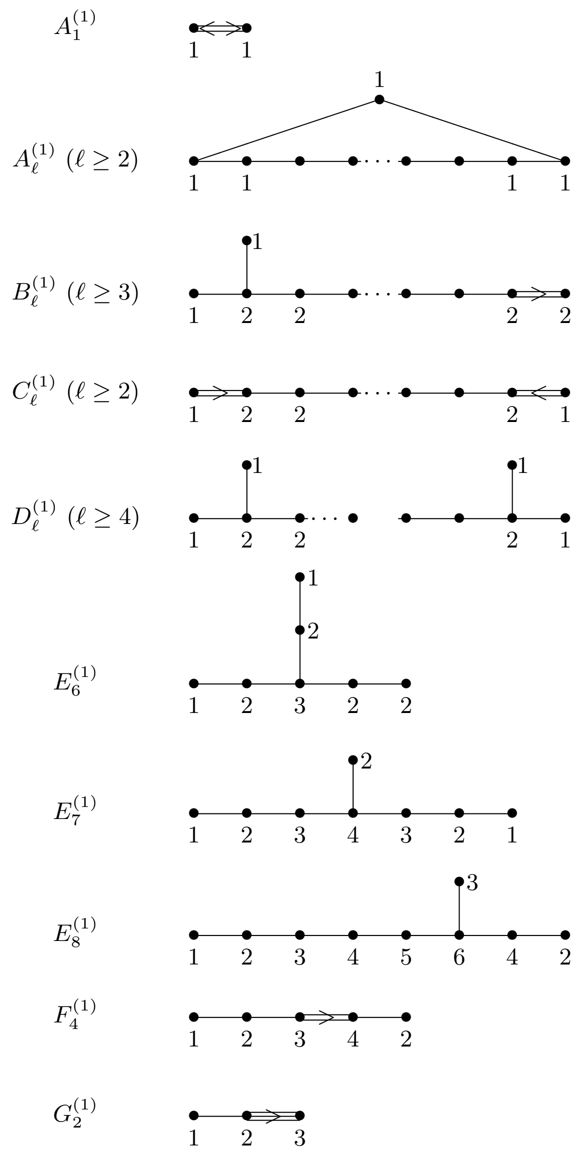


Fig. 4.2. Dynkin diagrams of untwisted affine GCMs

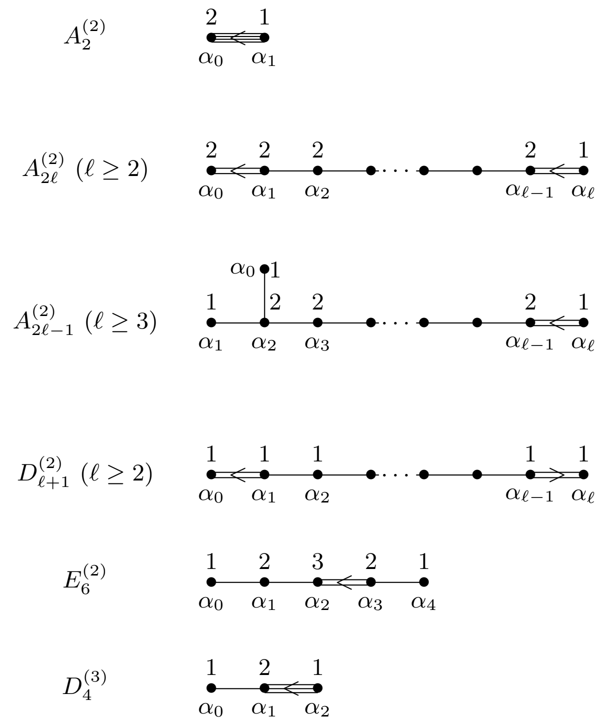


Fig. 4.3. Dynkin diagrams of twisted affine GCMs

5
Real and Imaginary Roots

5.1 Real roots

A root $\alpha \in \Delta$ is called *real* if there exists $w \in W$ such that $w(\alpha)$ is a simple root. Denote by Δ^{re} and Δ_+^{re} the sets of the real and positive real roots respectively. If A is of finite type, then induction on height shows that every root is real.

Let $\alpha \in \Delta^{\text{re}}$. Then $\alpha = w(\alpha_i)$ for some w and some i. Define the *dual real root* $\alpha^\vee \in (\Delta^\vee)$re by setting

$$\alpha^\vee = w(\alpha_i^\vee).$$

This definition is independent of the choice of the presentation $\alpha = w(\alpha_i)$. Indeed, we have to show that the equality $u(\alpha_i) = \alpha_j$ implies $u(\alpha_i^\vee) = \alpha_j^\vee$, but this has been proved in Lemma 3.3.1, see (3.8). Thus we have a canonical W-equivariant bijection bijection $\Delta^{\text{re}} \to (\Delta^\vee)^{\text{re}}$.

Define the reflection

$$r_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*,\ \lambda \mapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Since $\langle \alpha, \alpha^\vee \rangle = 2$, it is indeed a reflection. If $\alpha = w(\alpha_i)$, then $wr_iw^{-1} = r_\alpha$, so we have $r_\alpha \in W$.

Proposition 5.1.1 *Let $\alpha \in \Delta^{\text{re}}$. Then:*

(i) mult $\alpha = 1$*;*
(ii) *$k\alpha$ is a root if and only if $k = \pm 1$.*
(iii) *If $\beta \in \Delta$, then there exist non-negative integers p, q such that $p - q = \langle \beta, \alpha^\vee \rangle$ such that $\beta + k\alpha \in \Delta \cup \{0\}$ if and only if $-p \le k \le q,\ k \in \mathbb{Z}$.*
(iv) *Suppose that A is symmetrizable and let $(\cdot|\cdot)$ is the standard invariant bilinear form on $\mathfrak{g}$. Then*

(a) $(\alpha|\alpha) > 0$;
(b) $\alpha^\vee = 2\nu^{-1}(\alpha)/(\alpha|\alpha)$;
(c) *if* $\alpha = \sum_i k_i\alpha_i$, *then* $k_i(\alpha_i|\alpha_i) \in (\alpha|\alpha)\mathbb{Z}$.

(v) *if* $\pm\alpha \notin \Pi$, *then there exists* i *such that*

$$|\operatorname{ht} r_i(\alpha)| < |\operatorname{ht} \alpha|.$$

(vi) *if* $\alpha > 0$ *then* $\alpha^\vee > 0$.

Proof The proposition is true if α is a simple root, see (2.8), (2.10), and Proposition 3.1.5. Now (i)-(iii) follow from Proposition 3.2.1(ii), and (iv)(a),(b) from Proposition 3.2.4.

(iv)(c) follows from the fact that $\alpha^\vee \in \sum_i \mathbb{Z}\alpha_i^\vee$ and the formula

$$\alpha^\vee = \sum_i \frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} k_i\alpha_i^\vee, \tag{5.1}$$

which in turn follows from (iv)(b).

(v) Assume the statement does not hold. We may assume that $\alpha > 0$. Then $-\alpha \in C^\vee$, and by Proposition 3.4.1(iv) applied to dual root system, $-\alpha + w(\alpha) \geq 0$ for any $w \in W$. Taking w such that $w(\alpha) \in \Pi$ we get a contradiction.

(vi) Apply induction on $\operatorname{ht} \alpha$. For $\operatorname{ht} \alpha > 1$ we have by (v) that $\operatorname{ht} r_i\alpha < \operatorname{ht} \alpha$, for some i, and $r_i\alpha > 0$. By induction, $r_i(\alpha^\vee) = (r_i\alpha)^\vee > 0$, whence $\alpha^\vee > 0$. □

Lemma 5.1.2 *Assume that* A *is symmetrizable. Then the set of all* $\alpha = \sum_i k_i\alpha_i \in Q$ *such that*

$$k_i(\alpha_i|\alpha_i) \in (\alpha|\alpha)\mathbb{Z} \quad \text{for all } i \tag{5.2}$$

is W*-invariant.*

Proof It suffices to check that $r_i\alpha$ again satisfies (5.2), i.e.

$$(k_i - (\alpha|\alpha_i^\vee))(\alpha_i|\alpha_i) \in (\alpha|\alpha)\mathbb{Z},$$

or

$$2(\alpha|\alpha_i) \in (\alpha|\alpha)\mathbb{Z},$$

which follows from (5.2):

$$2(\alpha|\alpha_i) = \sum_j \frac{2(\alpha_j|\alpha_i)}{(\alpha_j|\alpha_j)} k_j(\alpha_j|\alpha_j) = \sum_j a_{ji}k_j(\alpha_j|\alpha_j) \in (\alpha|\alpha)\mathbb{Z}.$$

□

Let A be an indecomposable symmetrizable and $(\cdot|\cdot)$ be a standard invariant bilinear form. Then for a real root α we have $(\alpha|\alpha) = (\alpha_i|\alpha_i)$, where α_i is one of the simple roots. We call α a *short* (resp. *long*) root if $(\alpha|\alpha) = \min_i(\alpha_i|\alpha_i)$ (resp. $(\alpha|\alpha) = \max_i(\alpha_i|\alpha_i)$). This definition is independent of the choice of the standard form since α is a linear combination of simple roots.

Note that if A is symmetric then all simple roots are of the same length (so they are both short and long). If A is not symmetric and $S(A)$ has m arrows directed in the same direction then A has $m+1$ different lengths, as the arrow is directed from a longer to a shorter root. Hence if A is not symmetric in Figure 4.1 then every root is either long or short. Moreover, if A is not symmetric and affine and its type is not $A_{2\ell}^{(2)}$ for $\ell > 1$, then every real root is either short or long. In the exceptional case there are three root lengths for real roots. We use notation

$$\Delta_s^{\text{re}}, \quad \Delta_l^{\text{re}}, \quad \Delta_i^{\text{re}}$$

to denote the set of all short, long, and intermediate roots, repspectively.

Note that α is a short real root for $\mathfrak{g}(A)$ if and only if $\alpha^\vee$ is a long real root for $\mathfrak{g}(A^t)$. Indeed, by Proposition 5.1.1(iv)(b)

$$(\alpha^\vee|\alpha^\vee) = (\frac{2\nu^{-1}(\alpha)}{(\alpha|\alpha)}|\frac{2\nu^{-1}(\alpha)}{(\alpha|\alpha)}) = \frac{4}{(\alpha|\alpha)}. \tag{5.3}$$

Throughout this chapter: we normalize the form so that $(\alpha_i|\alpha_i)$ are mutually prime positive integers for each connected component of $S(A)$. In particular, if A is symmetric then $(\alpha_i|\alpha_i) = 1$ for all i.

5.2 Real roots for finite and affine types

Throughout this section we assume that A is finite or affine type.

If $\alpha = \sum_i k_i\alpha_i \in Q$ then $(\alpha|\alpha) = \sum_{i,j} k_ik_j(\alpha_i|\alpha_j)$. Now, $(\alpha_i|\alpha_j) \in \mathbb{Q}$ for all i, j. Thus there exists a positive integer d such that $(\alpha_i|\alpha_j) \in \frac{1}{d}\mathbb{Z}$ for all i, j. Thus if $(\alpha|\alpha) > 0$ then $(\alpha|\alpha) \geq \frac{1}{d}$. Hence there exists $m > 0$ such that

$$m = \min\{(\alpha|\alpha)|\alpha \in Q \text{ and } (\alpha|\alpha) > 0\}.$$

Lemma 5.2.1 *Let $\alpha = \sum_i k_i\alpha_i \in Q$.*

(i) *If $(\alpha|\alpha) = m$ then $\pm\alpha \in Q_+$.*
(ii) *If $k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ for all i then $\pm\alpha \in Q_+$.*

Proof If $\pm\alpha \notin Q_+$, then $\alpha = \beta - \gamma$ for $\beta, \gamma \in Q_+$ and $\operatorname{supp}\beta \cap \operatorname{supp}\gamma = \varnothing$. Hence $(\beta|\gamma) \leq 0$ and

$$(\alpha|\alpha) = (\beta|\beta) + (\gamma|\gamma) - 2(\beta|\gamma) \geq (\beta|\beta) + (\gamma|\gamma).$$

All proper principal minors of A have finite type, so, considering connected components $\beta_1, \ldots, \beta_r$ of β we have $(\beta|\beta) = (\beta_1|\beta_1) + \cdots + (\beta_r|\beta_r) > 0$. Hence $(\beta|\beta) \geq m$. Similarly $(\gamma|\gamma) \geq m$. Hence $(\alpha|\alpha) \geq 2m$, which proves (i).

Next,

$$\begin{aligned} \frac{(\beta|\beta)}{(\alpha|\alpha)} &= \frac{1}{(\alpha|\alpha)} \left(\sum_i k_i^2 (\alpha_i|\alpha_j) + \sum_{i<j} 2k_i k_j (\alpha_i|\alpha_j) \right) \\ &= \sum_i k_i \frac{k_i(\alpha_i|\alpha_i)}{(\alpha|\alpha)} + \sum_{i<j} a_{ij} k_j \left(\frac{k_i(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \right) \in \mathbb{Z}. \end{aligned}$$

Since $(\beta|\beta) \neq 0$, it follows that $(\beta|\beta) \geq (\alpha|\alpha)$. Similarly $(\gamma|\gamma) \geq (\alpha|\alpha)$. So $(\alpha|\alpha) \geq (\beta|\beta) + (\gamma|\gamma) \geq 2(\alpha|\alpha)$. This contradiction yields (ii). □

Proposition 5.2.2 *Let A be an indecomposable GCM of finite or affine type. Then*

$$\Delta_s^{\mathrm{re}} = \{\alpha \in Q \mid (\alpha|\alpha) = m\}.$$

Proof Suppose $\alpha \in Q$ satisfies $(\alpha|\alpha) = m$. By the previous lemma, we may assume that $\alpha \in Q_+$. Consider the set

$$\{w(\alpha) \mid w \in W\} \cap Q_+.$$

We choose an element $\beta = \sum k_i \alpha_i$ in this set with $\operatorname{ht}\beta$ minimal. Since $(\beta|\beta) = (\alpha|\alpha) = m$, we have

$$\sum_i k_i (\alpha_i|\beta) = m.$$

Since $k_i \geq 0$ and $m > 0$ there exists i with $(\alpha_i|\beta) > 0$. Then $\langle \beta, \alpha_i^\vee \rangle > 0$. So $r_i(\beta)$ has smaller height than β, whence $s_i(\beta) \in -Q_+$, using the previous lemma. It follows that $\beta = r\alpha_i$ for some positive integer r. Since $(r\alpha_i|r\alpha_i) \geq r^2 m$, we have $r = 1$. Hence $\beta \in \Delta_s^{\mathrm{re}}$ and $\alpha \in \Delta_s^{\mathrm{re}}$ also.

Conversely, if $\alpha \in \Delta_s^{\mathrm{re}}$ then $\alpha = w(\alpha_i)$ for some i and $(\alpha|\alpha) = (\alpha_i|\alpha_i)$. However, we have seen in the previous paragraph that the short simple roots have $(\alpha_i|\alpha_i) = m$, so $(\alpha|\alpha) = m$ also. □

Proposition 5.2.3 *Let A be an indecomposable GCM of finite or affine type, and*

$$M := \max\{(\alpha|\alpha) \mid \alpha \in \Delta^{\mathrm{re}}\}.$$

Then

$$\Delta_l^{\mathrm{re}} = \{\alpha = \sum_i k_i\alpha_i \in Q \mid (\alpha|\alpha) = M,\ k_i \frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z} \textit{ for all } i\}.$$

Proof Let $\alpha \in \Delta_l^{\mathrm{re}}$. Then it satisfies all the required conditions by Proposition 5.1.1(iv)(c). Conversely, suppose $\alpha \in Q$ satisfies the given conditions. Then $\alpha^\vee \in \mathbb{Z}Q^\vee$ by (5.1), and $(\alpha^\vee|\alpha^\vee) = 4/M$ by (5.3). Now $4/M$ is the minimal possible value of β for all $\beta \in (\Delta^\vee)^{\mathrm{re}}$. By Proposition 5.2.2, $\alpha^\vee \in (\Delta^\vee)_s^{\mathrm{re}}$, so $\alpha \in \Delta_l^{\mathrm{re}}$. □

Proposition 5.2.4 *Let $A = A_{2\ell}^{(2)}$ and $m' = (\alpha_i|\alpha_i)$ for $1 \le i < \ell$. Then*

$$\Delta_i^{\mathrm{re}} = \{\alpha \in Q \mid (\alpha|\alpha) = m'\}.$$

Proof Let $\alpha = \sum_{i=0}^{\ell} k_i\alpha_i \in Q$ satisfy $(\alpha|\alpha) = m'$. We first claim that

$$k_i \frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z} \quad \text{for all } i. \tag{5.4}$$

Indeed the condition $k_i \frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ is obvious for $i \neq 0$ since $(\alpha_i|\alpha_i) = m'$ for $i = 1, \dots, \ell - 1$ and $2m'$ for $i = \ell$. It just remains to show that k_0 is even. We have

$$\begin{aligned}
(\alpha|\alpha) &= k_0^2(\alpha_0|\alpha_0) + 2k_0k_1(\alpha_0|\alpha_1) + (\sum_{i=1}^{\ell}\alpha_i|\sum_{i=1}^{\ell}\alpha_i) \\
&= k_0^2(\alpha_0|\alpha_0) + k_0k_1a_{10}(\alpha_1|\alpha_1) + \sum_{i=1}^{\ell} k_i^2(\alpha_i|\alpha_i) \\
&\quad + \sum_{1\le i<j\le \ell} k_ik_ja_{ij}(\alpha_i|\alpha_i).
\end{aligned}$$

Thus $(\alpha|\alpha) \in k_0^2(\alpha_0|\alpha_0) + \mathbb{Z}m'$. But $(\alpha|\alpha) = m$, so $k_0^2(\alpha_0|\alpha_0) \in \mathbb{Z}m'$. Since $(\alpha_0|\alpha_0) = m'/2$, we have $k_0^2/2 \in \mathbb{Z}$, whence k_0 is even as required.

Now Lemma 5.2.1(ii) implies that $\pm\alpha \in Q$. We may assume that $\alpha \in Q$. Consider the set $\{w(\alpha) \mid w \in W\} \cap Q_+$. Let $\beta = \sum_{i=0}^{\ell} k_i'\alpha_i$ be an element of this set with minimal height. Since $(\beta|\beta) = m'$, we have $\sum_{i=0}^{\ell} k_i'(\alpha_i|\beta) = m'$. Now, $m' > 0$ and $k_i' \ge 0$, so there is i with $(\alpha_i|\beta) > 0$, and $\langle\beta, \alpha_i^\vee\rangle = 2\frac{(\alpha_i|\beta)}{(\alpha_i|\alpha_i)} > 0$. So $r_i(\beta) = \beta - \langle\beta, \alpha_i^\vee\rangle\alpha_i$ has smaller height, and $r_i(\beta) \notin Q_+$. But by Lemmas 5.2.1(ii) and 5.1.2, $\pm r_i(\beta) \in Q_+$, so $r_i(\beta) \in -Q_+$. Hence $\beta = k\alpha_i$ for some positive integer k. Thus $m' = (\beta|\beta) = r^2(\alpha_i|\alpha_i)$. However, $(\alpha_i|\alpha_i) \geq m'/2$, so $r = 1$. Hence $\beta = \alpha_i \in \Delta_i^{\mathrm{re}}$. It follows that $\alpha \in \Delta_i^{\mathrm{re}}$ also. □

Proposition 5.2.5

$$\Delta^{\mathrm{re}} = \{\alpha = \sum_i k_i\alpha_i \in Q \mid (\alpha|\alpha) > 0,\ \textit{and } k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z} \textit{ for all } i\}.$$

Proof "$\subset$" is obvious for the short roots, follows from Proposition 5.2.3 for long roots, and from (5.4) for intermediate roots. Conversely, let α be as in the right hand side. Then $w(\alpha) \in \pm Q_+$ for any $w \in W$ be Lemmas 5.1.2 and 5.2.1(ii). We may assume that $\alpha \in Q_+$, and let $\beta = \sum_{i=0}^{\ell} k_i'\alpha_i$ be an element of $\{w(\alpha) \mid w \in W\} \cap Q_+$ with minimal possible height. Since $(\beta|\beta) > 0$, we have $\sum_{i=0}^{\ell} k_i'(\alpha_i|\beta) > 0$. As all $k_i' \geq 0$, there is i with $(\alpha_i|\beta) > 0$, and $\langle\beta, \alpha_i^\vee\rangle = 2\frac{(\alpha_i|\beta)}{(\alpha_i|\alpha_i)} > 0$. So $r_i(\beta) = \beta - \langle\beta, \alpha_i^\vee\rangle\alpha_i$ has smaller height, and $r_i(\beta) \notin Q_+$. But by Lemmas 5.2.1(ii) and 5.1.2, $\pm r_i(\beta) \in Q_+$, so $r_i(\beta) \in -Q_+$. Hence $\beta = k\alpha_i$ for some positive integer k. Thus $m' = (\beta|\beta) = r^2(\alpha_i|\alpha_i)$. So $k_i'\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} = \frac{1}{r}$, whence $r = 1$, and we are done. □

5.3 Imaginary roots

If a root is not real it is called *imaginary.* Denote by Δ^{im} and Δ_+^{im} the sets of the imaginary and positive imaginary roots respectively.

Proposition 5.3.1

(i) *The set Δ_+^{im} is W-invariant.*
(ii) *For $\alpha \in \Delta_+^{\mathrm{im}}$ there exists a unique root $\beta \in -C^\vee$ which is W-conjugate to α.*
(iii) *If A is symmetrizable then the root α is imaginary if and only if $(\alpha|\alpha) \leq 0$.*

Proof (i) As $\Delta_+^{\mathrm{im}} \subset \Delta \setminus \Pi$ and the set $\Pi \setminus \{\alpha_i\}$ is r_i-invariant, it follows that Δ_+^{im} is W-invariant.

(ii) Let $\alpha \in \Delta_+^{\mathrm{im}}$ and β be the element of minimal height in $W \cdot \alpha \subset \Delta_+$. Then $\beta \in -C^\vee$. Indeed, if $\langle\beta, \alpha_i^\vee\rangle > 0$ then $r_i\beta \in \Delta_+$ has smaller height. Uniqueness of β follows from Proposition 3.4.1(ii).

(iii) If $\alpha \in \Delta^{\mathrm{im}}$. Since the form is W-invariant, as in (ii), we may assume that $\alpha = \sum_i k_i\alpha_i \in -C^\vee$ and $k_i \in \mathbb{Z}_+$. Then

$$(\alpha|\alpha) = \sum_i k_i(\alpha|\alpha_i) = \sum_i \frac{1}{2}|\alpha_i|^2\langle \alpha, \alpha_i^\vee\rangle \leq 0.$$

The converse follows from Proposition 5.1.1(iv)(a). □

For $\alpha = \sum_i k_i\alpha_i \in Q$ define the *support* of α, denoted $\operatorname{supp}\alpha$, as the subdiagram of $S(A)$ of the vertices i such that $k_i \neq 0$ and all edges connecting them. By Lemma 2.2.1, $\operatorname{supp}\alpha$ is connected. Set

$$K = \{\alpha \in Q_+ \setminus \{0\} \mid \langle \alpha, \alpha_i^\vee\rangle \leq 0 \text{ for all } i \text{ and } \operatorname{supp}\alpha \text{ is connected}\}.$$

Lemma 5.3.2 $K \subset \Delta_+^{\mathrm{im}}$.

Proof Let $\alpha = \sum_i k_i\alpha_i \in K$. Set

$$\Omega_\alpha = \{\gamma \in \Delta_+ \mid \gamma \leq \alpha\}.$$

The set Ω_α is finite, and it is non-empty since the simple roots appearing in decomposition of α belong to Ω_α. Let $\beta = \sum_i m_i\alpha_i$ be an element of maximal height in Ω_α. Note by definition

$$\beta + \alpha_i \notin \Delta_+ \quad \text{if } k_i > m_i. \tag{5.5}$$

Next,

$$\operatorname{supp}\beta = \operatorname{supp}\alpha.$$

Indeed, if some $i \in \operatorname{supp}\alpha \setminus \operatorname{supp}\beta$, we may assume that $\langle \beta, \alpha_i^\vee\rangle < 0$, whence $\beta + \alpha_i \in \Omega_\alpha$ by Proposition 3.1.5(v), giving a contradiction.

Let A_1 be the principal minor of A corresponding to the subset $\operatorname{supp}\alpha$. If A_1 is of finite type then $\langle \alpha, \alpha_i^\vee\rangle \leq 0$ for all i implies $\alpha = 0$ giving a contradiction (see the argument in the proof of Proposition 4.3.2). If A_1 is not of finite type, then by Proposition 4.3.2(vi),

$$P := \{j \in \operatorname{supp}\alpha \mid k_j = m_j\} \neq \varnothing.$$

We aim to first show that $P = \operatorname{supp}\alpha$, and so $\alpha = \beta \in \Delta_+$. Let R be a connected component of subdiagram $\operatorname{supp}\alpha \setminus P$. By (5.5) and Proposition 3.1.5(v),

$$\langle \beta, \alpha_i^\vee\rangle \geq 0 \quad \text{for all } i \in R. \tag{5.6}$$

Set $\beta' = \sum_{i \in R} m_i \alpha_i$. Then

$$\langle \beta', \alpha_i^\vee \rangle = \langle \beta, \alpha_i^\vee \rangle - \sum_{j \in \operatorname{supp} \alpha \setminus R} m_j a_{ij}.$$

Now (5.6) implies $\langle \beta', \alpha_i^\vee \rangle \geq 0$ for all $i \in R$ and $\langle \beta', \alpha_j^\vee \rangle > 0$ for some $j \in R$.

Let A_R be the principal minor corresponding to the subset R, and u be the column vector with entries m_j, $j \in R$. Since

$$\langle \beta', \alpha_i^\vee \rangle = \sum_{j \in R} a_{ij} m_j \qquad (i \in R),$$

we have $u > 0$, $A_M u \geq 0$, and $A_M u \neq 0$. It follows that A_M is not affine or indefinite type, hence it is finite type. Now let

$$\alpha' = \sum_{i \in R} (k_i - m_i) \alpha_i.$$

We have $k_i - m_i > 0$ for all $i \in R$, and $\alpha - \beta = \sum_{i \in \operatorname{supp} \alpha \setminus P} (k_i - m_i) \alpha_i$. Thus for $i \in R$ we have

$$\langle \alpha - \beta, \alpha_i^\vee \rangle = \sum_{j \in \operatorname{supp} \alpha \setminus P} (k_j - m_j) a_{ij} = \sum_{j \in R} (k_i - m_i) a_{ij} = \langle \alpha', \alpha_i^\vee \rangle,$$

since R is a connected component of $\operatorname{supp} \alpha \setminus P$. Thus

$$\langle \alpha', \alpha_i^\vee \rangle = \langle \alpha, \alpha_i^\vee \rangle - \langle \beta, \alpha_i^\vee \rangle \qquad (i \in R).$$

Now $\langle \alpha, \alpha_i^\vee \rangle \leq 0$ since $\alpha \in K$ and $\langle \beta, \alpha_i^\vee \rangle \geq 0$ by (5.6), so $\langle \alpha', \alpha_i^\vee \rangle \leq 0$ for all $i \in R$. Now let u be the column vector with coordinates $k_i - m_i$ for $i \in R$. Then we have $u > 0$ and $A_M u \leq 0$. Since A_M has finite type $A_M(-u) \geq 0$ implies $-u > 0$ or $-u = 0$, giving a contradiction. This completes the proof of the fact that $\alpha \in \Delta_+$.

Finally, 2α satisfies all the assumptions of the lemma, so $\alpha \in \Delta_+$, and by Proposition 5.1.1(ii), $\alpha \in \Delta_+^{\mathrm{im}}$. □

Theorem 5.3.3 $\Delta_+^{\mathrm{im}} = \bigcup_{w \in W} w(K)$.

Proof "$\supset$" follows from Lemma 5.3.2 and Proposition 5.3.1(i). The converse embedding holds in view of Proposition 5.3.1(i),(ii) and the fact that $\operatorname{supp} \alpha$ is connected for every root α. □

Proposition 5.3.4 *If $\alpha \in \Delta_+^{\mathrm{im}}$ and r a non-zero rational number such that $r\alpha \in Q$, then $r\alpha \in \Delta^{\mathrm{im}}$.*

Proof In view of Proposition 5.3.1(i),(ii) we may assume that $\alpha \in -C^\vee \cap Q_+$. Since α is a root, its support is connected, so $\alpha \in K$. Then $r\alpha \in K$ for any $r > 0$ as in the assumption. By Lemma 5.3.2, $r\alpha \in \Delta_+^{\mathrm{im}}$. □

Theorem 5.3.5 *Let A be indecomposable.*

(i) *If A is finite type then $\Delta^{\mathrm{im}} = \varnothing$.*
(ii) *If A is affine type then $\Delta_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$, where $\delta = \sum_{i=0}^{\ell} a_i\alpha_i$ and a_i are the marks in the Dynkin diagram.*
(iii) *If A is indefinite type then there exists a positive imaginary root $\alpha = \sum_i k_i\alpha_i$ such that $k_i > 0$ and $\langle \alpha, \alpha_i^\vee \rangle < 0$ for all $i = 1, \dots, n$.*

Proof By the definition of types and Remark 4.1.2, the set

$$\{\alpha \in Q_+ \mid \langle \alpha, \alpha_i \rangle \leq 0\}$$

is $\{0\}$ if A is finite type, is $\mathbb{Z}\delta$ is A is affine type, and there exists $\alpha = \sum_i k_i\alpha_i$ such that $k_i > 0$ and $\langle \alpha, \alpha_i^\vee \rangle < 0$ for all $i = 1, \dots, n$, if A is indefinite type. Now apply Theorem 5.3.3. □

We call a root α *null-root* if $\alpha|_{\mathfrak{h}'} = 0$, or equivalently $\langle \alpha, \alpha_i^\vee \rangle = 0$ for all i. It follows from Theorem 4.1.12 that if α is a null-root if and only if $\operatorname{supp}\alpha$ is affine type which represents a connected component of the diagram A and $\alpha = k\delta$ for $k \in \mathbb{Z}$. We call a root α *isotropic* if $(\alpha|\alpha) = 0$.

Proposition 5.3.6 *Let A be symmetrizable. A root α is isotropic if and only if it is W-conjugate to an imaginary root β such that* $\operatorname{supp}\beta$ *is a subdiagram of affine type in $S(A)$.*

Proof Let α be an isotropic root. We may assume that $\alpha > 0$. Then $\alpha \in \Delta_+^{\mathrm{im}}$ by Proposition 5.1.1(iv)(a), and α is W-conjugate to an imaginary root $\beta \in K$ such that $\langle \beta, \alpha_i^\vee \rangle \leq 0$ for all i, thanks to Proposition 5.3.1(ii). Let $\beta = \sum_{i \in P} k_i\alpha_i$ and $P = \operatorname{supp}\beta$. Then

$$(\beta|\beta) = \sum_{i \in P} k_i(\beta|\alpha_i) = 0,$$

where $k_i > 0$ and

$$(\beta|\alpha_i) = \frac{1}{2}|\alpha_i|^2\langle \beta, \alpha_i^\vee \rangle \leq 0$$

for all $i \in P$. So $\langle \beta, \alpha_i^\vee \rangle = 0$ for all $i \in P$, and P is an affine diagram.

Conversely, let $\beta = k\delta$ be an imaginary root for an affine diagram. Then

$$(\beta|\beta) = k^2(\delta|\delta) = k^2 \sum_i a_i(\delta|\alpha_i) = 0$$

since $\langle \delta, \alpha_i^\vee \rangle = 0$ for all i. □

6
Affine Algebras

6.1 Notation

Throughout we use the following notation in the affine case:

- A is an indecomposable GCM of affine type of order $\ell + 1$ and rank ℓ.
- $a_0, a_1, \dots, a_\ell$ are the marks of the diagram $S(A)$ (note that $a_0 = 1$, unless $A = A_{2\ell}^{(2)}$ in which case $a_0 = 2$).
- $a_0^\vee, a_1^\vee, \dots, a_\ell^\vee$ are the marks of the dual diagram $S(A^t)$ (this diagram is obtained from $S(A)$ by changing direction of all arrows and preserving the labels of the vertices). Note that in all cases $a_0^\vee = 1$.
- The numbers

$$h := \sum_{i=0}^{\ell} a_i, \qquad h^\vee := \sum_{i=0}^{\ell} a_i^\vee$$

  are *Coxeter* and *dual Coxeter* numbers.
- $r \in \{1, 2, 3\}$ refers to the number r in the type $X_N^{(r)}$.
- $c = \sum_{i=0}^{\ell} a_i^\vee \alpha_i^\vee$ is the *canonical central element*. By Proposition 1.4.6, the center $\mathfrak{c}$ of $\mathfrak{g}$ is $\mathbb{C}c$.
- $\delta = \sum_{i=0}^{\ell} a_i \alpha_i$. Then $\Delta^{\mathrm{im}} = \{\pm\delta, \pm 2\delta, \dots\}$, $\Delta_+^{\mathrm{im}} = \{\delta, 2\delta, \dots\}$, see Theorem 5.3.5.

6.2 Standard bilinear form

We know that A is symmetrizable. Moreover,

$$A = \mathrm{diag}(\frac{a_0}{a_0^\vee}, \dots, \frac{a_\ell}{a_\ell^\vee})B \tag{6.1}$$

for a symmetric matrix B. Indeed let $\delta = (a_0, \ldots, a_\ell)^t$ and $\delta^\vee = (a_0^\vee, \ldots, a_\ell^\vee)^t$. If $A = DB$ where D is a diagonal invertible matrix and B is a symmetric matrix then $B\delta = 0$, and hence $\delta^t B = 0$. On the other hand, $(\delta^\vee)^t A = 0$ implies $(\delta^\vee)^t DB = 0$, whence $BD\delta^\vee = 0$, and since $\dim \ker B = 1$, we get $D\delta^\vee$ is proportional to δ.

Fix an element $d \in \mathfrak{h}$ such that

$$\langle \alpha_i, d\rangle = 0 \quad \text{for } i = 1, \ldots, \ell, \qquad \langle \alpha_0, d\rangle = 1.$$

d is defined up to a summand proportional to c and is called *energy element*. Note that $\{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee, d\}$ is a basis of $\mathfrak{h}$. Indeed, we must show that d is not a linear combination of $\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_\ell^\vee$. Otherwise $d = \sum_{i=0}^{\ell} k_i \alpha_i^\vee$. Then $\langle \alpha_j, d\rangle = \sum_{i=0}^{\ell} k_i a_{ij}$. Hence

$$\sum_{i=0}^{\ell} k_i (a_{i0}, \ldots, a_{i\ell}) = (1, 0, \ldots, 0).$$

In particular,

$$\sum_{i=0}^{\ell} k_i (a_{i1}, \ldots, a_{i\ell}) = (0, \ldots, 0).$$

However, we also have

$$\sum_{i=0}^{\ell} a_i (a_{i1}, \ldots, a_{i\ell}) = (0, \ldots, 0).$$

Since the $(\ell+1)\times\ell$ matrix $(a_{ij})_{0\le i\le\ell, 1\le j\le\ell}$ has rank ℓ, this implies that $(k_0, \ldots, k_\ell)$ is a scalar multiple of $(a_0, \ldots, a_\ell)$. But this would imply $(k_0, \ldots, k_\ell)A = (0, \ldots, 0)$, giving a contradiction.

Note that

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}d.$$

Define the non-degenerate symmetric bilinear form $(\cdot|\cdot)$ on $\mathfrak{h}$ by

$$\begin{aligned} (\alpha_i^\vee | \alpha_j^\vee) &= \frac{a_j}{a_j^\vee} a_{ij} \qquad (0 \le i, j \le \ell); \\ (\alpha_i^\vee | d) &= \delta_{i,0} a_0 \qquad (0 \le i \le \ell); \\ (d|d) &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} (c|\alpha_i^\vee) &= 0 \qquad (0 \le i \le \ell); \\ (c|c) &= 0; \\ (c|d) &= a_0. \end{aligned}$$

By Theorem 2.2.3, this form can be uniquely extended to $\mathfrak{g}$ so that all conditions of that theorem hold. The extended form $(\cdot|\cdot)$ will be referred to as the *normalized invariant form*.

Next define $\Lambda_0 \in \mathfrak{h}^*$ by

$$\langle \Lambda_0, \alpha_i^\vee \rangle = \delta_{i0}, \quad \langle \Lambda_0, d \rangle = 0.$$

Then

$$\{\alpha_0, \dots, \alpha_1, \Lambda_0\}$$

is a basis of $\mathfrak{h}^*$, since the matrix obtained by applying these elements to $\{\alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_\ell^\vee, d\}$ is

$$\begin{pmatrix} 2 & * & 1 \\ * & \mathring{A} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where $\mathring{A}$ is Cartan matrix of finite type.

The isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ defined by the form $(\cdot|\cdot)$ is given by

$$\nu : \alpha_i^\vee \mapsto \frac{a_i}{a_i^\vee}\alpha_i,$$
$$\nu : d \mapsto a_0\Lambda_0.$$

We also have that

$$\nu : c \mapsto \delta.$$

Indeed, for $j = 0, \dots, \ell$, we have

$$\langle \tfrac{a_i}{a_i^\vee}\alpha_i, \alpha_j^\vee \rangle = \tfrac{a_i}{a_i^\vee} a_{ji} = (\alpha_i^\vee | \alpha_j^\vee),$$
$$\langle \tfrac{a_i}{a_i^\vee}\alpha_i, d \rangle = \tfrac{a_i}{a_i^\vee}\delta_{i0} = \delta_{i0}\tfrac{a_0}{a_0^\vee} = \delta_{i0}a_0 = (\alpha_i^\vee | d),$$

using the fact that $a_0^\vee = 1$. Now $\nu(c) = \nu(\sum a_i^\vee \alpha_i^\vee) = \sum_i a_i\alpha_i = \delta$. The transported form $(\cdot|\cdot)$ on $\mathfrak{h}^*$ has the following properties:

$$\begin{aligned}
(\alpha_i|\alpha_j) &= \frac{a_i^\vee}{a_i} a_{ij} \qquad (0 \le i, j \le \ell); \\
(\alpha_i|\Lambda_0) &= \delta_{i0} a_0^{-1} \qquad (0 \le i \le \ell); \\
(\Lambda_0|\Lambda_0) &= 0; \\
(\delta|\alpha_i) &= 0 \qquad (0 \le i \le \ell); \\
(\delta|\delta) &= 0; \\
(\delta|\Lambda_0) &= 1.
\end{aligned}$$

It follows that there is an isometry of lattices

$$Q^\vee(A) \cong Q(A^t). \tag{6.2}$$

Denote by $\overset{\circ}{\mathfrak{h}}$ (resp. $\overset{\circ}{\mathfrak{h}}_{\mathbb{R}}$) the $\mathbb{C}$-span (resp. $\mathbb{R}$-span) of $\alpha_1^\vee, \dots, \alpha_\ell^\vee$. The dual notions $\overset{\circ}{\mathfrak{h}}{}^*$ and $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$ are defined as similar linear combinations of $\alpha_1, \dots, \alpha_\ell$. Then we have decompositions into orthogonal direct sums

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus (\mathbb{C}c + \mathbb{C}d), \quad \mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}{}^* \oplus (\mathbb{C}\delta + \mathbb{C}\Lambda_0).$$

Set

$$\mathfrak{h}_{\mathbb{R}} := \overset{\circ}{\mathfrak{h}}_{\mathbb{R}} + \mathbb{R}c + \mathbb{R}d, \qquad \mathfrak{h}^*_{\mathbb{R}} = \overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}} + \mathbb{R}\Lambda_0 + \mathbb{R}\delta.$$

By Theorem 4.2.4, the restriction of the bilinear form $(\cdot|\cdot)$ to $\overset{\circ}{\mathfrak{h}}_R$ and $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}}$ (resp. $\overset{\circ}{\mathfrak{h}}_R + \mathbb{R}c$ and $\overset{\circ}{\mathfrak{h}}{}^*_{\mathbb{R}} + \mathbb{R}\delta$) is positive definite (resp. positive semidefinite with kernels $\mathbb{R}c$ and $\mathbb{R}\delta$).

For a subset $S \subset \mathfrak{h}^*$ denote by $\bar{S}$ the orthogonal projection of S onto $\overset{\circ}{\mathfrak{h}}{}^*$. We have

$$\lambda - \bar{\lambda} = \langle \lambda, c \rangle \Lambda_0 + \frac{|\lambda|^2 - |\bar{\lambda}|^2}{2\langle \lambda, c \rangle}\delta \qquad (\lambda \in \mathfrak{h}^*,\ \langle \lambda, c \rangle \neq 0). \tag{6.3}$$

Indeed, $\lambda - \bar{\lambda} = b_1\Lambda_0 + b_2\delta$. Applying $(\cdot|\delta)$, we deduce that $b_1 = (\lambda|\delta) = \langle \lambda, c \rangle$. Now, $|\lambda|^2 = |\bar{\lambda}|^2 + 2b_1b_2$, which implies the required expression for b_2. The following closely related formula is proved similarly:

$$\lambda = \bar{\lambda} + \langle \lambda, c \rangle \Lambda_0 + (\lambda|\Lambda_0)\delta. \tag{6.4}$$

Define $\rho \in \mathfrak{h}^*$ by

$$\langle \rho, d \rangle = 0, \quad \langle \rho, \alpha_i^\vee \rangle = 1 \qquad (0 \le i \le \ell).$$

Then (6.4) gives

$$\rho = \bar{\rho} + h^\vee \Lambda_0. \tag{6.5}$$

6.3 Roots of affine algebra

Denote by $\overset{\circ}{\mathfrak{g}}$ the subalgebra of $\mathfrak{g}$ generated by e_i and f_i for $i = 1, \dots, \ell$. This subalgebra is isomorphic to $\mathfrak{g}(\overset{\circ}{A})$ where $\overset{\circ}{A}$ is obtained from A by removing 0th row and 0th column. This is a finite dimensional simple Lie algebra whose Dynkin diagram comes from $S(A)$ by deleting the 0th vertex, see Proposition 4.3.2.

Indeed, let

$$\overset{\circ}{\Pi}=\{\alpha_1,\dots,\alpha_\ell\},\quad \overset{\circ}{\Pi}{}^\vee=\{\alpha_1^\vee,\dots,\alpha_\ell^\vee\}.$$

Then $\overset{\circ}{\mathfrak{h}},\overset{\circ}{\Pi},\overset{\circ}{\Pi}{}^\vee$ is a realization of $\overset{\circ}{A}$, and since $[e_i,f_i]=\alpha_i^\vee$, $\overset{\circ}{\mathfrak{g}}$ is generated by e_i,f_i for $i=1,\dots,\ell$ and $\overset{\circ}{\mathfrak{h}}$, and the relations (1.12)-(1.15) hold. So there is a homomorphism from $\tilde{\mathfrak{g}}(\overset{\circ}{A})$ onto $\overset{\circ}{\mathfrak{g}}$. We claim that $\overset{\circ}{\mathfrak{g}}$ has no non-trivial ideals which have trivial intersection with $\overset{\circ}{\mathfrak{h}}$. Otherwise, if $\mathfrak{i}$ is such an ideal let $x\in\mathfrak{i}$ be a non-zero element of weight $\alpha\neq 0$. Then $\alpha\in\overset{\circ}{\Delta}$, where

$$\overset{\circ}{\Delta}=\Delta\cap\overset{\circ}{\mathfrak{h}}{}^*. \tag{6.6}$$

We may assume that α is the smallest positive root for which such x exists. Then

$$[f_i,x]=0\qquad(i=1,\dots,\ell).$$

But it is also clear from the relations that $[f_0,x]=0$. By Lemma 1.4.5, $x=0$. This contradiction proves that there is a homomorphism from $\mathfrak{g}(\overset{\circ}{A})$ onto $\overset{\circ}{\mathfrak{g}}$. Since $\mathfrak{g}(\overset{\circ}{A})$ is simple by Proposition 1.4.8(i), this homomorphism must be an isomorphism.

We will also use the notations

$$\overset{\circ}{\Delta}_+:=\overset{\circ}{\Delta}\cap\Delta_+,\quad \overset{\circ}{Q}=\mathbb{Z}\,\overset{\circ}{\Delta},\quad \overset{\circ}{Q}{}^\vee=\mathbb{Z}\overset{\circ}{\Delta}{}^\vee.$$

$\overset{\circ}{\Delta}_s$ and $\overset{\circ}{\Delta}_l$ for the sets of short and long roots in $\overset{\circ}{\Delta}$, respectively, and $\overset{\circ}{W}$ for the Weyl group for $\overset{\circ}{\Delta}$.

Note that $a_0^\vee=1$ implies

$$Q^\vee=\overset{\circ}{Q}{}^\vee\oplus\mathbb{Z}c\qquad\text{(orthogonal direct sum).} \tag{6.7}$$

We denote by Δ_s^{re} and Δ_l^{re} the sets of short and long real roots, respectively. For type $A_{2\ell}^{(2)}$ we denote by Δ_i^{re} the set of real roots of intermediate length.

Proposition 6.3.1

(i) *If* $r=1$ *then* $\Delta^{\mathrm{re}}=\{\alpha+n\delta\mid\alpha\in\overset{\circ}{\Delta},\ n\in\mathbb{Z}\}$, *and* $\alpha+n\delta\in\Delta^{\mathrm{re}}$ *is short if and only if* $\alpha\in\overset{\circ}{\Delta}$ *is short.*

(ii) *If* $r = 2$ *or* 3 *and* $A \neq A_{2\ell}^{(2)}$ *then*

$$\begin{aligned}\Delta_s^{\mathrm{re}} &= \{\alpha + n\delta \mid \alpha \in \mathring{\Delta}_s,\ n \in \mathbb{Z}\},\\ \Delta_l^{\mathrm{re}} &= \{\alpha + nr\delta \mid \alpha \in \mathring{\Delta}_l,\ n \in \mathbb{Z}\}.\end{aligned}$$

(iii) *If* $A = A_{2\ell}^{(2)}$ *for* $\ell > 1$ *then*

$$\begin{aligned}\Delta_s^{\mathrm{re}} &= \{\frac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in \mathring{\Delta}_l,\ n \in \mathbb{Z}\},\\ \Delta_i^{\mathrm{re}} &= \{\alpha + n\delta \mid \alpha \in \mathring{\Delta}_s,\ n \in \mathbb{Z}\},\\ \Delta_l^{\mathrm{re}} &= \{\alpha + 2n\delta \mid \alpha \in \mathring{\Delta}_l,\ n \in \mathbb{Z}\}.\end{aligned}$$

(iv) *If* $A = A_2^{(2)}$ *then*

$$\begin{aligned}\Delta_s^{\mathrm{re}} &= \{\frac{1}{2}(\alpha + (2n-1)\delta) \mid \alpha \in \mathring{\Delta},\ n \in \mathbb{Z}\},\\ \Delta_l^{\mathrm{re}} &= \{\alpha + 2n\delta \mid \alpha \in \mathring{\Delta},\ n \in \mathbb{Z}\}.\end{aligned}$$

(v) $\Delta^{\mathrm{re}} + r\delta = \Delta^{\mathrm{re}}$.

(vi) $\Delta_+^{\mathrm{re}} = \{\alpha \in \Delta^{\mathrm{re}}$ *with* $n > 0\} \cup \mathring{\Delta}_+$.

Proof (v),(vi) follow from (i)-(iv).

Suppose that $A \neq A_{2\ell}^{(2)}$. Then $\mathring{\Delta}_s \subset \Delta_s^{\mathrm{re}}$. Let $\alpha \in \mathring{\Delta}_s$. Then $(\alpha|\alpha) = m$. Hence for $n \in Z$ we have $(\alpha + n\delta|\alpha + n\delta) = m$. By Proposition 5.2.2, $\alpha + n\delta \in \Delta_s^{\mathrm{re}}$.

Conversely, let $\beta = \sum_{i=0}^{\ell} k_i\alpha_i \in \Delta_s^{\mathrm{re}}$. By Proposition 5.2.2,

$$(\beta - k_0\delta|\beta - k_0\delta) = (\beta|\beta) = m.$$

Since $a_0 = 1$, we have $\beta - k_0\delta = \sum_{i=1}^{\ell}(k_i - k_0a_i)\alpha_i$. So by (6.6) and Proposition 5.2.2 again, we deduce that $\beta - k_0\delta \in \mathring{\Delta}_s$, thus the short roots have the required form.

We now consider the long roots. Note that $\mathring{\Delta}_l \subset \Delta_l^{\mathrm{re}}$. Let $\alpha = \sum_{i=1}^{n} k_i\alpha_i \in \mathring{\Delta}_l$. Then $(\alpha + n\delta|\alpha + n\delta) = (\alpha|\alpha) = M$. By Proposition 5.2.3, we have $k_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ for $i = 1, \dots, \ell$. By the same proposition $\alpha + n\delta \in \Delta_l^{\mathrm{re}}$ if and only $na_i\frac{(\alpha_i|\alpha_i)}{(\alpha|\alpha)} \in \mathbb{Z}$ for $i = 0, \dots, \ell$. Now $(\alpha_i|\alpha_i) = \frac{2a_i^\vee}{a_i}$, so the condition is $n\frac{2a_i^\vee}{(\alpha|\alpha)} \in \mathbb{Z}$. Note also that $(\alpha_0|\alpha_0) = 2$.

First suppose that α_0 is a long root, i.e. we are in the case (i). Then $(\alpha|\alpha) = 2$, and so $n\frac{2a_i^\vee}{(\alpha|\alpha)} \in \mathbb{Z}$. Hence $\alpha + n\delta \in \Delta_l^{\mathrm{re}}$ for all $n \in \mathbb{Z}$.

Conversely, let $\beta = \sum_{i=0}^{\ell} k_i\alpha_i \in \Delta_l^{\mathrm{re}}$. By Proposition 5.2.3, $(\beta - k_0\delta|\beta - k_0\delta) = (\beta|\beta) = M$, and $k_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} \in \mathbb{Z}$ for $i = 0, \dots, \ell$. We have $k_0 a_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} \in \mathbb{Z}$ also since $(\alpha_i|\alpha_i) = \frac{2a_i^\vee}{a_i}$, and $(\beta|\beta) = 2$. We now conclude that have $\beta - k_0\delta = \sum_{i=1}^{\ell}(k_i - k_0 a_i)\alpha_i \in \mathring{\Delta}_l$ by (6.6) and Proposition 5.2.3 again, and so the long roots have the required form.

Now suppose that α_0 is a short root, i.e. we are in the case (ii). Note that $r = \frac{(\alpha|\alpha)}{(\alpha_0|\alpha_0)}$. Thus

$$n\frac{2a_i^\vee}{(\alpha|\alpha)} = n\frac{a_i^\vee}{r},$$

since $(\alpha|\alpha) = 2r$. Since $a_0^\vee = 1$ this lies in $\mathbb{Z}$ for all $i = 0, \dots, \ell$ if and only if n is divisible by r. Thus by Proposition 5.2.3, $\alpha + rn\delta \in \Delta_l^{\mathrm{re}}$ for all $n \in \mathbb{Z}$.

Conversely, let $\beta = \sum_{i=0}^{\ell} k_i\alpha_i \in \Delta_l^{\mathrm{re}}$. By Proposition 5.2.3, $(\beta - k_0\delta|\beta - k_0\delta) = (\beta|\beta) = M$, and $k_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} \in \mathbb{Z}$ for $i = 0, \dots, \ell$. In particular, $k_0\frac{(\alpha_0|\alpha_0)}{(\beta|\beta)} = \frac{k_0}{r} \in \mathbb{Z}$. We have

$$k_0 a_i\frac{(\alpha_i|\alpha_i)}{(\beta|\beta)} = a_i^\vee\frac{k_0}{r} \in \mathbb{Z}$$

for $i = 1, \dots, \ell$, as $(\alpha_i|\alpha_i) = \frac{2a_i^\vee}{a_i}$ and $(\beta|\beta) = 2r$. Thus by Proposition 5.2.3, $\beta - k_0\delta = \sum_{i=1}^{\ell}(k_i - k_0 a_i)\alpha_i \in \mathring{\Delta}_l$ by (6.6) and Proposition 5.2.3 again, and so the long roots have the required form.

The proof of (iii) and (iv) is similar. □

Proposition 6.3.1 will also follow from explicit constructions of affine Lie algebras given in the next chapter.

Remark 6.3.2 $\bar{\Delta} = \mathring{\Delta} \setminus \{0\}$ in all cases except $A_{2\ell}^2$ in which case the root system $\bar{\Delta}$ is not reduced, and $\mathring{\Delta}$ is the corresponding reduced root system.

Introduce the element

$$\theta := \delta - a_0\alpha_0 = \sum_{i=1}^{\ell} a_i\alpha_i \in \mathring{Q}\ .$$

We have

$$(\theta|\theta) = (\delta - a_0\alpha_0|\delta - a_0\alpha_0) = a_0^2(\alpha_0|\alpha_0) = 2a_0.$$

Thus $(\theta|\theta) = M$ if $r = 1$ or $A = A_{2\ell}^{(2)}$, and $(\theta|\theta) = m$ otherwise. In all cases it follows from Propositions 5.2.2 and 5.2.3 that $\theta \in \overset{\circ}{\Delta}_+$. Moreover,

$$\theta^\vee = 2\frac{\nu^{-1}(\theta)}{(\theta|\theta)} = \frac{1}{a_0}\nu^{-1}(\theta) = \frac{1}{a_0}\sum_{i=1}^{\ell} a_i^\vee \alpha_i^\vee.$$

$$(\theta^\vee|\theta^\vee) = \frac{2}{a_0},$$

$$\alpha_0^\vee = \nu^{-1}(\delta - \theta) = c - a_0\theta^\vee.$$

Proposition 6.3.3 *If $r = 1$ or $A = A_{2\ell}^{(2)}$, then $\theta \in (\overset{\circ}{\Delta}_+)_l$ and θ is the unique root in $\overset{\circ}{\Delta}$ of maximal height $(= h - a_0)$. Otherwise $\theta \in (\overset{\circ}{\Delta}_+)_s$ and θ is the unique root in $\overset{\circ}{\Delta}_s$ of maximal height $(= h - 1)$.*

Proof First we show that $\theta \in \overset{\circ}{\Delta}$. If $a_0 = 1$ and α_0 is a long root, then $(\theta|\theta) = (\alpha_0|\alpha_0) = 2$. Also $a_i\frac{(\alpha_i|\alpha_i)}{(\theta|\theta)} = a_i^\vee \in \mathbb{Z}$, and by Proposition 5.2.3, $\theta \in \overset{\circ}{\Delta}_l$. If $a_0 = 2$ then $(\theta|\theta) = 4 = M$. Also, $a_i\frac{(\alpha_i|\alpha_i)}{(\theta|\theta)} = \frac{a_i^\vee}{2} \in \mathbb{Z}$ for $i = 1, \ldots, \ell$, and by Proposition 5.2.3 again, $\theta \in \overset{\circ}{\Delta}_l$. Finally, if $a_0 = 1$ and α_0 is a short root, then $(\theta|\theta) = m$, and $\theta \in \overset{\circ}{\Delta}_s$ by Proposition 5.2.2.

One checks that all simple roots in $\overset{\circ}{\Delta}$ of the same length are $\overset{\circ}{W}$-conjugate (this is essentially a type A_2 argument). Hence $\overset{\circ}{\Delta}_s$ and $\overset{\circ}{\Delta}_l$ are the orbits of $\overset{\circ}{W}$ on $\overset{\circ}{\Delta}$. Moreover,

$$\langle \theta, \alpha_i^\vee \rangle = \langle \delta - a_0\alpha_0, \alpha_i^\vee \rangle = -a_0 a_{i0} \geq 0 \qquad (1 \leq i \leq \ell).$$

Hence θ is in the fundamental domain of $\overset{\circ}{W}$, which determines the short or long root uniquely. The height of θ is easy to compute from the definition. Finally, θ is the only root of its length in $\overset{\circ}{\Delta}$ such that $\theta + \alpha_i$ is not a root for all i, hence it is the only root of maximal height of its length. □

If A is a matrix of *finite* type, we assume that the standard invariant form $(\cdot|\cdot)$ on $\mathfrak{g}(A)$ is normalized by the condition $(\alpha|\alpha) = 2$ for $\alpha \in \Delta_l$, and call it the *normalized invariant form*.

Corollary 6.3.4 *Let $\mathfrak{g}$ be an affine algebra of type $X_N^{(r)}$. Then the ratio of the restriction to the subalgebra $\overset{\circ}{\mathfrak{g}}$ of the normalized invariant on $\mathfrak{g}$ to the normalized invariant form on $\overset{\circ}{\mathfrak{g}}$ is equal to r.*

6.4 Affine Weyl Group

Since $\langle \delta, \alpha_i^\vee \rangle = 0$ for all i, we have $w(\delta) = \delta$ for all $w \in W$. Denote by $\overset{\circ}{W}$ the subgroup of W generated by $r_1, \dots, r_\ell$. As $r_i(\Lambda_0) = \Lambda_0$ for $i \geq 1$, $\overset{\circ}{W}$ acts trivially on $\mathbb{C}\Lambda_0 + \mathbb{C}\delta$. It is also clear that $\overset{\circ}{\mathfrak{h}}{}^*$ is $\overset{\circ}{W}$-invariant. So the action of $\overset{\circ}{W}$ on $\overset{\circ}{\mathfrak{h}}{}^*$ is faithful, and we can identify $\overset{\circ}{W}$ with the Weyl group of of $\overset{\circ}{\mathfrak{g}}$ also acting on $\overset{\circ}{\mathfrak{h}}{}^*$. Hence $\overset{\circ}{W}$ is finite.

We have

$$r_0 r_\theta(\lambda) = \lambda + \langle \lambda, c\rangle \nu(\theta^\vee) - \big(\langle \lambda, \theta^\vee \rangle + \frac{1}{2}(\theta^\vee|\theta^\vee)\langle \lambda, c\rangle\big)\delta. \tag{6.8}$$

Indeed,

$$\begin{aligned}
r_0 r_\theta(\lambda) &= r_0(\lambda - \langle \lambda, \theta^\vee\rangle\theta) \\
&= \lambda - \langle \lambda, \alpha_0^\vee\rangle\alpha_0 - \langle \lambda, \theta^\vee\rangle(\theta - \langle \theta, \alpha_0^\vee\rangle\alpha_0) \\
&= \lambda - \langle \lambda, \alpha_0^\vee\rangle\frac{1}{a_0}(\delta - \theta) - \langle \lambda, \theta^\vee\rangle\theta + \langle \lambda, \theta^\vee\rangle\langle \theta, \alpha_0^\vee\rangle\frac{1}{a_0}(\delta - \theta) \\
&= \lambda + \Big(\frac{\langle \lambda, \alpha_0^\vee\rangle}{a_0} - \langle \lambda, \theta^\vee\rangle - \frac{\langle \lambda, \theta^\vee\rangle\langle \theta, \alpha_0^\vee\rangle}{a_0}\Big)\theta \\
&\quad + \Big(-\frac{\langle \lambda, \alpha_0^\vee\rangle}{a_0} + \frac{\langle \lambda, \theta^\vee\rangle\langle \theta, \alpha_0^\vee\rangle}{a_0}\Big)\delta \\
&= \lambda + (\langle \lambda, \alpha_0^\vee\rangle - a_0\langle \lambda, \theta^\vee\rangle - \langle \lambda, \theta^\vee\rangle\langle \theta, \alpha_0^\vee\rangle)\frac{1}{a_0}\theta \\
&\quad - \Big(\frac{\langle \lambda, \alpha_0^\vee\rangle}{a_0} - \frac{\langle \lambda, \theta^\vee\rangle\langle \theta, \alpha_0^\vee\rangle}{a_0}\Big)\delta \\
&= \lambda + (\langle \lambda, \alpha_0^\vee\rangle - a_0\langle \lambda, \theta^\vee\rangle - \langle \lambda, \theta^\vee\rangle\langle \theta, \alpha_0^\vee\rangle)\frac{1}{a_0}\theta \\
&\quad - \Big(\frac{\langle \lambda, \alpha_0^\vee\rangle}{a_0} - \frac{\langle \lambda, \theta^\vee\rangle\langle \theta, \alpha_0^\vee\rangle}{a_0}\Big)\delta \\
&= \lambda + (\langle \lambda, c - a_0\theta^\vee\rangle - a_0\langle \lambda, \theta^\vee\rangle - \langle \lambda, \theta^\vee\rangle\langle \theta, c - a_0\theta^\vee\rangle)\nu(\theta^\vee) \\
&\quad - \Big(\frac{\langle \lambda, c - a_0\theta^\vee\rangle}{a_0} - \frac{\langle \lambda, \theta^\vee\rangle\langle \theta, c - a_0\theta^\vee\rangle}{a_0}\Big)\delta,
\end{aligned}$$

which easily implies (6.8).

Bibliography

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