REPRESENTATIONS OF SHIFTED YANGIANS

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Abstract. We study highest weight representations of shifted Yangians over an algebraically closed field of characteristic 0. In particular, we classify the finite dimensional irreducible representations and explain how to compute their Gelfand-Tsetlin characters in terms of known characters of standard modules and certain Kazhdan-Lusztig polynomials. Our approach exploits the relationship between shifted Yangians and the finite W-algebras associated to nilpotent orbits in general linear Lie algebras. At the end of the introduction, we formulate some conjectures.

1. Introduction

Following work of Premet, there has been renewed interest recently in the representation theory of certain algebras that are associated to nilpotent orbits in complex semisimple Lie algebras. We refer to these algebras as *finite W-algebras*. They should be viewed as analogues of universal enveloping algebras for the Slodowy slice through the nilpotent orbit in question. Actually, in the special cases considered in this article, the definition of these algebras first appeared in 1979 in the Ph.D. thesis of Lynch [Ly], extending the celebrated work of Kostant [Ko2] treating regular nilpotent orbits. However, despite quite a lot of attention by a number of authors since then, see e.g. [Ka, M, Ma, BT, VD, GG, P1, P2], there is still surprisingly little concrete information about the representation theory of these algebras to be found in the literature. The goal in this article is to undertake a thorough study of finite dimensional representations of the finite W-algebras associated to nilpotent orbits in the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$. We are able to make progress in this case thanks largely to the relationship between finite W-algebras and *shifted Yangians* first noticed in [RS, BR] and developed in full generality in [BK5].

Fix for the remainder of the introduction a partition $\lambda = (p_1 \leq \cdots \leq p_n)$ of N. We draw the Young diagram of λ in a slightly unconventional way, so that there are p_i boxes in the ith row, numbering rows $1, \dots, n$ from top to bottom in order of

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increasing length. Also number the non-empty columns of this diagram by $1, \dots, l$ from left to right, and let q_i denote the number of boxes in the ith column, so $\lambda' = (q_1 \geq \cdots \geq q_l)$ is the transpose partition to λ. For example, if $(p_1, p_2, p_3) = (2, 3, 4)$ then the Young diagram of λ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 4 \\ \cline{1-3} 2 & 5 & 7 \\ \hline 3 & 6 & 8 & 9 \\ \hline \end{array}$$

and $(q_1, q_2, q_3, q_4) = (3, 3, 2, 1)$. We number the boxes of the diagram by $1, 2, \dots, N$ down columns from left to right, and let $\mathrm{row}(i)$ and $\mathrm{col}(i)$ denote the row and column numbers of the ith box.

Writing $e_{i,j}$ for the ij-matrix unit in the Lie algebra $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{C})$, let e denote the matrix $\sum_{i,j} e_{i,j}$ summing over all $1 \leq i, j \leq N$ such that $\mathrm{row}(i) = \mathrm{row}(j)$ and $\mathrm{col}(i) = \mathrm{col}(j) - 1$. This is a nilpotent matrix of Jordan type λ. For instance, if λ is as above, then $e = e_{1,4} + e_{2,5} + e_{5,7} + e_{3,6} + e_{6,8} + e_{8,9}$. Define a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ of the Lie algebra $\mathfrak{g}$ by declaring that each $e_{i,j}$ is of degree $(\mathrm{col}(j) - \mathrm{col}(i))$. This is a *good grading* for $e \in \mathfrak{g}_1$ in the sense of [KRW] (see also [EK] for the full classification). However, it is not the usual Dynkin grading arising from an $\mathfrak{sl}_2$-triple unless all the parts of λ are equal. Actually, in the main body of the article, we work with more general good gradings than the one described here, replacing the Young diagram of λ with a more general diagram called a *pyramid* and denoted by the symbol π; see §3.1. When the pyramid π is left-justified, it coincides with the Young diagram of λ. We have chosen to focus just on this case in the introduction, since it plays a distinguished role in the theory.

Now we give a formal definition of the finite W-algebra $W(\lambda)$ associated to this data. Let $\mathfrak{p}$ denote the parabolic subalgebra $\bigoplus_{j \geq 0} \mathfrak{g}_j$ of $\mathfrak{g}$ with Levi factor $\mathfrak{h} = \mathfrak{g}_0$, and let $\mathfrak{m}$ denote the opposite nilradical $\bigoplus_{j<0} \mathfrak{g}_j$. Taking the trace form with e defines a one dimensional representation $\chi : \mathfrak{m} \to \mathbb{C}$. Let $\mathrm{pr}_\chi : U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{p})$ denote the projection along the direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{p}) \oplus U(\mathfrak{g})I_\chi$, where $U(\mathfrak{g})I_\chi$ is the left ideal of $U(\mathfrak{g})$ generated by the elements $\{x - \chi(x) \,|\, x \in \mathfrak{m}\}$. Define a twisted action of $x \in \mathfrak{m}$ on $U(\mathfrak{p})$ by setting $x \cdot u := \mathrm{pr}_\chi([x, u])$ for all $u \in U(\mathfrak{p})$. Then, $W(\lambda)$ is the subalgebra $U(\mathfrak{p})^{\mathfrak{m}}$ of all twisted $\mathfrak{m}$-invariants in $U(\mathfrak{p})$; see §3.2. For example, if the Young diagram of λ consists of a single column and e is the zero matrix, $W(\lambda)$ coincides with the entire universal enveloping algebra $U(\mathfrak{g})$. At the other extreme, if the Young diagram of λ consists of a single row and e is a regular nilpotent element, the work of Kostant [Ko2] shows that $W(\lambda)$ is isomorphic to the center of $U(\mathfrak{g})$, in particular it is commutative.

Let $\xi : U(\mathfrak{p}) \to U(\mathfrak{h})$ be the algebra homomorphism induced by the natural projection $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$. Let $\eta : U(\mathfrak{h}) \to U(\mathfrak{h})$ be the automorphism mapping a matrix unit $e_{i,j} \in \mathfrak{h}$ to $e_{i,j} + \delta_{i,j}(q_{\mathrm{col}(i)+1} + \cdots + q_l)$; this simply provides a convenient renormalization of the generators. The restriction of the composition $\eta \circ \xi$ to $W(\lambda)$ defines an *injective* algebra homomorphism $\mu : W(\lambda) \hookrightarrow U(\mathfrak{h})$ which we call the *Miura transform*; see §3.6. To explain its importance in the theory, we note that $\mathfrak{h} = \mathfrak{gl}_{q_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{q_l}(\mathbb{C})$, so $U(\mathfrak{h})$ is naturally identified with the tensor product $U(\mathfrak{gl}_{q_1}(\mathbb{C})) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l}(\mathbb{C}))$. Given $\mathfrak{gl}_{q_k}(\mathbb{C})$-modules M_k for each $k = 1, \dots, l$, the outer tensor product $M_1 \boxtimes \cdots \boxtimes M_l$

is therefore a $U(\mathfrak{h})$-module in the natural way. Hence, via the Miura transform, $M_1 \boxtimes \cdots \boxtimes M_l$ is a $W(\lambda)$-module too. This construction plays the role of tensor product in the representation theory of $W(\lambda)$.

Next we want to recall the connection between $W(\lambda)$ and shifted Yangians. Let σ be the upper triangular $n \times n$ matrix with ij-entry $(p_j - p_i)$ for each $1 \leq i \leq j \leq n$. The *shifted Yangian* $Y_n(\sigma)$ associated to σ is the associative algebra over $\mathbb{C}$ with generators $D_i^{(r)}$ $(1 \leq i \leq n, r > 0), E_i^{(r)}$ $(1 \leq i < n, r > p_{i+1} - p_i)$ and $F_i^{(r)}$ $(1 \leq i < n, r > 0)$ subject to certain relations recorded explicitly in §2.1. In the case that σ is the zero matrix, i.e. all parts of λ are equal, $Y_n(\sigma)$ is precisely the usual Yangian Y_n associated to the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$ and the defining relations are a variation on the Drinfeld presentation of [D]; see [BK4]. In general, the presentation of $Y_n(\sigma)$ is adapted to its natural *triangular decomposition*, allowing us to study representations in terms of highest weight theory. In particular, the subalgebra generated by all the elements $D_i^{(r)}$ is a maximal commutative subalgebra which we call the *Gelfand-Tsetlin subalgebra*. We often work with the generating functions

$$D_i(u) = 1 + D_i^{(1)} u^{-1} + D_i^{(2)} u^{-2} + \cdots \in Y_n(\sigma)[[u^{-1}]].$$

The main result of [BK5] shows that the finite W-algebra $W(\lambda)$ is isomorphic to the quotient of $Y_n(\sigma)$ by the two-sided ideal generated by all $D_1^{(r)}$ $(r > p_1)$. A precise identification of $W(\lambda)$ with this quotient is explained in §3.4, using explicit formulae for invariants in $U(\mathfrak{p})^{\mathfrak{m}}$. Under this identification, the tensor product construction defined using the Miura transform in the previous paragraph is induced by the comultiplication on the Hopf algebra Y_n; see §2.5.

We are ready to describe the first results about representation theory. We call a vector v in a $Y_n(\sigma)$-module M a *highest weight vector* if it is annihilated by all $E_i^{(r)}$ and each $D_i^{(r)}$ acts on v by a scalar. A critical point is that if v is a highest weight vector in a $W(\lambda)$-module, viewed as a $Y_n(\sigma)$-module via the map $Y_n(\sigma) \twoheadrightarrow W(\lambda)$, then in fact $D_i^{(r)} v = 0$ for all $r > p_i$. This is obvious for $i = 1$, since the image of $D_1^{(r)}$ in $W(\lambda)$ is zero by the definition of the map for all $r > p_1$. For $i > 1$, it follows from the following fundamental result proved in §3.7: for any i and $r > p_i$, the image of $D_i^{(r)}$ in $W(\lambda)$ is congruent to zero modulo the left ideal generated by all $E_j^{(s)}$. Hence, if v is a highest weight vector in a $W(\lambda)$-module, then there exist scalars $(a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq p_i}$ such that

$$\begin{aligned} u^{p_1} D_1(u) v &= (u + a_{1,1})(u + a_{1,2}) \cdots (u + a_{1,p_1}) v, \\ (u-1)^{p_2} D_2(u-1) v &= (u + a_{2,1})(u + a_{2,2}) \cdots (u + a_{2,p_2}) v, \\ &\vdots \\ (u-n+1)^{p_n} D_n(u-n+1) v &= (u + a_{n,1})(u + a_{n,2}) \cdots (u + a_{n,p_n}) v. \end{aligned}$$

Let A be the λ-tableau obtained by writing the scalars $a_{i,1}, \dots, a_{i,p_i}$ into the boxes on the ith row of the Young diagram of λ. In this way, the set of highest weights that can arise in $W(\lambda)$-modules are parametrized by the set $\mathrm{Row}(\lambda)$ of *row symmetrized λ-tableaux*, i.e. tableaux of shape λ with entries from $\mathbb{C}$ viewed up to row equivalence. Conversely, given any row symmetrized λ-tableau $A \in \mathrm{Row}(\lambda)$, there exists a (non-zero) universal highest weight module $M(A) \in W(\lambda)$-mod generated by such a highest weight vector; see §6.1. We call $M(A)$ the *Verma module* of type A. By familiar arguments, $M(A)$ has a unique irreducible quotient $L(A)$. The modules $\{L(A) \mid A \in \mathrm{Row}(\lambda)\}$ are all irreducible highest weight modules for $W(\lambda)$ up to isomorphism.

There is a natural abelian category $\mathcal{M}(\lambda)$ which is an analogue of the BGG category $\mathcal{O}$ for the algebra $W(\lambda)$; see §7.5. (Actually, $\mathcal{M}(\lambda)$ is more like the category $\mathcal{O}^\infty$ obtained by weakening the hypothesis that a Cartan subalgebra acts semisimply in the usual definition of $\mathcal{O}$.) All objects in $\mathcal{M}(\lambda)$ are of finite length, and the simple objects are precisely the irreducible highest weight modules, hence the isomorphism classes $\{[L(A)] \mid A \in \mathrm{Row}(\lambda)\}$ give a canonical basis for the Grothendieck group $[\mathcal{M}(\lambda)]$ of the category $\mathcal{M}(\lambda)$. Verma modules belong to $\mathcal{M}(\lambda)$, and it is natural to consider the *composition multiplicities* $[M(A) : L(B)]$ for $A, B \in \mathrm{Row}(\lambda)$. We will formulate a precise combinatorial conjecture for these, in the spirit of the Kazhdan-Lusztig conjecture, later on in the introduction. For now, we just record the following basic result about the structure of Verma modules; see §6.3. For the statement, let $\leq$ denote the Bruhat ordering on row symmetrized λ-tableaux; see §4.1.

Theorem A (Linkage principle). *For $A, B \in \mathrm{Row}(\lambda)$, the composition multiplicity $[M(A) : L(A)]$ is equal to 1, and $[M(A) : L(B)] \neq 0$ if and only if $B \leq A$ in the Bruhat ordering.*

Hence, $\{[M(A)] \mid A \in \mathrm{Row}(\lambda)\}$ is another natural basis for the Grothendieck group $[\mathcal{M}(\lambda)]$. We want to say a few words about the proof of Theorem A, since it involves an important technique. Modules in the category $\mathcal{M}(\lambda)$ possess *Gelfand-Tsetlin characters*; see §5.2. This is a formal notion that keeps track of the dimensions of the generalized weight space decomposition of a module with respect to the Gelfand-Tsetlin subalgebra of $Y_n(\sigma)$, similar in spirit to the q-characters of Frenkel and Reshetikhin [FR]. The map sending a module to its Gelfand-Tsetlin character induces an embedding of the Grothendieck group $[\mathcal{M}(\lambda)]$ into a certain completion of the ring of Laurent polynomials $\mathbb{Z}[y_{i,a}^{\pm 1} \mid i = 1, \dots, n, a \in \mathbb{C}]$, for indeterminates $y_{i,a}$. The key step in our proof of Theorem A is the computation of the Gelfand-Tsetlin character of the Verma module $M(A)$ itself; see §6.2 for the precise statement. In general, $\mathrm{ch}\, M(A)$ is an infinite sum of monomials in the $y_{i,a}^{\pm 1}$'s involving both positive and negative powers, but the highest weight vector of $M(A)$ contributes just the positive monomial

$$y_{1,a_{1,1}} y_{1,a_{1,2}} \cdots y_{1,a_{1,p_1}} y_{2,a_{2,1}} y_{2,a_{2,2}} \cdots y_{2,a_{2,p_2}} \cdots y_{n,a_{n,1}} y_{n,a_{n,2}} \cdots y_{n,a_{n,p_n}},$$

where $a_{i,1}, \dots, a_{i,p_i}$ are the entries in the ith row of A as above. The highest weight vector of any composition factor contributes a similar such positive monomial. So by analyzing the positive monomials appearing in the formula for $\mathrm{ch}\, M(A)$, we get information about the possible $L(B)$'s that can be composition factors of $M(A)$. The Bruhat ordering on tableaux emerges naturally out of these considerations.

Another important property of Verma modules has to do with tensor products. Let $A \in \mathrm{Row}(\lambda)$ be a row equivalence class of λ-tableaux. Pick a representative for A and let A_k denote its kth column with entries $a_{k,1}, \dots, a_{k,q_k}$ read from top to bottom, for each $k = 1, \dots, l$. Let $M(A_k)$ denote the Verma module for the Lie algebra $\mathfrak{gl}_{q_k}(\mathbb{C})$ generated by a highest weight vector m_+ which is annihilated by all strictly upper triangular matrices and on which $e_{i,i}$ acts as the scalar $(a_{k,i}+i-1)$ for each $1 \leq i \leq q_k$. Via the Miura transform, the tensor product $M(A_1) \boxtimes \cdots \boxtimes M(A_l)$ is then naturally a $W(\lambda)$-module as explained above, and the vector $m_+ \otimes \cdots \otimes m_+$ is an obvious highest weight vector in this tensor product of highest weight A. In fact, our formula for the Gelfand-Tsetlin character of $M(A)$ implies that

$$[M(A)] = [M(A_1) \boxtimes \cdots \boxtimes M(A_l)],$$

equality in the Grothendieck group $[\mathcal{M}(\lambda)]$. Generically, $M(A)$ is irreducible, hence it is actually isomorphic to $M(A_1) \boxtimes \cdots \boxtimes M(A_l)$ as a module. But there are examples when, although equal in the Grothendieck group, $M(A)$ is never isomorphic to $M(A_1) \boxtimes \cdots \boxtimes M(A_l)$, for any choice of representative for A. The first part of next theorem, proved in §6.4, is a consequence of this equality in the Grothendieck group; the second part is an application of [FO].

Theorem B (Structure of center). *The restriction of the linear map* $\mathrm{pr}_\chi : U(\mathfrak{g}) \to U(\mathfrak{p})$ *defines an algebra isomorphism* $Z(U(\mathfrak{g})) \to Z(W(\lambda))$ *between the center of* $U(\mathfrak{g})$ *and the center of* $W(\lambda)$. *Moreover,* $W(\lambda)$ *is free as a module over its center.*

Now we switch our attention to finite dimensional $W(\lambda)$-modules. Let $\mathcal{F}(\lambda)$ denote the category of all finite dimensional $W(\lambda)$-module, viewed as a subcategory of the category $\mathcal{M}(\lambda)$. The problem of classifying all finite dimensional irreducible $W(\lambda)$-modules reduces to determining precisely which $A \in \mathrm{Row}(\lambda)$ have the property that $L(A)$ is finite dimensional. To formulate the final result, we need one more definition. Call a λ-tableau A with entries in $\mathbb{C}$ *column strict* if in every column the entries belong to the same coset of $\mathbb{C}$ modulo $\mathbb{Z}$ and are strictly increasing from bottom to top. Let $\mathrm{Col}(\lambda)$ denote the set of all such column strict λ-tableaux. There is an obvious map

$$R : \mathrm{Col}(\lambda) \to \mathrm{Row}(\lambda)$$

mapping a λ-tableau to its row equivalence class. Let $\mathrm{Dom}(\lambda)$ denote the image of this map, the set of all *dominant* row symmetrized λ-tableaux.

Theorem C (Finite dimensional irreducible representations). *For* $A \in \mathrm{Row}(\lambda)$*, the irreducible highest weight module* $L(A)$ *is finite dimensional if and only if* A *is dominant. Hence, the modules* $\{L(A) \mid A \in \mathrm{Dom}(\lambda)\}$ *form a complete set of pairwise non-isomorphic finite dimensional irreducible* $W(\lambda)$*-modules.*

To prove this, there are two steps: one needs to show first that each $L(A)$ with $A \in \mathrm{Dom}(\lambda)$ is finite dimensional, and second that all other $L(A)$'s are infinite dimensional. Let us explain the argument for the first step. Given $A \in \mathrm{Col}(\lambda)$ and $k = 1, \dots, l$, let A_k denote its kth column and let $L(A_k)$ denote the irreducible highest weight module for the Lie algebra $\mathfrak{gl}_{q_k}(\mathbb{C})$ of the same highest weight as the Verma module $M(A_k)$ introduced above. Because A is column strict, each $L(A_k)$ is finite dimensional. Hence we obtain a finite dimensional $W(\lambda)$-module

$$V(A) = L(A_1) \boxtimes \cdots \boxtimes L(A_l),$$

which we call the *standard module* corresponding to $A \in \mathrm{Col}(\lambda)$. It contains an obvious highest weight vector of highest weight equal to the image of A in $\mathrm{Row}(\lambda)$. This simple construction is enough to finish the first step of the proof. The second step is actually much harder, and is an extension of the proof due to Tarasov [T1, T2] and Drinfeld [D] of the classification of finite dimensional irreducible representations of the Yangian Y_n by *Drinfeld polynomials*. It is based on the remarkable fact that when $n = 2$, i.e. the Young diagram of λ has just two rows, *every* $L(A)\,(A \in \mathrm{Row}(\lambda))$ can be expressed as an irreducible tensor product; see §7.1.

Amongst all the standard modules, there are some special ones which are highest weight modules, and whose isomorphism classes form a basis for the Grothendieck group of the category $\mathcal{F}(\lambda)$. To formulate the key result here, we need one more combinatorial definition. Let $A \in \mathrm{Col}(\lambda)$ be a column strict λ-tableau with entries $a_{i,1}, \dots, a_{i,p_i}$ in its ith row read from left to right. We say that A is *standard* if $a_{i,j} \leq a_{i,k}$ for every $1 \leq i \leq n$ and $1 \leq j < k \leq p_i$ such that $a_{i,j}$ and $a_{i,k}$ belong to the same coset of $\mathbb{C}$ modulo $\mathbb{Z}$. If all entries of A are integers, this is the usual definition of a standard tableau: entries increase strictly up columns and weakly along rows. Our proof of the next theorem is based on an argument due to Chari [C2] in the context of quantum affine algebras; see §7.3.

Theorem D (Highest weight standard modules). *If $A \in \mathrm{Col}(\lambda)$ is standard, then the standard module $V(A)$ is a highest weight module of highest weight equal to the row equivalence class of A.*

Most of the results so far are analogous to well known results in the representation theory of Yangians and quantum affine algebras, and do not exploit the finite W-algebra side of the picture in any significant way. To remedy this, we need to apply *Skryabin's theorem* from [Sk]; see §8.1. This gives an explicit equivalence of categories between the category of all $W(\lambda)$-modules and the category $\mathcal{W}(\lambda)$ of all *generalized Whittaker modules*, namely, all $\mathfrak{g}$-modules on which $(x - \chi(x))$ acts locally nilpotently for all $x \in \mathfrak{m}$. For any finite dimensional $\mathfrak{g}$-module V, it is obvious that the functor $? \otimes V$ maps objects in $\mathcal{W}(\lambda)$ to objects in $\mathcal{W}(\lambda)$. Transporting through Skryabin's equivalence of categories, we obtain a functor $? \circledast V$ on $W(\lambda)$-mod itself; see §8.2. In this way, one can introduce *translation functors* on the categories $\mathcal{M}(\lambda)$ and $\mathcal{F}(\lambda)$. Actually, we just need some special translation functors, peculiar to the type A theory and denoted e_i, f_i for $i \in \mathbb{C}$, which arise from $\circledast$'ing with the natural $\mathfrak{gl}_N(\mathbb{C})$-module and its dual. These functors fit into the axiomatic framework developed by Chuang and Rouquier [CR]; see §8.3.

Now recall the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with Levi factor $\mathfrak{h}$. We let $\mathcal{O}(\lambda)$ denote the corresponding parabolic category $\mathcal{O}$, the category of all finitely generated $\mathfrak{g}$-modules on which $\mathfrak{p}$ acts locally finitely and $\mathfrak{h}$ acts semisimply. For $A \in \mathrm{Col}(\lambda)$ with entry a_i in its ith box, we let $N(A) \in \mathcal{O}(\lambda)$ denote the *parabolic Verma module* generated by a highest weight vector m_+ that is annihilated by all strictly upper triangular matrices in $\mathfrak{g}$ and on which $e_{i,i}$ acts as the scalar $(a_i + i - 1)$ for each $i = 1, \dots, N$. Let $K(A)$ denote the unique irreducible quotient of $N(A)$. Both of the sets $\{[N(A)] \mid A \in \mathrm{Col}(\lambda)\}$ and $\{[K(A)] \mid A \in \mathrm{Col}(\lambda)\}$ form natural bases for the

Grothendieck group $[\mathcal{O}(\lambda)]$. There is a remarkable functor

$$F : \mathcal{O}(\lambda) \to \mathcal{F}(\lambda)$$

which is essentially the same as the Whittaker functor studied by Kostant and Lynch; see §8.4. It is an exact functor preserving central characters and commuting with translation functors. Moreover, it maps the parabolic Verma module $N(A)$ to the standard module $V(A)$ for every $A \in \mathrm{Col}(\lambda)$. The culmination of this article is the following theorem.

Theorem E (Construction of irreducible modules). *The functor $F : \mathcal{O}(\lambda) \to \mathcal{F}(\lambda)$ sends irreducible modules to irreducible modules or zero. More precisely, take any $A \in \mathrm{Col}(\lambda)$ and let $B \in \mathrm{Row}(\lambda)$ be its row equivalence class. Then,*

$$F(K(A)) \cong \begin{cases} L(B) & \text{if } A \text{ is standard,} \\ 0 & \text{otherwise.} \end{cases}$$

Every finite dimensional irreducible $W(\lambda)$-module arises in this way.

There are three main ingredients to the proof of this theorem. First, we need detailed information about the translation functors e_i, f_i, much of which is provided by [CR] as an application of the representation theory of degenerate affine Hecke algebras. Second, we need to know that the standard modules $V(A)$ have simple cosocle whenever A is standard, which follows from Theorem D. Finally, we need to apply the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$ in order to determine exactly when $F(K(A))$ is non-zero.

Let us discuss some of the combinatorial consequences of Theorem E in more detail. For this, we at last restrict our attention just to modules having integral central character. Let $\mathrm{Col}_0(\lambda), \mathrm{Dom}_0(\lambda)$ and $\mathrm{Row}_0(\lambda)$ denote the subsets of $\mathrm{Col}(\lambda), \mathrm{Dom}(\lambda)$ and $\mathrm{Row}(\lambda)$ consisting of the tableaux all of whose entries are integers. Also let $\mathrm{Std}_0(\lambda)$ denote the set of all standard tableaux in $\mathrm{Col}_0(\lambda)$. So elements of $\mathrm{Std}_0(\lambda)$ are tableaux of shape λ having entries from $\mathbb{Z}$, such that entries are weakly increasing along rows from left to right and strictly increasing up columns from bottom to top. The map R actually gives a bijection between the set $\mathrm{Std}_0(\lambda)$ and $\mathrm{Dom}_0(\lambda)$. Let $\mathcal{O}_0(\lambda)$, $\mathcal{F}_0(\lambda)$ and $\mathcal{M}_0(\lambda)$ denote the full subcategories of $\mathcal{O}(\lambda)$, $\mathcal{F}(\lambda)$ and $\mathcal{M}(\lambda)$ that consist of objects all of whose composition factors are of the form $\{K(A) \mid A \in \mathrm{Col}_0(\lambda)\}$, $\{L(A) \mid A \in \mathrm{Dom}_0(\lambda)\}$ and $\{L(A) \mid A \in \mathrm{Row}_0(\lambda)\}$, respectively. The isomorphism classes of these three sets of objects give canonical bases for the Grothendieck groups $[\mathcal{O}_0(\lambda)], [\mathcal{F}_0(\lambda)]$ and $[\mathcal{M}_0(\lambda)]$, as do the isomorphism classes of the parabolic Verma modules $\{N(A) \mid A \in \mathrm{Col}_0(\lambda)\}$, the standard modules $\{V(A) \mid A \in \mathrm{Std}_0(\lambda)\}$, and the Verma modules $\{M(A) \mid A \in \mathrm{Row}_0(\lambda)\}$, respectively.

The functor F above restricts to an exact functor $F : \mathcal{O}_0(\lambda) \to \mathcal{F}_0(\lambda)$, and we also have the natural embedding I of the category $\mathcal{F}_0(\lambda)$ into $\mathcal{M}_0(\lambda)$. At the level of Grothendieck groups, these functors induce maps

$$[\mathcal{O}_0(\lambda)] \stackrel{F}{\twoheadrightarrow} [\mathcal{F}_0(\lambda)] \stackrel{I}{\hookrightarrow} [\mathcal{M}_0(\lambda)].$$

The translation functors e_i, f_i for $i \in \mathbb{Z}$ (and more generally their divided powers $e_i^{(r)}, f_i^{(r)}$ defined as in [CR]) induce maps also denoted e_i, f_i on all these Grothendieck groups. The resulting maps satisfy the relations of the Chevalley generators (and their divided powers) for the Kostant $\mathbb{Z}$-form $U_\mathbb{Z}$ of the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_\infty(\mathbb{C})$, that is, the Lie algebra of matrices with rows and columns labelled by $\mathbb{Z}$ all but finitely many of which are zero. The maps F and I are then $U_\mathbb{Z}$-module homomorphisms with respect to these actions.

Now the point is that all of this categorifies a well known situation in linear algebra. Let $V_\mathbb{Z}$ denote the natural $U_\mathbb{Z}$-module, on basis v_i ($i \in \mathbb{Z}$). We write $\bigwedge^{\lambda'}(V_\mathbb{Z})$ for the tensor product $\bigwedge^{q_1}(V_\mathbb{Z}) \otimes \cdots \otimes \bigwedge^{q_l}(V_\mathbb{Z})$ and $S^\lambda(V_\mathbb{Z})$ for the tensor product $S^{p_1}(V_\mathbb{Z}) \otimes \cdots \otimes S^{p_n}(V_\mathbb{Z})$. These free $\mathbb{Z}$-modules have natural monomial bases denoted $\{N_A \mid A \in \mathrm{Col}_0(\lambda)\}$ and $\{M_A \mid A \in \mathrm{Row}_0(\lambda)\}$, respectively; see §4.2. A well known consequence of the Littlewood-Richardson rule is that the space

$$\mathrm{Hom}_{U_\mathbb{Z}}(\textstyle\bigwedge^{\lambda'}(V_\mathbb{Z}), S^\lambda(V_\mathbb{Z}))$$

is a free $\mathbb{Z}$-module of rank one; indeed, there is a canonical $U_\mathbb{Z}$-module homomorphism $F : \bigwedge^{\lambda'}(V_\mathbb{Z}) \to S^\lambda(V_\mathbb{Z})$ that generates the space of all such maps. The image of this map F is $P^\lambda(V_\mathbb{Z})$, the familiar *irreducible polynomial representation* of $U_\mathbb{Z}$ labelled by the partition λ. Recall $P^\lambda(V_\mathbb{Z})$ also possesses a standard monomial basis $\{V_A \mid A \in \mathrm{Std}_0(\lambda)\}$, defined from $V_A = F(N_A)$. Finally, let $i : \bigwedge^{\lambda'}(V_\mathbb{Z}) \to [\mathcal{O}_0(\lambda)]$, $j : P^\lambda(V_\mathbb{Z}) \to [\mathcal{F}_0(\lambda)]$ and $k : S^\lambda(V_\mathbb{Z}) \to [\mathcal{M}_0(\lambda)]$ be the $\mathbb{Z}$-module homomorphisms sending $N_A \mapsto N(A), V_A \mapsto [V(A)]$ and $M_A \mapsto [M(A)]$ for $A \in \mathrm{Col}_0(\lambda)$, $A \in \mathrm{Std}_0(\lambda)$ and $A \in \mathrm{Row}_0(\lambda)$, respectively.

Theorem F (Categorification of polynomial functors). *The maps i, j, k are all isomorphisms of $U_\mathbb{Z}$-modules, and the following diagram commutes:*

$$\begin{array}{ccccc}
\bigwedge^{\lambda'}(V_\mathbb{Z}) & \xrightarrow{F} & P^\lambda(V_\mathbb{Z}) & \xrightarrow{I} & S^\lambda(V_\mathbb{Z}) \\
{\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle k} \\
[\mathcal{O}_0(\lambda)] & \xrightarrow[F]{} & [\mathcal{F}_0(\lambda)] & \xrightarrow[I]{} & [\mathcal{M}_0(\lambda)].
\end{array}$$

Moreover, setting $L_A = j^{-1}([L(A)])$ for $A \in \mathrm{Dom}_0(\lambda)$, the basis $\{L_A \mid A \in \mathrm{Dom}_0(\lambda)\}$ of $P^\lambda(V_\mathbb{Z})$ coincides with Lusztig's dual canonical basis.

Again, the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$ plays the central role in the proof of this theorem. Actually, we use the following increasingly well known reformulation of the Kazhdan-Lusztig conjecture in type A: setting $K_A = i^{-1}([K(A)])$, the basis $\{K_A | A \in \mathrm{Col}_0(\lambda)\}$ coincides with Lusztig's dual canonical basis for the space $\bigwedge^{\lambda'}(V_\mathbb{Z})$. In particular, this implies that the decomposition numbers $[V(A) : L(B)]$ for $A \in \mathrm{Std}_0(\lambda)$ and $B \in \mathrm{Dom}_0(\lambda)$ can be computed in terms of certain Kazhdan-Lusztig polynomials associated to the symmetric group S_N evaluated at $q = 1$. From a special case, one can also recover the analogous result for the Yangian Y_n itself. We mention this, because it is interesting to compare the strategy followed here with that of Arakawa [A1], who also computes the decomposition matrices of the Yangian in terms of Kazhdan-Lusztig polynomials starting from the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$, via [AS]. There should also be a geometric approach to representation theory of shifted Yangians in the spirit of [V].

As promised earlier in the introduction, let us now formulate a precise conjecture that explains how to compute the decomposition numbers $[M(A) : L(B)]$ of Verma modules for $A, B \in \mathrm{Row}_0(\lambda)$, also in terms of Kazhdan-Lusztig polynomials associated to the symmetric group S_N. Setting $L_A = k^{-1}([L(A)])$ for any $A \in \mathrm{Row}_0(\lambda)$, we conjecture that $\{L_A \mid A \in \mathrm{Row}_0(\lambda)\}$ coincides with Lusztig's dual canonical basis for the space $S^\lambda(V_\mathbb{Z})$; see §7.5. This is a purely combinatorial reformulation in type A of the conjecture of de Vos and van Driel [VD] for arbitrary finite W-algebras, and is consistent with an idea of Premet that there should be an equivalence of categories between the category $\mathcal{M}(\lambda)$ here and a certain category $\mathcal{N}(\lambda)$ considered by Miličíc and Soergel [MS]. Our conjecture is known to be true in the special case that the Young diagram of λ consists of a single column: in that case it is precisely the Kazhdan-Lusztig conjecture for the Lie algebra $\mathfrak{gl}_N(\mathbb{C})$. It is also true if the Young diagram of λ has at most two rows, as can be verified by comparing the explicit construction of the simple highest weight modules in the two row case from §7.1 with the explicit description of the dual canonical basis in this case from [B, Theorem 20]. Finally, Theorem E would be an easy consequence of this conjecture.

Despite the unreasonable length of this article, we seem to have only just broken the surface of this remarkably rich subject, and only in type A at that. There are a number of exciting directions for future research. In a subsequent paper [BK6], we will investigate two natural subcategories $\mathcal{P}(\lambda)$ and $\mathcal{R}(\lambda)$ of $\mathcal{F}(\lambda)$, consisting of *polynomial* and *rational* representations of $W(\lambda)$, respectively. Let us make a few further, informal and in part conjectural, remarks about this forthcoming work that might help the reader to put this subject into context.

First, the category $\mathcal{P}(\lambda)$ is defined from $\mathcal{P}(\lambda) = \bigoplus_{d \geq 0} \mathcal{P}_d(\lambda)$, where $\mathcal{P}_d(\lambda)$ is the category of all finite dimensional $W(\lambda)$-modules that are annihilated by the annihilator of the tensor space $\mathbb{C}_\rho \circledast V^{\otimes d}$. Here, V is the natural $\mathfrak{gl}_N(\mathbb{C})$-module and $\mathbb{C}_\rho$ is the trivial $W(\lambda)$-module; see §3.6. Assuming at least d parts of λ are equal to l, we can prove a generalization of Schur-Weyl duality giving an equivalence of categories between the category $\mathcal{P}_d(\lambda)$ of polynomial representations of $W(\lambda)$ of degree d and the cyclotomic quotients of the degenerate affine Hecke algebra H_d associated to the weight $\sum_{i=1}^{l} \Lambda_{n-q_i}$ for the affine Lie algebra $\widehat{\mathfrak{gl}}_\infty(\mathbb{C})$. In fact, one can recover the irreducible integrable highest weight representation of $\widehat{\mathfrak{gl}}_\infty(\mathbb{C})$ labelled by this level l dominant integral weight by considering the Grothendieck group of the category $\mathcal{P}(\lambda)$ and taking a natural limit $n \to \infty$. This can be viewed as a categorification of the semi-infinite wedge construction for higher levels.

Second, the category $\mathcal{R}(\lambda)$ of rational representations consists of all $W(\lambda)$-modules M with the property that $M \circledast \det^p$ is polynomial for some $p \gg 0$. We expect, but have not yet been able to prove, that the functor $F : \mathcal{O}_0(\lambda) \to \mathcal{F}_0(\lambda)$ restricts to give an essentially surjective quotient functor $F : \mathcal{O}_0(\lambda) \to \mathcal{R}(\lambda)$. Let us reformulate this statement in a way which should be compared with [Ba, S]. Let $P(\lambda) = \bigoplus_{A \in \mathrm{Std}_0(\lambda)} P(A)$, where $P(A)$ is the projective cover of $K(A)$ in the category $\mathcal{O}_0(\lambda)$. These are precisely the self-dual PIMs in $\mathcal{O}_0(\lambda)$; see [I, MSt]. Let $C(\lambda) = \mathrm{End}_\mathfrak{g}(P(\lambda))^{\mathrm{op}}$. Then, our statement is equivalent to saying that there is an equivalence of categories $G : \mathcal{R}(\lambda) \to C(\lambda)\text{-mof}$ such that $G \circ F = \mathrm{Hom}_\mathfrak{g}(P(\lambda), ?)$, equality of functors from $\mathcal{O}_0(\lambda)$ to $C(\lambda)$-mof. If true, this could have implications for *vertex W-algebras*, like in [A2].

Notation. Throughout we work over an algebraically closed field $\mathbb{F}$ of characteristic 0. Let $\geq$ denote the partial order on $\mathbb{F}$ defined by $x \geq y$ if $(x-y) \in \mathbb{N}$, where $\mathbb{N}$ denotes $\{0, 1, 2, \dots\} \subset \mathbb{F}$. We write simply $\mathfrak{gl}_n$ for the Lie algebra $\mathfrak{gl}_n(\mathbb{F})$.

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2. Shifted Yangians

In this preliminary chapter, we collect some basic definitions and results about shifted Yangians, most of which are taken from [BK5].

2.1. Generators and relations. By a *shift matrix* we mean a matrix $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ of non-negative integers such that

$$s_{i,j} + s_{j,k} = s_{i,k} \tag{2.1}$$

whenever $|i-j|+|j-k| = |i-k|$. Note this means that $s_{1,1} = \dots = s_{n,n} = 0$, and the matrix σ is completely determined by the upper diagonal entries $s_{1,2}, s_{2,3}, \dots, s_{n-1,n}$ and the lower diagonal entries $s_{2,1}, s_{3,2}, \dots, s_{n,n-1}$.

The *shifted Yangian* associated to the matrix σ is the algebra $Y_n(\sigma)$ over $\mathbb{F}$ defined by generators

$$\{D_i^{(r)} \mid 1 \leq i \leq n, r > 0\}, \tag{2.2}$$

$$\{E_i^{(r)} \mid 1 \leq i < n, r > s_{i,i+1}\}, \tag{2.3}$$

$$\{F_i^{(r)} \mid 1 \leq i < n, r > s_{i+1,i}\} \tag{2.4}$$

subject to certain relations. In order to write down these relations, let

$$D_i(u) := \sum_{r \geq 0} D_i^{(r)} u^{-r} \in Y_n(\sigma)[[u^{-1}]] \tag{2.5}$$

where $D_i^{(0)} := 1$, and then define some new elements $\widetilde{D}_i^{(r)}$ of $Y_n(\sigma)$ from the equation

$$\widetilde{D}_i(u) = \sum_{r \geq 0} \widetilde{D}_i^{(r)} u^{-r} := -D_i(u)^{-1}. \tag{2.6}$$

With this notation, the relations are as follows.

$$[D_i^{(r)}, D_j^{(s)}] = 0, \tag{2.7}$$

$$[E_i^{(r)}, F_j^{(s)}] = \delta_{i,j} \sum_{t=0}^{r+s-1} \widetilde{D}_i^{(t)} D_{i+1}^{(r+s-1-t)}, \tag{2.8}$$

$$[D_i^{(r)}, E_j^{(s)}] = (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} D_i^{(t)} E_j^{(r+s-1-t)}, \tag{2.9}$$

$$[D_i^{(r)}, F_j^{(s)}] = (\delta_{i,j+1} - \delta_{i,j}) \sum_{t=0}^{r-1} F_j^{(r+s-1-t)} D_i^{(t)}, \tag{2.10}$$

$$[E_i^{(r)}, E_i^{(s+1)}] - [E_i^{(r+1)}, E_i^{(s)}] = E_i^{(r)} E_i^{(s)} + E_i^{(s)} E_i^{(r)}, \tag{2.11}$$

$$[F_i^{(r+1)}, F_i^{(s)}] - [F_i^{(r)}, F_i^{(s+1)}] = F_i^{(r)} F_i^{(s)} + F_i^{(s)} F_i^{(r)}, \tag{2.12}$$

$$[E_i^{(r)}, E_{i+1}^{(s+1)}] - [E_i^{(r+1)}, E_{i+1}^{(s)}] = -E_i^{(r)} E_{i+1}^{(s)}, \tag{2.13}$$

$$[F_i^{(r+1)}, F_{i+1}^{(s)}] - [F_i^{(r)}, F_{i+1}^{(s+1)}] = -F_{i+1}^{(s)} F_i^{(r)}, \tag{2.14}$$

$$[E_i^{(r)}, E_j^{(s)}] = 0 \qquad \text{if } |i-j| > 1, \tag{2.15}$$

$$[F_i^{(r)}, F_j^{(s)}] = 0 \qquad \text{if } |i-j| > 1, \tag{2.16}$$

$$[E_i^{(r)}, [E_i^{(s)}, E_j^{(t)}]] + [E_i^{(s)}, [E_i^{(r)}, E_j^{(t)}]] = 0 \qquad \text{if } |i-j| = 1, \tag{2.17}$$

$$[F_i^{(r)}, [F_i^{(s)}, F_j^{(t)}]] + [F_i^{(s)}, [F_i^{(r)}, F_j^{(t)}]] = 0 \qquad \text{if } |i-j| = 1, \tag{2.18}$$

for all meaningful r, s, t, i, j. (For example, the relation (2.13) should be understood to hold for all $i = 1, \dots, n-2$, $r > s_{i,i+1}$ and $s > s_{i+1,i+2}$.)

It is often helpful to view $Y_n(\sigma)$ as an algebra graded by the root lattice Q_n associated to the Lie algebra $\mathfrak{gl}_n$. Let $\mathfrak{d}_n$ be the (abelian) Lie subalgebra of $Y_n(\sigma)$ spanned by the elements $D_1^{(1)}, \dots, D_n^{(1)}$. Let $\varepsilon_1, \dots, \varepsilon_n$ be the basis for $\mathfrak{d}_n^*$ dual to the basis $D_1^{(1)}, \dots, D_n^{(1)}$. We refer to elements of $\mathfrak{d}_n^*$ as *weights* and elements of

$$P_n := \bigoplus_{i=1}^{n} \mathbb{Z}\varepsilon_i \subset \mathfrak{d}_n^* \tag{2.19}$$

as *integral weights*. The *root lattice* associated to the Lie algebra $\mathfrak{gl}_n$ is then the $\mathbb{Z}$-submodule Q_n of P_n spanned by the *simple roots* $\varepsilon_i - \varepsilon_{i+1}$ for $i = 1, \dots, n-1$. We have the usual *dominance ordering* on $\mathfrak{d}_n^*$ defined by $\alpha \geq \beta$ if $(\alpha - \beta)$ is a sum of simple roots. With this notation set up, the relations imply that we can define a Q_n-grading

$$Y_n(\sigma) = \bigoplus_{\alpha \in Q_n} (Y_n(\sigma))_\alpha \tag{2.20}$$

of the algebra $Y_n(\sigma)$ by declaring that the generators $D_i^{(r)}, E_i^{(r)}$ and $F_i^{(r)}$ are of degrees $0, \varepsilon_i - \varepsilon_{i+1}$ and $\varepsilon_{i+1} - \varepsilon_i$, respectively.

2.2. **PBW theorem.** For $1 \leq i < j \leq n$ and $r > s_{i,j}$ resp. $r > s_{j,i}$, we inductively define the *higher root elements* $E_{i,j}^{(r)}$ resp. $F_{i,j}^{(r)}$ of $Y_n(\sigma)$ from the formulae

$$E_{i,i+1}^{(r)} := E_i^{(r)}, \qquad E_{i,j}^{(r)} := [E_{i,j-1}^{(r-s_{j-1,j})}, E_{j-1}^{(s_{j-1,j}+1)}], \tag{2.21}$$

$$F_{i,i+1}^{(r)} := F_i^{(r)}, \qquad F_{i,j}^{(r)} := [F_{j-1}^{(s_{j,j-1}+1)}, F_{i,j-1}^{(r-s_{j,j-1})}]. \tag{2.22}$$

Introduce the *canonical filtration* $\mathrm{F}_0 Y_n(\sigma) \subseteq \mathrm{F}_1 Y_n(\sigma) \subseteq \cdots$ of $Y_n(\sigma)$ by declaring that all $D_i^{(r)}, E_{i,j}^{(r)}$ and $F_{i,j}^{(r)}$ are of degree r, i.e. $\mathrm{F}_d Y_n(\sigma)$ is the span of all monomials in these elements of total degree $\leq d$. Then [BK5, Theorem 5.2] shows that the associated graded algebra $\operatorname{gr} Y_n(\sigma)$ is free commutative on generators

$$\{\operatorname{gr}_r D_i^{(r)} \mid 1 \leq i \leq n, s_{i,i} < r\}, \tag{2.23}$$

$$\{\operatorname{gr}_r E_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{i,j} < r\}, \tag{2.24}$$

$$\{\operatorname{gr}_r F_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{j,i} < r\}. \tag{2.25}$$

It follows immediately that the monomials in the elements

$$\{D_i^{(r)} \mid 1 \leq i \leq n, s_{i,i} < r\}, \tag{2.26}$$

$$\{E_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{i,j} < r\}, \tag{2.27}$$

$$\{F_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{j,i} < r\} \tag{2.28}$$

taken in some fixed order give a basis for the algebra $Y_n(\sigma)$. Moreover, letting $Y_{(1^n)}$ resp. $Y_{(1^n)}^+(\sigma)$ resp. $Y_{(1^n)}^-(\sigma)$ denote the subalgebra of $Y_n(\sigma)$ generated by the $D_i^{(r)}$'s resp. the $E_i^{(r)}$'s resp. the $F_i^{(r)}$'s, the monomials just in the elements (2.26) resp. (2.27) resp. (2.28) taken in some fixed order give bases for these subalgebras; see [BK5, Theorem 2.3]. These basis theorems imply in particular that multiplication defines a vector space isomorphism

$$Y_{(1^n)}^-(\sigma) \otimes Y_{(1^n)} \otimes Y_{(1^n)}^+(\sigma) \stackrel{\sim}{\longrightarrow} Y_n(\sigma), \tag{2.29}$$

giving us a *triangular decomposition* of the shifted Yangian. Also define the *positive* and *negative Borel subalgebras* $Y_{(1^n)}^\sharp(\sigma) := Y_{(1^n)} Y_{(1^n)}^+(\sigma)$ and $Y_{(1^n)}^\flat(\sigma) := Y_{(1^n)}^-(\sigma) Y_{(1^n)}$. By the relations, these are indeed subalgebras of $Y_n(\sigma)$. Moreover, there are obvious surjective homomorphisms

$$Y_{(1^n)}^\sharp(\sigma) \twoheadrightarrow Y_{(1^n)}, \qquad Y_{(1^n)}^\flat(\sigma) \twoheadrightarrow Y_{(1^n)} \tag{2.30}$$

with kernels $K_{(1^n)}^\sharp(\sigma)$ and $K_{(1^n)}^\flat(\sigma)$ generated by all $E_{i,j}^{(r)}$ and all $F_{i,j}^{(r)}$, respectively.

We now introduce a new basis for $Y_n(\sigma)$, which will play a central role in this article. First, define the power series

$$E_{i,j}(u) := \sum_{r > s_{i,j}} E_{i,j}^{(r)} u^{-r}, \qquad F_{i,j}(u) := \sum_{r > s_{j,i}} F_{i,j}^{(r)} u^{-r} \tag{2.31}$$

for $1 \leq i < j \leq n$, and set $E_{i,i}(u) = F_{i,i}(u) := 1$ by convention. Recalling (2.5), let $D(u)$ denote the $n \times n$ diagonal matrix with ii-entry $D_i(u)$ for $1 \leq i \leq n$, let $E(u)$ denote the $n \times n$ upper triangular matrix with ij-entry $E_{i,j}(u)$ for $1 \leq i \leq j \leq n$, and let $F(u)$ denote the $n \times n$ lower triangular matrix with ji-entry $F_{i,j}(u)$ for $1 \leq i \leq j \leq n$. Consider the product

$$T(u) = F(u) D(u) E(u) \tag{2.32}$$

of matrices with entries in $Y_n(\sigma)[[u^{-1}]]$. The ij-entry of the matrix $T(u)$ defines a power series

$$T_{i,j}(u) = \sum_{r \geq 0} T_{i,j}^{(r)} u^{-r} := \sum_{k=1}^{\min(i,j)} F_{k,i}(u) D_k(u) E_{k,j}(u) \tag{2.33}$$

for some new elements $T_{i,j}^{(r)} \in \mathrm{F}_r Y_n(\sigma)$. Note that $T_{i,j}^{(0)} = \delta_{i,j}$ and $T_{i,j}^{(r)} = 0$ for $0 < r \leq s_{i,j}$.

Lemma 2.1. *The associated graded algebra* $\operatorname{gr} Y_n(\sigma)$ *is free commutative on generators* $\{\operatorname{gr}_r T_{i,j}^{(r)} \mid 1 \leq i,j \leq n, s_{i,j} < r\}$. *Hence, the monomials in the elements* $\{T_{i,j}^{(r)} \mid 1 \leq i,j \leq n, s_{i,j} < r\}$ *taken in some fixed order form a basis for* $Y_n(\sigma)$.

Proof. Recall that $T_{i,j}^{(r)} = 0$ for $0 < r \leq s_{i,j}$. Given this, it is easy to see, e.g. by solving the equation (2.32) in terms of quasi-determinants as in [BK4, (5.2)–(5.4)], that each of the elements $D_i^{(r)}, E_{i,j}^{(r)}$ and $F_{i,j}^{(r)}$ of $Y_n(\sigma)$ can be written as a linear combination of monomials of total degree r in the elements $\{T_{i,j}^{(s)} \mid 1 \leq i,j \leq n, s_{i,j} < s\}$. Since we already know that $\operatorname{gr} Y_n(\sigma)$ is free commutative on the generators (2.23)–(2.25), it follows that the elements $\{\operatorname{gr}_r T_{i,j}^{(r)} \mid 1 \leq i,j \leq n, s_{i,j} < r\}$ also generate $\operatorname{gr} Y_n(\sigma)$. Now the lemma follows by dimension considerations. □

2.3. **Some automorphisms.** Let $\dot\sigma = (\dot s_{i,j})_{1 \leq i,j \leq n}$ be another shift matrix such that $\dot s_{i,i+1} + \dot s_{i+1,i} = s_{i,i+1} + s_{i+1,i}$ for all $i = 1, \dots, n-1$. Then the defining relations imply that there is a unique algebra isomorphism $\iota : Y_n(\sigma) \to Y_n(\dot\sigma)$ defined on generators by the equations

$$\iota(D_i^{(r)}) = D_i^{(r)}, \quad \iota(E_i^{(r)}) = E_i^{(r - s_{i,i+1} + \dot s_{i,i+1})}, \quad \iota(F_i^{(r)}) = F_i^{(r - s_{i+1,i} + \dot s_{i+1,i})}. \tag{2.34}$$

Another useful map is the anti-isomorphism $\tau : Y_n(\sigma) \to Y_n(\sigma^t)$ defined by

$$\tau(D_i^{(r)}) = D_i^{(r)}, \qquad \tau(E_i^{(r)}) = F_i^{(r)}, \qquad \tau(F_i^{(r)}) = E_i^{(r)}, \tag{2.35}$$

where σ^t denotes the *transpose* of the shift matrix σ. Note from (2.21)–(2.22) and (2.33) that

$$\tau(E_{i,j}^{(r)}) = F_{i,j}^{(r)}, \qquad \tau(F_{i,j}^{(r)}) = E_{i,j}^{(r)}, \qquad \tau(T_{i,j}^{(r)}) = T_{j,i}^{(r)}. \tag{2.36}$$

Finally for any power series $f(u) \in 1 + u^{-1}\mathbb{F}[[u^{-1}]]$, it is easy to check from the relations that there is an automorphism $\mu_f : Y_n(\sigma) \to Y_n(\sigma)$ fixing each $E_i^{(r)}$ and $F_i^{(r)}$ and mapping the power series $D_i(u)$ to the product $f(u)D_i(u)$, i.e.

$$\mu_f(D_i^{(r)}) = \sum_{s=0}^{r} a_s D_i^{(r-s)}, \qquad \mu_f(E_i^{(r)}) = E_i^{(r)}, \qquad \mu_f(F_i^{(r)}) = F_i^{(r)}, \tag{2.37}$$

if $f(u) = \sum_{s \geq 0} a_s u^{-s}$.

2.4. **Parabolic generators.** In this section, we recall some more complicated *parabolic presentations* of $Y_n(\sigma)$ from [BK5]. Actually the parabolic generators defined here will be needed later on only in §3.7. By a *shape* we mean a tuple $\nu = (\nu_1, \dots, \nu_m)$ of positive integers summing to n, which we think of as the shape of the standard Levi subalgebra $\mathfrak{gl}_{\nu_1} \oplus \cdots \oplus \mathfrak{gl}_{\nu_m}$ of $\mathfrak{gl}_n$. We say that a shape $\nu = (\nu_1, \dots, \nu_m)$ is *admissible* (for σ) if $s_{i,j} = 0$ for all $\nu_1 + \cdots + \nu_{a-1} + 1 \le i, j \le \nu_1 + \cdots + \nu_a$ and $a = 1, \dots, m$, in which case we define

$$s_{a,b}(\nu) := s_{\nu_1+\cdots+\nu_a, \nu_1+\cdots+\nu_b} \tag{2.38}$$

for $1 \le a, b \le m$. An important role is played by the *minimal admissible shape* (for σ), namely, the admissible shape whose length m is as small as possible.

Suppose that we are given an admissible shape $\nu = (\nu_1, \dots, \nu_m)$. Writing $e_{i,j}$ for the ij-matrix unit in the space $M_{r,s}$ of $r \times s$ matrices over $\mathbb{F}$, define

$${}^\nu T_{a,b}(u) := \sum_{\substack{1 \le i \le \nu_a \\ 1 \le j \le \nu_b}} e_{i,j} \otimes T_{\nu_1+\cdots+\nu_{a-1}+i, \nu_1+\cdots+\nu_{b-1}+j}(u) \in M_{\nu_a,\nu_b} \otimes Y_n(\sigma)[[u^{-1}]] \tag{2.39}$$

for each $1 \le a, b \le m$. Let ${}^\nu T(u)$ denote the $m \times m$ matrix with ab-entry ${}^\nu T_{a,b}(u)$. Generalizing (2.32) (which is the special case $\nu = (1^n)$ of the present definition), consider the Gauss factorization

$${}^\nu T(u) = {}^\nu F(u) {}^\nu D(u) {}^\nu E(u) \tag{2.40}$$

where ${}^\nu D(u)$ is an $m \times m$ diagonal matrix with aa-entry denoted ${}^\nu D_a(u) \in M_{\nu_a,\nu_a} \otimes Y_n(\sigma)[[u^{-1}]]$ for $1 \le a \le m$, ${}^\nu E(u)$ is an $m \times m$ upper unitriangular matrix with ab-entry denoted ${}^\nu E_{a,b}(u) \in M_{\nu_a,\nu_b} \otimes Y_n(\sigma)[[u^{-1}]]$ and ${}^\nu F(u)$ is an $m \times m$ lower unitriangular matrix with ba-entry denoted ${}^\nu F_{a,b}(u) \in M_{\nu_b,\nu_a} \otimes Y_n(\sigma)[[u^{-1}]]$. So, ${}^\nu E_{a,a}(u)$ and ${}^\nu F_{a,a}(u)$ are both the identity and

$${}^\nu T_{a,b}(u) = \sum_{c=1}^{\min(a,b)} {}^\nu F_{c,a}(u) {}^\nu D_c(u) {}^\nu E_{c,b}(u). \tag{2.41}$$

Also for $1 \le a \le m$ let

$${}^\nu \widetilde{D}_a(u) := -{}^\nu D_a(u)^{-1}, \tag{2.42}$$

inverse computed in the algebra $M_{\nu_a,\nu_a} \otimes Y_n(\sigma)[[u^{-1}]]$. We expand

$${}^\nu D_a(u) = \sum_{1 \le i,j \le \nu_a} e_{i,j} \otimes {}^\nu D_{a;i,j}(u) = \sum_{\substack{1 \le i,j \le \nu_a \\ r \ge 0}} e_{i,j} \otimes {}^\nu D_{a;i,j}^{(r)} u^{-r}, \tag{2.43}$$

$${}^\nu \widetilde{D}_a(u) = \sum_{1 \le i,j \le \nu_a} e_{i,j} \otimes {}^\nu \widetilde{D}_{a;i,j}(u) = \sum_{\substack{1 \le i,j \le \nu_a \\ r \ge 0}} e_{i,j} \otimes {}^\nu \widetilde{D}_{a;i,j}^{(r)} u^{-r}, \tag{2.44}$$

$${}^\nu E_{a,b}(u) = \sum_{\substack{1 \le i \le \nu_a \\ 1 \le j \le \nu_b}} e_{i,j} \otimes {}^\nu E_{a,b;i,j}(u) = \sum_{\substack{1 \le i \le \nu_a \\ 1 \le j \le \nu_b \\ r > s_{a,b}(\nu)}} e_{i,j} \otimes {}^\nu E_{a,b;i,j}^{(r)} u^{-r}, \tag{2.45}$$

$$ {}^{\nu}F_{a,b}(u) = \sum_{\substack{1\leq i\leq \nu_b\\ 1\leq j\leq \nu_a}} e_{i,j}\otimes {}^{\nu}F_{a,b;i,j}(u) = \sum_{\substack{1\leq i\leq \nu_b\\ 1\leq j\leq \nu_a\\ r>s_{b,a}(\nu)}} e_{i,j}\otimes {}^{\nu}F^{(r)}_{a,b;i,j}u^{-r}, \tag{2.46}$$

where ${}^{\nu}D_{a;i,j}(u), {}^{\nu}\widetilde{D}_{a;i,j}(u), {}^{\nu}E_{a,b;i,j}(u)$ and ${}^{\nu}F_{a,b;i,j}(u)$ are power series in $Y_n(\sigma)[[u^{-1}]]$, and ${}^{\nu}D^{(r)}_{a;i,j}$, ${}^{\nu}\widetilde{D}^{(r)}_{a;i,j}$, ${}^{\nu}E^{(r)}_{a,b;i,j}$ and ${}^{\nu}F^{(r)}_{a,b;i,j}$ are elements of $Y_n(\sigma)$. We will usually omit the superscript ν, writing simply $D^{(r)}_{a;i,j}$, $\widetilde{D}^{(r)}_{a;i,j}$, $E^{(r)}_{a,b;i,j}$ and $F^{(r)}_{a,b;i,j}$, and also abbreviate $E^{(r)}_{a,a+1;i,j}$ by $E^{(r)}_{a;i,j}$ and $F^{(r)}_{a,a+1;i,j}$ by $F^{(r)}_{a;i,j}$. Note finally that the anti-isomorphism τ from (2.35) satisfies

$$\tau(D^{(r)}_{a;i,j}) = D^{(r)}_{a;j,i},\quad \tau(E^{(r)}_{a,b;i,j}) = F^{(r)}_{a,b;j,i},\quad \tau(F^{(r)}_{a,b;i,j}) = E^{(r)}_{a,b;j,i}, \tag{2.47}$$

as follows from (2.41) and (2.36).

In [BK5, §3], we proved that $Y_n(\sigma)$ is generated by the elements

$$\{D^{(r)}_{a;i,j}\mid a=1,\dots,m, 1\leq i,j\leq \nu_a, r>0\}, \tag{2.48}$$

$$\{E^{(r)}_{a;i,j}\mid a=1,\dots,m-1, 1\leq i\leq \nu_a, 1\leq j\leq \nu_{a+1}, r>s_{a,a+1}(\nu)\}, \tag{2.49}$$

$$\{F^{(r)}_{a;i,j}\mid a=1,\dots,m-1, 1\leq i\leq \nu_{a+1}, 1\leq j\leq \nu_a, r>s_{a+1,a}(\nu)\} \tag{2.50}$$

subject to certain relations recorded explicitly in [BK5, (3.3)–(3.14)]. Moreover, the monomials in the elements

$$\{D^{(r)}_{a;i,j}\mid 1\leq a\leq m, 1\leq i,j\leq \nu_a, s_{a,a}(\nu)<r\}, \tag{2.51}$$

$$\{E^{(r)}_{a,b;i,j}\mid 1\leq a<b\leq m, 1\leq i\leq \nu_a, 1\leq j\leq \nu_b, s_{a,b}(\nu)<r\}, \tag{2.52}$$

$$\{F^{(r)}_{a,b;i,j}\mid 1\leq a<b\leq m, 1\leq i\leq \nu_b, 1\leq j\leq \nu_a, s_{b,a}(\nu)<r\} \tag{2.53}$$

taken in some fixed order form a basis for $Y_n(\sigma)$. Actually the definition of the higher root elements $E^{(r)}_{a,b;i,j}$ and $F^{(r)}_{a,b;i,j}$ given here is different from the definition given in [BK5]. The equivalence of the two definitions is verified by the following lemma.

Lemma 2.2. *For $1\leq a<b-1<m$, $1\leq i\leq \nu_a, 1\leq j\leq \nu_b$ and $r>s_{a,b}(\nu)$, we have that*

$$E^{(r)}_{a,b;i,j} = [E^{(r-s_{b-1,b}(\nu))}_{a,b-1;i,k}, E^{(s_{b-1,b}(\nu)+1)}_{b-1;k,j}]$$

for any $1\leq k\leq \nu_{b-1}$. Similarly, for $1\leq a<b-1<m$, $1\leq i\leq \nu_b, 1\leq j\leq \nu_a$ and $r>s_{b,a}(\nu)$, we have that

$$F^{(r)}_{a,b;i,j} = [F^{(s_{b,b-1}(\nu)+1)}_{b-1;i,k}, F^{(r-s_{b,b-1}(\nu))}_{a,b-1;k,j}]$$

for any $1\leq k\leq \nu_{b-1}$.

Proof. We just prove the statement about the E's; the statement about the F's then follows on applying the anti-isomorphism τ. Proceed by downward induction on the length of the admissible shape $\nu=(\nu_1,\dots,\nu_m)$. The base case $m=n$ is the definition (2.21), so suppose $m<n$. Pick $1\leq p\leq m$ and $x,y>0$ such that $\nu_p=x+y$, then let $\mu=(\nu_1,\dots,\nu_{p-1},x,y,\nu_{p+1},\dots,\nu_m)$, an admissible shape of strictly longer length. A matrix calculation from the definitions shows for each $1 \leq a < b \leq m$, $1 \leq i \leq \nu_a$ and $1 \leq j \leq \nu_b$ that

$$
{}^\nu E_{a,b;i,j}(u) = \begin{cases} {}^\mu E_{a,b;i,j}(u) & \text{if } b < p; \\ {}^\mu E_{a,b;i,j}(u) & \text{if } b = p, j \leq x; \\ {}^\mu E_{a,b+1;i,j-x}(u) & \text{if } b = p, j > x; \\ {}^\mu E_{a,b+1;i,j}(u) & \text{if } a < p, b > p; \\ {}^\mu E_{a,b+1;i,j}(u) - \sum_{h=1}^{y} {}^\mu E_{a,a+1;i,h}(u) {}^\mu E_{a+1,b+1;h,j}(u) & \text{if } a = p, i \leq x; \\ {}^\mu E_{a+1,b+1;i-x,j}(u) & \text{if } a = p, i > x; \\ {}^\mu E_{a+1,b+1;i,j}(u) & \text{if } a > p. \end{cases}
$$

Now suppose that $b > a+1$. We need to prove that

$$
{}^\nu E_{a,b;i,j}(u) = [{}^\nu E_{a,b-1;i,k}(u), {}^\nu E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}] u^{-s_{b-1,b}(\nu)}
$$

for each $1 \leq k \leq \nu_{b-1}$. The strategy is as follows: rewrite both sides of the identity we are trying to prove in terms of the ${}^\mu E$'s and then use the induction hypothesis, which asserts that

$$
{}^\mu E_{a,b;i,j}(u) = [{}^\mu E_{a,b-1;i,k}(u), {}^\mu E_{b-1;k,j}^{(s_{b-1,b}(\mu)+1)}] u^{-s_{b-1,b}(\mu)}
$$

for each $1 \leq a < b-1 \leq m, 1 \leq i \leq \mu_a, 1 \leq j \leq \mu_b$ and $1 \leq k \leq \mu_{b-1}$. Most of the cases follow at once on doing this; we just discuss the more difficult ones in detail below.

Case one: $b < p$. Easy.

Case two: $b = p, j \leq x$. Easy.

Case three: $b = p, j > x$. We have by induction that

$$
{}^\mu E_{a,b+1;i,j-x}(u) = [{}^\mu E_{a,b;i,h}(u), {}^\mu E_{b;h,j-x}^{(s_{b,b+1}(\mu)+1)}] u^{-s_{b,b+1}(\mu)}
$$

for $1 \leq h \leq x$. Noting that $s_{b,b+1}(\mu) = 0$ and that ${}^\mu E_{b;h,j-x}^{(1)} = {}^\nu D_{b;h,j}^{(1)}$, this shows that ${}^\nu E_{a,b;i,j}(u) = [{}^\nu E_{a,b;i,h}(u), {}^\nu D_{b;h,j}^{(1)}]$. Using the cases already considered and the relations, we get that

$$
\begin{aligned}
[{}^\nu E_{a,b;i,h}(u), {}^\nu D_{b;h,j}^{(1)}] &= [[{}^\nu E_{a,b-1;i,k}(u), {}^\nu E_{b-1;k,h}^{(s_{b-1,b}(\nu)+1)}], {}^\nu D_{b;h,j}^{(1)}] u^{-s_{b-1,b}(\nu)} \\
&= [{}^\nu E_{a,b-1;i,k}(u), {}^\nu E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}] u^{-s_{b-1,b}(\nu)}
\end{aligned}
$$

for any $1 \leq k \leq \nu_{b-1}$.

Case four: $a < p, b > p$. Easy if $b > p+1$ or if $b = p+1$ and $k > x$. Now suppose that $b = p+1$ and $k \leq x$. We know already that

$$
{}^\nu E_{a,b;i,j}(u) = [{}^\nu E_{a,b-1;i,x+1}(u), {}^\nu E_{b-1;x+1,j}^{(s_{b-1,b}(\nu)+1)}] u^{-s_{b-1,b}(\nu)}.
$$

Using the cases already considered to express ${}^\nu E_{a,b-1;i,k}^{(r)}$ as a commutator then using the relation [BK5, (3.11)], we have that $[{}^\nu E_{a,b-1;i,k}^{(r)}, {}^\nu E_{b-1;x+1,j}^{(s)}] = 0$. Bracketing with ${}^\nu D_{b-1;k,x+1}^{(1)}$ and using the relations one deduces that $[{}^\nu E_{a,b-1;i,x+1}^{(r)}, {}^\nu E_{b-1;x+1,j}^{(s)}] = [{}^\nu E_{a,b-1;i,k}^{(r)}, {}^\nu E_{b-1;k,j}^{(s)}]$. Hence,

$$
[{}^\nu E_{a,b-1;i,x+1}(u), {}^\nu E_{b-1;x+1,j}^{(s_{b-1,b}(\nu)+1)}] = [{}^\nu E_{a,b-1;i,k}(u), {}^\nu E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}].
$$

Using this we do indeed get that ${}^{\nu}E_{a,b;i,j}(u) = [{}^{\nu}E_{a,b-1;i,k}(u), {}^{\nu}E^{(s_{b-1,b}(\nu)+1)}_{b-1;k,j}]u^{-s_{b-1,b}(\nu)}$.
Case five: $a = p, i \leq x$. The left hand side of the identity we are trying to prove is equal to

$${}^{\mu}E_{a,b+1;i,j}(u) - \sum_{h=1}^{y} {}^{\mu}E_{a,a+1;i,h}(u){}^{\mu}E_{a+1,b+1;h,j}(u).$$

The right hand side equals

$$[{}^{\mu}E_{a,b;i,k}(u) - \sum_{h=1}^{y} {}^{\mu}E_{a,a+1;i,h}(u){}^{\mu}E_{a+1,b;h,k}(u), {}^{\mu}E^{(s_{b,b+1}(\mu)+1)}_{b;k,j}]u^{-s_{b,b+1}(\mu)}.$$

Now apply the induction hypothesis together with the fact from the relations that ${}^{\mu}E_{a,a+1;i,h}(u)$ and ${}^{\mu}E^{(s_{b,b+1}(\mu)+1)}_{b;k,j}$ commute.
Case six: $a = p, i > x$. Easy.
Case seven: $a > p$. Easy. □

We also introduce here one more family of elements of $Y_n(\sigma)$ needed in §3.7. Continue with $\nu = (\nu_1, \dots, \nu_m)$ being a fixed admissible shape for σ. Recalling that ${}^{\nu}E_{a,a}(u)$ and ${}^{\nu}F_{a,a}(u)$ are both the identity, we define

$${}^{\nu}\bar{E}_{a,b}(u) := {}^{\nu}E_{a,b}(u) - \sum_{c=a}^{b-1} {}^{\nu}E_{a,c}(u){}^{\nu}E^{(s_{c,b}(\nu)+1)}_{c,b}u^{-s_{c,b}(\nu)-1}, \tag{2.54}$$

$${}^{\nu}\bar{F}_{a,b}(u) := {}^{\nu}F_{a,b}(u) - \sum_{c=a}^{b-1} {}^{\nu}F^{(s_{b,c}(\nu)+1)}_{c,b}{}^{\nu}F_{a,c}(u)u^{-s_{b,c}(\nu)-1}, \tag{2.55}$$

for $1 \leq a \leq b \leq m$. As in (2.45)–(2.46), we expand

$${}^{\nu}\bar{E}_{a,b}(u) = \sum_{\substack{1\leq i\leq \nu_a\\ 1\leq j\leq \nu_b}} e_{i,j} \otimes {}^{\nu}\bar{E}_{a,b;i,j}(u) = \sum_{\substack{1\leq i\leq \nu_a\\ 1\leq j\leq \nu_b\\ r>s_{a,b}(\nu)+1}} e_{i,j} \otimes {}^{\nu}\bar{E}^{(r)}_{a,b;i,j}u^{-r}, \tag{2.56}$$

$${}^{\nu}\bar{F}_{a,b}(u) = \sum_{\substack{1\leq i\leq \nu_b\\ 1\leq j\leq \nu_a}} e_{i,j} \otimes {}^{\nu}\bar{F}_{a,b;i,j}(u) = \sum_{\substack{1\leq i\leq \nu_b\\ 1\leq j\leq \nu_a\\ r>s_{b,a}(\nu)+1}} e_{i,j} \otimes {}^{\nu}\bar{F}^{(r)}_{a,b;i,j}u^{-r}, \tag{2.57}$$

where ${}^{\nu}\bar{E}_{a,b;i,j}(u)$ and ${}^{\nu}\bar{F}_{a,b;i,j}(u)$ are power series in $Y_n(\sigma)[[u^{-1}]]$, and ${}^{\nu}\bar{E}^{(r)}_{a,b;i,j}$ and ${}^{\nu}\bar{F}^{(r)}_{a,b;i,j}$ are elements of $Y_n(\sigma)$. We usually drop the superscript ν from this notation.

Lemma 2.3. *For $1 \leq a < b-1 < m$, $1 \leq i \leq \nu_a, 1 \leq j \leq \nu_b$ and $r > s_{a,b}(\nu)+1$, we have that*

$$\bar{E}^{(r)}_{a,b;i,j} = [E^{(r-s_{b-1,b}(\nu)-1)}_{a,b-1;i,k}, E^{(s_{b-1,b}(\nu)+2)}_{b-1;k,j}]$$

for any $1 \leq k \leq \nu_{b-1}$. Similarly, for $1 \leq a < b-1 < m$, $1 \leq i \leq \nu_b, 1 \leq j \leq \nu_a$ and $r > s_{b,a}(\nu)+1$, we have that

$$\bar{F}^{(r)}_{a,b;i,j} = [F^{(s_{b,b-1}(\nu)+2)}_{b-1;i,k}, F^{(r-s_{b,b-1}(\nu)-1)}_{a,b-1;k,j}]$$

for any $1 \leq k \leq \nu_{b-1}$.

Proof. We just prove the statement about the E's; the statement for the F's then follows on applying the anti-isomorphism τ. We need to prove that

$$\bar{E}_{a,b;i,j}(u) = [E_{a,b-1;i,k}(u), E_{b-1;k,j}^{(s_{b-1,b}(\nu)+2)}]u^{-s_{b-1,b}(\nu)-1}.$$

Proceed by induction on $b = a+2, \dots, m$. For the base case $b = a+2$, we have by the relation [BK5, (3.9)] that

$$[E_{a,b-1;i,k}(u), E_{b-1;k,j}^{(s_{b-1,b}(\nu)+2)}] - [E_{a,b-1;i,k}(u), E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}]u =$$
$$- [E_{a,b-1;i,k}^{(s_{a,b-1}(\nu)+1)}, E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}]u^{-s_{a,b-1}(\nu)} - \sum_{h=1}^{\nu_{b-1}} E_{a,b-1;i,h}(u)E_{b-1;h,j}^{(s_{b-1,b}(\nu)+1)}.$$

Multiplying by $u^{-s_{b-1,b}(\nu)-1}$ and using Lemma 2.2, this shows that

$$[E_{a,b-1;i,k}(u), E_{b-1;k,j}^{(s_{b-1,b}(\nu)+2)}]u^{-s_{b-1,b}(\nu)-1} = E_{a,b;i,j}(u)$$
$$- E_{a,b;i,j}^{(s_{a,b}(\nu)+1)}u^{-s_{a,b}(\nu)-1} - \sum_{h=1}^{\nu_{b-1}} E_{a,b-1;i,h}(u)E_{b-1;h,j}^{(s_{b-1,b}(\nu)+1)}u^{-s_{b-1,b}(\nu)-1}.$$

The right hand side is exactly the definition (2.54) of $\bar{E}_{a,b;i,j}(u)$ in this case. Now assume that $b > a+2$ and calculate using Lemma 2.2, relations [BK5, (3.9)] and [BK5, (3.11)] and the induction hypothesis:

$$\begin{aligned}
&[E_{a,b-1;i,k}(u), E_{b-1;k,j}^{(s_{b-1,b}(\nu)+2)}]u^{-1}\\
&=[[E_{a,b-2;i,1}(u), E_{b-2;1,k}^{(s_{b-2,b-1}(\nu)+1)}], E_{b-1;k,j}^{(s_{b-1,b}(\nu)+2)}]u^{-s_{b-2,b-1}(\nu)-1}\\
&=[E_{a,b-2;i,1}(u), [E_{b-2;1,k}^{(s_{b-2,b-1}(\nu)+1)}, E_{b-1;k,j}^{(s_{b-1,b}(\nu)+2)}]]u^{-s_{b-2,b-1}(\nu)-1}\\
&=[E_{a,b-2;i,1}(u), [E_{b-2;1,k}^{(s_{b-2,b-1}(\nu)+2)}, E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}]]u^{-s_{b-2,b-1}(\nu)-1}\\
&\quad - \sum_{h=1}^{\nu_{b-1}} [E_{a,b-2;i,1}(u), E_{b-2;1,h}^{(s_{b-2,b-1}(\nu)+1)}E_{b-1;h,j}^{(s_{b-1,b}(\nu)+1)}]u^{-s_{b-2,b-1}(\nu)-1}\\
&=[[E_{a,b-2;i,1}(u), E_{b-2;1,k}^{(s_{b-2,b-1}(\nu)+2)}], E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}]u^{-s_{b-2,b-1}(\nu)-1}\\
&\quad - \sum_{h=1}^{\nu_{b-1}} [E_{a,b-2;i,1}(u), E_{b-2;1,h}^{(s_{b-2,b-1}(\nu)+1)}]E_{b-1;h,j}^{(s_{b-1,b}(\nu)+1)}u^{-s_{b-2,b-1}(\nu)-1}\\
&=[\bar{E}_{a,b-1;i,k}(u), E_{b-1;k,j}^{(s_{b-1,b}(\nu)+1)}] - \sum_{h=1}^{\nu_{b-1}} E_{a,b-1;i,h}(u)E_{b-1;h,j}^{(s_{b-1,b}(\nu)+1)}u^{-1}.
\end{aligned}$$

Multiplying both sides by $u^{-s_{b-1,b}(\nu)}$ and using the definition (2.54) together with Lemma 2.2 once more gives the conclusion. □

2.5. **Hopf algebra structure.** In the special case that the shift matrix σ is the zero matrix, we denote $Y_n(\sigma)$ simply by Y_n. Observe that the parabolic generators $D_{1;i,j}^{(r)}$ of Y_n defined from (2.40) relative to the admissible shape $\nu = (n)$ are simply equal to the elements $T_{i,j}^{(r)}$ from (2.33). Hence the parabolic presentation from [BK5, (3.3)–(3.14)] asserts in this case that the elements $\{T_{i,j}^{(r)} \,|\, 1 \leq i,j \leq n, r > 0\}$ generate Y_n subject only to the relations

$$[T_{i,j}^{(r)}, T_{h,k}^{(s)}] = \sum_{t=0}^{\min(r,s)-1} \left(T_{i,k}^{(r+s-1-t)} T_{h,j}^{(t)} - T_{i,k}^{(t)} T_{h,j}^{(r+s-1-t)} \right) \tag{2.58}$$

for every $1 \leq h,i,j,k \leq n$ and $r,s > 0$, where $T_{i,j}^{(0)} = \delta_{i,j}$. This is precisely the RTT presentation for the *Yangian* associated to the Lie algebra $\mathfrak{gl}_n$ originating in the work of Faddeev, Reshetikhin and Takhtadzhyan [FRT]; see also [D] and [MNO, §1]. It is well known that the Yangian Y_n is actually a Hopf algebra with comultiplication $\Delta : Y_n \to Y_n \otimes Y_n$ and counit $\varepsilon : Y_n \to \mathbb{F}$ defined in terms of the generating function (2.33) by

$$\Delta(T_{i,j}(u)) = \sum_{k=1}^{n} T_{i,k}(u) \otimes T_{k,j}(u), \tag{2.59}$$

$$\varepsilon(T_{i,j}(u)) = \delta_{i,j}. \tag{2.60}$$

Note also that the algebra anti-automorphism $\tau : Y_n \to Y_n$ from (2.36) is a coalgebra anti-automorphism, i.e. we have that

$$\Delta \circ \tau = P \circ (\tau \otimes \tau) \circ \Delta \tag{2.61}$$

where P denotes the permutation operator $x \otimes y \mapsto y \otimes x$.

It is usually difficult to compute the comultiplication $\Delta : Y_n \to Y_n \otimes Y_n$ in terms of the generators $D_i^{(r)}, E_i^{(r)}$ and $F_i^{(r)}$. At least the case $n = 2$ can be worked out explicitly like in [M1, Definition 2.24]: we have that

$$\Delta(D_1(u)) = D_1(u) \otimes D_1(u) + D_1(u)E_1(u) \otimes F_1(u)D_1(u), \tag{2.62}$$

$$\Delta(D_2(u)) = D_2(u) \otimes D_2(u) + \sum_{k \geq 1} (-1)^k D_2(u)E_1(u)^k \otimes F_1(u)^k D_2(u), \tag{2.63}$$

$$\Delta(E_1(u)) = 1 \otimes E_1(u) + \sum_{k \geq 1} (-1)^k E_1(u)^k \otimes \widetilde{D}_1(u)F_1(u)^{k-1} D_2(u), \tag{2.64}$$

$$\Delta(F_1(u)) = F_1(u) \otimes 1 + \sum_{k \geq 1} (-1)^k D_2(u)E_1(u)^{k-1} \widetilde{D}_1(u) \otimes F_1(u)^k, \tag{2.65}$$

as can be checked directly from (2.59) and (2.32). The next lemma gives some further information about Δ for $n > 2$; cf. [CP2, Lemma 2.1]. To formulate the lemma precisely, recall from (2.20) how Y_n is viewed as a Q_n-graded algebra; the elements $T_{i,j}^{(r)}$ are of degree $(\varepsilon_i - \varepsilon_j)$ for this grading. For any $s \geq 0$ and $m \geq 1$ with $m + s \leq n$ there is an algebra embedding

$$\psi_s : Y_m \hookrightarrow Y_n, \quad D_i^{(r)} \mapsto D_{i+s}^{(r)}, E_i^{(r)} \mapsto E_{i+s}^{(r)}, F_i^{(r)} \mapsto F_{i+s}^{(r)}. \tag{2.66}$$

A different description of this map in terms of the generators $T_{i,j}^{(r)}$ of Y_n is given in [BK4, (4.2)]. The map ψ_s is *not* a Hopf algebra embedding; in particular, the maps $\Delta \circ \psi_s$ and $(\psi_s \otimes \psi_s) \circ \Delta$ from Y_m to $Y_n \otimes Y_n$ are definitely different if $m < n$.

Lemma 2.4. *For any $x \in Y_m$ such that $\psi_s(x) \in (Y_n)_\alpha$ for some $\alpha \in Q_n$, we have that*

$$\Delta(\psi_s(x)) - (\psi_s \otimes \psi_s)(\Delta(x)) \in \sum_{0 \neq \beta \in Q_n^+} (Y_n)_\beta \otimes (Y_n)_{\alpha-\beta}$$

where Q_n^+ here denotes the set of all elements $\sum_{i=1}^{n-1} c_i(\varepsilon_i - \varepsilon_{i+1})$ of the root lattice Q_n such that $c_i \geq 0$ for all $i \in \{1, \dots, s\} \cup \{m+s, \dots, n-1\}$.

Proof. It suffices to prove the lemma in the two special cases $s = 0$ and $m+s = n$. Consider first the case that $s = 0$. Then $\psi_s : Y_m \hookrightarrow Y_n$ is just the map sending $T_{i,j}^{(r)} \in Y_m$ to $T_{i,j}^{(r)} \in Y_n$ for $1 \leq i, j \leq m$ and $r > 0$. For these elements the statement of the lemma is clear from the explicit formula for Δ from (2.59). It follows in general since Y_m is generated by these elements and Q_n^+ is closed under addition.

Instead suppose that $m + s = n$. Let $\widetilde{T}_{i,j}^{(r)} := -S(T_{i,j}^{(r)})$ where S is the antipode. Then by [BK4, (4.2)], $\psi_s : Y_m \hookrightarrow Y_n$ is the map sending $\widetilde{T}_{i,j}^{(r)} \in Y_m$ to $\widetilde{T}_{i+s,j+s}^{(r)} \in Y_n$ for $1 \leq i, j \leq m, r > 0$. Since (2.59) implies that

$$\Delta(\widetilde{T}_{i,j}^{(r)}) = -\sum_{k=1}^{n} \sum_{t=0}^{r} \widetilde{T}_{k,j}^{(t)} \otimes \widetilde{T}_{i,k}^{(r-t)},$$

the proof can now be completed as in the previous paragraph. □

Now we can formulate a very useful result describing the effect of Δ on the generators of Y_n in general. Recall from (2.30) that $K_{(1^n)}^\sharp(\sigma)$ resp. $K_{(1^n)}^\flat(\sigma)$ denotes the two-sided ideal of the Borel subalgebra $Y_{(1^n)}^\sharp(\sigma)$ resp. $Y_{(1^n)}^\flat(\sigma)$ generated by the $E_i^{(r)}$ resp. the $F_i^{(r)}$; in the case σ is the zero matrix, we denote these simply by $K_{(1^n)}^\sharp$ and $K_{(1^n)}^\flat$. Also define

$$H_i(u) = \sum_{r \geq 0} H_i^{(r)} u^{-r} := \widetilde{D}_i(u) D_{i+1}(u) \tag{2.67}$$

for each $i = 1, \dots, n-1$. Since $\widetilde{D}_i(u) = -D_i(u)^{-1}$, we have that $H_i^{(0)} = -1$.

Theorem 2.5. *The comultiplication $\Delta : Y_n \to Y_n \otimes Y_n$ has the following properties:*

(i) $\Delta(D_i^{(r)}) \equiv \sum_{s=0}^r D_i^{(s)} \otimes D_i^{(r-s)} \pmod{K_{(1^n)}^\sharp \otimes K_{(1^n)}^\flat}$;

(ii) $\Delta(E_i^{(r)}) \equiv 1 \otimes E_i^{(r)} - \sum_{s=1}^r E_i^{(s)} \otimes H_i^{(r-s)} \pmod{(K_{(1^n)}^\sharp)^2 \otimes K_{(1^n)}^\flat}$;

(iii) $\Delta(F_i^{(r)}) \equiv F_i^{(r)} \otimes 1 - \sum_{s=1}^r H_i^{(r-s)} \otimes F_i^{(s)} \pmod{K_{(1^n)}^\sharp \otimes (K_{(1^n)}^\flat)^2}$.

Proof. This follows from Lemma 2.4, (2.62)–(2.65) and [BK5, Corollary 11.11]. □

Returning to the general case, there is for any shift matrix $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ a canonical embedding $Y_n(\sigma) \hookrightarrow Y_n$ such that the generators $D_i^{(r)}, E_i^{(r)}$ and $F_i^{(r)}$ of $Y_n(\sigma)$ from (2.2)–(2.4) map to the elements of Y_n with the same name. However, the higher root elements $E_{i,j}^{(r)}$ and $F_{i,j}^{(r)}$ of $Y_n(\sigma)$ do not in general map to the elements of Y_n with the same name under this embedding, and the elements $T_{i,j}^{(r)}$ of $Y_n(\sigma)$ do not in general map to the elements $T_{i,j}^{(r)}$ of Y_n. In particular, if $\sigma \neq 0$ we do *not* know a full set of relations for the generators $T_{i,j}^{(r)}$ of $Y_n(\sigma)$.

Write $\sigma = \sigma' + \sigma''$ where σ' is strictly lower triangular and σ'' is strictly upper triangular. Embedding the shifted Yangians $Y_n(\sigma)$, $Y_n(\sigma')$ and $Y_n(\sigma'')$ into Y_n in the canonical way, the first part of [BK5, Theorem 11.9] asserts that the comultiplication $\Delta : Y_n \to Y_n \otimes Y_n$ restricts to a map

$$\Delta : Y_n(\sigma) \to Y_n(\sigma') \otimes Y_n(\sigma''). \tag{2.68}$$

Also the restriction of the counit $\varepsilon : Y_n \to \mathbb{F}$ gives us the *trivial representation*

$$\varepsilon : Y_n(\sigma) \to \mathbb{F} \tag{2.69}$$

of the shifted Yangian, with $\varepsilon(D_i(u)) = 1$ and $\varepsilon(E_i(u)) = \varepsilon(F_i(u)) = 0$.

2.6. **The center of $\boldsymbol{Y_n(\sigma)}$.** Let us finally describe the center $Z(Y_n(\sigma))$ of $Y_n(\sigma)$. Recalling the notation (2.5), let

$$C_n(u) = \sum_{r \geq 0} C_n^{(r)} u^{-r} := D_1(u) D_2(u-1) \cdots D_n(u-n+1) \in Y_n(\sigma)[[u^{-1}]]. \tag{2.70}$$

In the case of the Yangian Y_n itself, there is a well known alternative description of the power series $C_n(u)$ in terms of quantum determinants due to Drinfeld [D] (see also [BK4, Theorem 8.6]). To recall this, given an $n \times n$ matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ with entries in some (not necessarily commutative) ring, set

$$\operatorname{rdet} A := \sum_{w \in S_n} \operatorname{sgn}(w) a_{1,w1} a_{2,w2} \cdots a_{n,wn}, \tag{2.71}$$

$$\operatorname{cdet} A := \sum_{w \in S_n} \operatorname{sgn}(w) a_{w1,1} a_{w2,2} \cdots a_{wn,n}, \tag{2.72}$$

where S_n is the symmetric group. Then, working in $Y_n[[u^{-1}]]$, we have that

$$C_n(u) = \operatorname{rdet} \begin{pmatrix} T_{1,1}(u-n+1) & T_{1,2}(u-n+1) & \cdots & T_{1,n}(u-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n-1,1}(u-1) & T_{n-1,2}(u-1) & \cdots & T_{n-1,n}(u-1) \\ T_{n,1}(u) & T_{n,2}(u) & \cdots & T_{n,n}(u) \end{pmatrix} \tag{2.73}$$

$$= \operatorname{cdet} \begin{pmatrix} T_{1,1}(u) & T_{1,2}(u-1) & \cdots & T_{1,n}(u-n+1) \\ \vdots & \vdots & \ddots & \vdots \\ T_{n-1,1}(u) & T_{n-1,2}(u-1) & \cdots & T_{n-1,n}(u-n+1) \\ T_{n,1}(u) & T_{n,2}(u-1) & \cdots & T_{n,n}(u-n+1) \end{pmatrix}. \tag{2.74}$$

In particular, in view of this alternative description, [MNO, Proposition 2.19] shows that

$$\Delta(C_n(u)) = C_n(u) \otimes C_n(u). \tag{2.75}$$

Theorem 2.6. *The elements $C_n^{(1)}, C_n^{(2)}, \dots$ are algebraically independent and generate $Z(Y_n(\sigma))$.*

Proof. Exploiting the embedding $Y_n(\sigma) \hookrightarrow Y_n$, it is known by [MNO, Theorem 2.13] that the elements $C_n^{(1)}, C_n^{(2)}, \dots$ are algebraically independent and generate $Z(Y_n)$ (see also [BK4, Theorem 7.2] for a slight variation on this argument). So they certainly belong to $Z(Y_n(\sigma))$. The fact that $Z(Y_n(\sigma))$ is no larger than $Z(Y_n)$ may be proved by passing to the associated graded algebra $\mathrm{gr}^{\mathrm{L}}\, Y_n(\sigma)$ from [BK5, Theorem 2.1] and using a variation on the trick from the proof of [MNO, Theorem 2.13]. We omit the details since we give an alternative argument in Corollary 6.11 below. $\square$

Recall the automorphisms $\mu_f : Y_n(\sigma) \to Y_n(\sigma)$ from (2.37). Define

$$SY_n(\sigma) := \{x \in Y_n(\sigma) \mid \mu_f(x) = x \text{ for all } f(u) \in 1 + u^{-1}\mathbb{F}[[u^{-1}]]\}. \tag{2.76}$$

Like in [MNO, Proposition 2.16], one can show that multiplication defines an algebra isomorphism

$$Z(Y_n(\sigma)) \otimes SY_n(\sigma) \stackrel{\sim}{\to} Y_n(\sigma). \tag{2.77}$$

Recalling (2.67), ordered monomials in the elements $\{H_i^{(r)} \mid i = 1, \dots, n-1, r > 0\}$, $\{E_{i,j}^{(r)} \mid 1 \le i < j \le n, r > s_{i,j}\}$ and $\{F_{i,j}^{(r)} \mid 1 \le i < j \le n, r > s_{j,i}\}$ form a basis for $SY_n(\sigma)$.

3. Finite W-algebras

In this chapter we review the definition of the finite W-algebras associated to nilpotent orbits in the Lie algebra $\mathfrak{gl}_N$, then explain their connection to the shifted Yangians. Again, much of this material is taken from [BK5], though there are some important new results too.

3.1. **Pyramids.** By a *pyramid* we mean a sequence $\pi = (q_1, \dots, q_l)$ of integers such that

$$0 < q_1 \le \cdots \le q_k, \quad q_{k+1} \ge \cdots \ge q_l > 0 \tag{3.1}$$

for some $0 \le k \le l$. We visualize π by means of a diagram consisting of q_1 bricks stacked in the first column, q_2 bricks stacked in the second column, ..., q_l bricks stacked in the lth column, where columns are numbered $1, 2, \dots, l$ from left to right. The *level* of the pyramid π is then the number l of non-empty columns and the *height* is the number $\max(q_1, \dots, q_l)$ of non-empty rows in the diagram. For example, the pyramid $\pi = (1, 2, 4, 3, 1)$ is of level 5 and height 4, and its diagram is

$$\begin{array}{ccccc} & & \boxed{4} & & \\ & & \boxed{5} & \boxed{8} & \\ & \boxed{2} & \boxed{6} & \boxed{9} & \\ \boxed{1} & \boxed{3} & \boxed{7} & \boxed{10} & \boxed{11} \end{array}\,. \tag{3.2}$$

As in this example, we always number the bricks of the diagram $1, 2, \dots, N$ down columns starting with the first column. Let $\mathrm{col}(i)$ denote the number of the column containing the entry $\boxed{i}$ in the diagram. We say that the pyramid is *left-justified* if $q_1 \ge \cdots \ge q_l$ and *right-justified* if $q_1 \le \cdots \le q_l$.

We often need to make the following two additional choices:

(a) an integer n greater than or equal to the height of π;
(b) an integer k with $0 \leq k \leq l$ such that $q_1 \leq \cdots \leq q_k, q_{k+1} \geq \cdots \geq q_l$ as in (3.1).

Having made such choices, we read off various further pieces of data from π. First, numbering the rows of the diagram of π by $1, 2, \dots, n$ from top to bottom so that the nth row is the last row containing l bricks, let p_i denote the number of bricks on the ith row. This defines the tuple $(p_1, \dots, p_n)$ of *row lengths*, with

$$0 \leq p_1 \leq \cdots \leq p_n = l. \tag{3.3}$$

Write row(i) for the number of the row containing the entry $\boxed{i}$ in the diagram of π. Next, define a shift matrix $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ from the equation

$$s_{i,j} := \begin{cases} \#\{c = 1, \dots, k \mid i > n - q_c \geq j\} & \text{if } i \geq j, \\ \#\{c = k+1, \dots, l \mid i \leq n - q_c < j\} & \text{if } i \leq j. \end{cases} \tag{3.4}$$

To make sense of this formula, we just point out that the pyramid π (hence all the other data) can easily be recovered given just this shift matrix σ and the level l, since its diagram consists of $p_i = l - s_{n,i} - s_{i,n}$ bricks on the ith row indented $s_{n,i}$ columns from the left edge and $s_{i,n}$ columns from the right edge. Finally, let

$$S_{i,j} := s_{i,j} + p_{\min(i,j)}. \tag{3.5}$$

3.2. **Finite *W*-algebras.** Let $\mathfrak{g}$ denote the Lie algebra $\mathfrak{gl}_N$, equipped with the trace form $(.,.)$. Given a pyramid $\pi = (q_1, \dots, q_l)$ with N bricks, there is a corresponding $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ defined by declaring that the ij-matrix unit $e_{i,j}$ is of degree $(\text{col}(j) - \text{col}(i))$ for each $1 \leq i, j \leq N$. Let $\mathfrak{h} := \mathfrak{g}_0, \mathfrak{p} := \bigoplus_{j \geq 0} \mathfrak{g}_j$ and $\mathfrak{m} := \bigoplus_{j<0} \mathfrak{g}_j$. Thus $\mathfrak{p}$ is a standard parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{h} \cong \mathfrak{gl}_{q_1} \oplus \cdots \oplus \mathfrak{gl}_{q_l}$, and $\mathfrak{m}$ is the opposite nilradical. Let $e \in \mathfrak{p}$ denote the nilpotent matrix

$$e = \sum_{i,j} e_{i,j} \tag{3.6}$$

summing over all pairs $\boxed{i}\boxed{j}$ of adjacent entries in the diagram; for example if π is as in (3.2) then $e = e_{5,8} + e_{2,6} + e_{6,9} + e_{1,3} + e_{3,7} + e_{7,10} + e_{10,11}$. The $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j$ is then a *good grading for* $e \in \mathfrak{g}_1$ in the sense of [KRW, EK].

The map $\chi : \mathfrak{m} \to \mathbb{F}, x \mapsto (x, e)$ is a Lie algebra homomorphism. Let I_χ denote the kernel of the associated homomorphism $U(\mathfrak{m}) \to \mathbb{F}$ (here $U(.)$ denotes universal enveloping algebra). We will work with the *twisted action* of $\mathfrak{m}$ on $U(\mathfrak{p})$ defined by setting

$$x \cdot u := \text{pr}_\chi([x, u]) \tag{3.7}$$

for $x \in \mathfrak{m}, u \in U(\mathfrak{p})$, where $\text{pr}_\chi : U(\mathfrak{g}) \to U(\mathfrak{p})$ denotes the projection along the direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{p}) \oplus U(\mathfrak{g})I_\chi$. Following [BK5, §8], we define the *finite W-algebra* corresponding to the pyramid π to be the subalgebra

$$W(\pi) := U(\mathfrak{p})^{\mathfrak{m}} = \{u \in U(\mathfrak{p}) \mid [x, u] \in U(\mathfrak{g})I_\chi \text{ for all } x \in \mathfrak{m}\} \tag{3.8}$$

of $U(\mathfrak{p})$. In this form, the definition of the algebra $W(\pi)$ goes back to Kostant [Ko2] and Lynch [Ly]; it is a special case of the construction due to Premet [P1] then

Gan and Ginzburg [GG] of non-commutative filtered deformations of the coordinate algebra of the Slodowy slice associated to the nilpotent orbit containing e.

To make the last statement precise, we need to introduce the *Kazhdan filtration* $\mathrm{F}_0 U(\mathfrak{p}) \subseteq \mathrm{F}_1 U(\mathfrak{p}) \subseteq \cdots$ of $U(\mathfrak{p})$. This can be defined simply by declaring that each matrix unit $e_{i,j} \in \mathfrak{p}$ is of degree

$$\deg(e_{i,j}) := (1 + \mathrm{col}(j) - \mathrm{col}(i)), \tag{3.9}$$

i.e. $\mathrm{F}_d U(\mathfrak{p})$ is the span of all monomials $e_{i_1,j_1} \cdots e_{i_r,j_r}$ in $U(\mathfrak{p})$ with $\deg(e_{i_1,j_1}) + \cdots + \deg(e_{i_r,j_r}) \leq d$. The associated graded algebra $\mathrm{gr}\, U(\mathfrak{p})$ is obviously identified with the symmetric algebra $S(\mathfrak{p})$ viewed as a graded algebra via (3.9) once more. We get induced a filtration $\mathrm{F}_0 W(\pi) \subseteq \mathrm{F}_1 W(\pi) \subseteq \cdots$ of $W(\pi)$, also called the Kazhdan filtration, by setting $\mathrm{F}_d W(\pi) := W(\pi) \cap \mathrm{F}_d U(\mathfrak{p})$; so $\mathrm{gr}\, W(\pi)$ is naturally a graded subalgebra of $\mathrm{gr}\, U(\mathfrak{p}) = S(\mathfrak{p})$. Let $\mathfrak{c}_g(e)$ denote the centralizer of e in $\mathfrak{g}$ and $\mathfrak{p}^\perp$ denote the nilradical of $\mathfrak{p}$. Also define elements $h \in \mathfrak{g}_0$ and $f \in \mathfrak{g}_{-1}$ so that (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ (taking $h = f = 0$ in the degenerate case $e = 0$). By [BK5, Lemma 8.1(ii)], we have that

$$\mathfrak{p} = \mathfrak{c}_{\mathfrak{g}}(e) \oplus [\mathfrak{p}^\perp, f]. \tag{3.10}$$

The projection $\mathfrak{p} \twoheadrightarrow \mathfrak{c}_{\mathfrak{g}}(e)$ along this direct sum decomposition induces a homomorphism $S(\mathfrak{p}) \twoheadrightarrow S(\mathfrak{c}_{\mathfrak{g}}(e))$. Now the precise statement is that the restriction of this homomorphism to $\mathrm{gr}\, W(\pi)$ is an isomorphism $\mathrm{gr}\, W(\pi) \xrightarrow{\sim} S(\mathfrak{c}_{\mathfrak{g}}(e))$ of graded algebras; see [Ly, Theorem 2.3].

3.3. **Invariants.** Given a pyramid $\pi = (q_1, \dots, q_l)$, choose integers n and k and use these choices to read off the extra data $(p_1, \dots, p_n)$ and $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ as in §3.1. Let $\rho : U(\mathfrak{p}) \to \mathbb{F}$ be the homomorphism with

$$\rho(e_{i,j}) = \delta_{i,j}(q_{\mathrm{col}(j)} + q_{\mathrm{col}(j)+1} + \cdots + q_l - n). \tag{3.11}$$

The corresponding one dimensional $\mathfrak{p}$-module will be denoted $\mathbb{F}_\rho$. We stress that this depends on our fixed choice of n, as do many subsequent definitions. For $1 \leq i, j \leq N$, introduce the shorthand

$$\tilde{e}_{i,j} := (-1)^{\mathrm{col}(j)-\mathrm{col}(i)} e_{i,j} + \delta_{i,j}(n - q_{\mathrm{col}(i)} - q_{\mathrm{col}(i)+1} - \cdots - q_l). \tag{3.12}$$

So, $\tilde{e}_{i,j}$ acts as zero on the module $\mathbb{F}_\rho$. For $1 \leq i, j \leq n$, $0 \leq x \leq n$ and $r \geq 1$ define

$$T_{i,j;x}^{(r)} := \sum_{s=1}^{r} \sum_{\substack{i_1,\dots,i_s \\ j_1,\dots,j_s}} (-1)^{\#\{t=1,\dots,s-1 \,|\, \mathrm{row}(j_t) \leq x\}} \tilde{e}_{i_1,j_1} \cdots \tilde{e}_{i_s,j_s} \tag{3.13}$$

where the second sum is over all $1 \leq i_1, \dots, i_s, j_1, \dots, j_s \leq N$ such that

(a) $\deg(e_{i_1,j_1}) + \cdots + \deg(e_{i_s,j_s}) = r$ (recall (3.9));
(b) $\mathrm{col}(i_t) \leq \mathrm{col}(j_t)$ for each $t = 1, \dots, s$;
(c) if $\mathrm{row}(j_t) > x$ then $\mathrm{col}(j_t) < \mathrm{col}(i_{t+1})$ for each $t = 1, \dots, s-1$;
(d) if $\mathrm{row}(j_t) \leq x$ then $\mathrm{col}(j_t) \geq \mathrm{col}(i_{t+1})$ for each $t = 1, \dots, s-1$;
(e) $\mathrm{row}(i_1) = i$, $\mathrm{row}(j_s) = j$;
(f) $\mathrm{row}(j_t) = \mathrm{row}(i_{t+1})$ for each $t = 1, \dots, s-1$.

Also set

$$T_{i,j;x}^{(0)} := \begin{cases} 1 & \text{if } i = j > x, \\ -1 & \text{if } i = j \leq x, \\ 0 & \text{if } i \neq j, \end{cases} \tag{3.14}$$

and introduce the generating function

$$T_{i,j;x}(u) := \sum_{r \geq 0} T_{i,j;x}^{(r)} u^{-r} \in U(\mathfrak{p})[[u^{-1}]]. \tag{3.15}$$

These remarkable elements were first introduced in [BK5, (9.6)]. As we will explain in the next section, certain $T_{i,j;x}^{(r)}$ in fact generate the subalgebra $W(\pi) = U(\mathfrak{p})^{\mathfrak{m}}$ of twisted $\mathfrak{m}$-invariants.

Here is a quite different description of the elements $T_{i,j;0}^{(r)}$ in the spirit of [BK5, (12.6)]. If either the ith or the jth row of the diagram is empty then we have simply that $T_{i,j;0}(u) = \delta_{i,j}$. Otherwise, let $a \in \{1, \dots, l\}$ be minimal such that $i > n - q_a$ and let $b \in \{2, \dots, l+1\}$ be maximal such that $j > n - q_{b-1}$. Using the shorthand $\pi(r,c)$ for the entry $(q_1 + \dots + q_c + r - n)$ in the rth row and the cth column of the diagram of π (which makes sense only if $r > n - q_c$), we have that

$$T_{i,j;0}(u) \;=\; u^{-S_{i,j}} \sum_{m=1}^{S_{i,j}} \sum_{\substack{r_0,\dots,r_m \\ c_0,\dots,c_m}} \prod_{t=1}^{m} \left(\tilde{e}_{\pi(r_{t-1},c_{t-1}),\pi(r_t,c_t-1)} \;+\; \delta_{\pi(r_{t-1},c_{t-1}),\pi(r_t,c_t-1)} u \right) \tag{3.16}$$

where the second summation is over all rows $r_0, \dots, r_m$ and columns $c_0, \dots, c_m$ such that $a = c_0 < \dots < c_m = b$, $r_0 = i$ and $r_m = j$, and $\max(n - q_{c_t - 1}, n - q_{c_t}) < r_t \leq n$ for each $t = 1, \dots, m-1$. This identity is proved by multiplying out the parentheses and comparing with the original definition (3.13).

3.4. **Finite W-algebras are quotients of shifted Yangians.** Now we can formulate the main theorem from [BK5] precisely. Continue with the notation from §3.3; in particular, we have made a fixed choice $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ of shift matrix corresponding to the pyramid π. Then, [BK5, Theorem 10.1] asserts that the elements

$$\{T_{i,i;i-1}^{(r)} \mid i = 1, \dots, n, r > s_{i,i}\}, \tag{3.17}$$

$$\{T_{i,i+1;i}^{(r)} \mid i = 1, \dots, n-1, r > s_{i,i+1}\}, \tag{3.18}$$

$$\{T_{i+1,i;i}^{(r)} \mid i = 1, \dots, n-1, r > s_{i+1,1}\} \tag{3.19}$$

of $U(\mathfrak{p})$ from (3.13) generate the subalgebra $W(\pi) = U(\mathfrak{p})^{\mathfrak{m}}$. Moreover, there is a unique surjective homomorphism

$$\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi) \tag{3.20}$$

under which the generators (2.2)–(2.4) of $Y_n(\sigma)$ map to the corresponding generators (3.17)–(3.19) of $W(\pi)$, i.e. $\kappa(D_i^{(r)}) = T_{i,i;i-1}^{(r)}, \kappa(E_i^{(r)}) = T_{i,i+1;i}^{(r)}$ and $\kappa(F_i^{(r)}) = T_{i+1,i;i}^{(r)}$. The kernel of κ is the two-sided ideal of $Y_n(\sigma)$ generated by the elements $\{D_1^{(r)} \mid r > p_1\}$. Finally, viewing $Y_n(\sigma)$ as a filtered algebra via the canonical filtration and $W(\pi)$ as a filtered algebra via the Kazhdan filtration, we have that $\kappa(\mathrm{F}_d Y_n(\sigma)) = \mathrm{F}_d W(\pi)$.

From now onwards we will abuse notation by using exactly the same notation for the elements of $Y_n(\sigma)$ (or $Y_n(\sigma)[[u^{-1}]]$) introduced in chapter 2 as for their images in $W(\pi)$ (or $W(\pi)[[u^{-1}]]$) under the map κ, relying on context to decide which we mean. So in particular we will denote the invariants (3.17)–(3.19) from now on just by $D_i^{(r)}, E_i^{(r)}$ and $F_i^{(r)}$. Thus, $W(\pi)$ is generated by these elements subject only to the relations (2.7)–(2.18) together with the one additional relation

$$D_1^{(r)} = 0 \qquad\qquad \text{for } r > p_1. \tag{3.21}$$

More generally, given an admissible shape $\nu = (\nu_1, \dots, \nu_m)$ for σ, $W(\pi)$ is generated by the parabolic generators (2.48)–(2.50) subject only to the relations from [BK5, (3.3)–(3.14)] together with the one additional relation

$$D_{1;i,j}^{(r)} = 0 \qquad\qquad \text{for } 1 \leq i, j \leq \nu_1 \text{ and } r > p_1. \tag{3.22}$$

These parabolic generators of $W(\pi)$ are also equal to certain of the $T_{i,j;x}^{(r)}$'s; see [BK5, Theorem 9.3] for the precise statement here.

We should also mention the special case that the pyramid π is an $n \times l$ rectangle, when the nilpotent matrix e consists of n Jordan blocks all of the same size l and the shift matrix σ is the zero matrix. In this case, the relation (3.22) implies that $W(\pi)$ is the quotient of the usual Yangian Y_n from §2.5 by the two-sided ideal generated by $\{T_{i,j}^{(r)} \mid 1 \leq i, j \leq n, r > l\}$. Hence $W(\pi)$ is isomorphic to the *Yangian of level l* introduced by Cherednik [C1, C2], as was first noticed by Ragoucy and Sorba [RS].

3.5. **More automorphisms.** The isomorphism type of the algebra $W(\pi)$ only actually depends on the conjugacy class of the nilpotent matrix $e \in \mathfrak{gl}_N$, i.e. on the row lengths $(p_1, \dots, p_n)$ of π, not on the pyramid π itself. To be precise, suppose that $\dot\pi$ is another pyramid with the same row lengths as π, and choose a shift matrix $\dot\sigma = (\dot s_{i,j})_{1 \leq i,j \leq n}$ corresponding to $\dot\pi$. Then, viewing $W(\pi)$ as a quotient of $Y_n(\sigma)$ and $W(\dot\pi)$ as a quotient of $Y_n(\dot\sigma)$, the automorphism ι from (2.34) obviously factors through the quotients to induce an isomorphism

$$\iota : W(\pi) \to W(\dot\pi). \tag{3.23}$$

In a similar fashion, the map τ from (2.35) induces an anti-isomorphism

$$\tau : W(\pi) \to W(\pi^t), \tag{3.24}$$

where here π^t denotes the *transpose pyramid* $(q_l, \dots, q_1)$ obtained by reversing the order of the columns of π.

Here is a useful invariant definition of τ. Recalling (3.11), let $\bar\rho : U(\mathfrak{p}) \to \mathbb{F}$ be the homomorphism with

$$\bar\rho(e_{i,j}) = \delta_{i,j}(q_1 + q_2 + \dots + q_{\mathrm{col}(j)} - n). \tag{3.25}$$

The corresponding one dimensional $\mathfrak{p}$-module will be denoted $\mathbb{F}_{\bar\rho}$. Also let $w_\pi \in S_N$ denote the permutation which when applied to the entries of the diagram π numbered in the standard way down columns from left to right gives the numbering down columns from right to left. For example, if π is as in (3.2) then

$w_\pi = (1\,11)(2\,9\,3\,10\,4\,5\,6\,7\,8)$:

$$\begin{array}{|c|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 4 & \multicolumn{2}{c}{} \\ \cline{3-4} \multicolumn{2}{c|}{} & 5 & 8 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 2 & 6 & 9 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 7 & 10 & 11 \\ \hline \end{array} \quad \xrightarrow{w_\pi} \quad \begin{array}{|c|c|c|c|c|}\cline{3-3} \multicolumn{2}{c|}{} & 5 & \multicolumn{2}{c}{} \\ \cline{3-4} \multicolumn{2}{c|}{} & 6 & 2 & \multicolumn{1}{c}{} \\ \cline{2-4} \multicolumn{1}{c|}{} & 9 & 7 & 3 & \multicolumn{1}{c}{} \\ \hline 11 & 10 & 8 & 4 & 1 \\ \hline \end{array}\,.$$

Let $\mathfrak{p}^\tau$ denote the parabolic subalgebra of $\mathfrak{gl}_N$ associated to the pyramid π^t. This is related to the usual transpose $\mathfrak{p}^t$ of the subalgebra $\mathfrak{p}$ by $\mathfrak{p}^\tau = w_\pi \mathfrak{p}^t w_\pi^{-1}$, viewing w_π here as a permutation matrix in the usual way. Define an algebra anti-isomorphism

$$\tau : U(\mathfrak{p}) \to U(\mathfrak{p}^\tau) \tag{3.26}$$

mapping $x \in \mathfrak{p}$ to $w_\pi x^t w_\pi^{-1} + (\rho - \bar{\rho})(x) \in U(\mathfrak{p}^\tau)$. In other words, the elements $\tilde{e}_{i,j} \in U(\mathfrak{p})$ from (3.12) map to the analogously defined elements $\tilde{e}_{w_\pi(j),w_\pi(i)} \in U(\mathfrak{p}^\tau)$. Considering the form of the definition (3.13) explicitly, one deduces from this that

$$\tau(T^{(r)}_{i,j;x}) = T^{(r)}_{j,i;x} \tag{3.27}$$

for all $1 \le i, j \le n, 0 \le x \le n$ and $r \ge 0$. Combining this with the results of §3.4, it follows that $\tau : U(\mathfrak{p}) \to U(\mathfrak{p}^\tau)$ maps the subalgebra $W(\pi)$ of $U(\mathfrak{p})$ to the subalgebra $W(\pi^t)$ of $U(\mathfrak{p}^\tau)$, and moreover its restriction to $W(\pi)$ coincides with the map (3.24).

There is one more useful automorphism of $W(\pi)$. For a scalar $c \in \mathbb{F}$, let $\eta_c : U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N)$ be the algebra automorphism mapping $e_{i,j} \mapsto e_{i,j} + \delta_{i,j} c$ for each $1 \le i, j \le N$. It is obvious from the definitions in §3.2 that this leaves the subalgebra $W(\pi)$ of $U(\mathfrak{gl}_N)$ invariant, hence it restricts to an algebra automorphism

$$\eta_c : W(\pi) \to W(\pi) \tag{3.28}$$

The following lemma gives a description of η_c in terms of the generators of $W(\pi)$.

Lemma 3.1. *For any $c \in \mathbb{F}$, the following equations hold:*

(i) $\eta_c(u^{p_i} D_i(u)) = (u+c)^{p_i} D_i(u+c)$ *for* $1 \le i \le n$*;*

(ii) $\eta_c(u^{s_{i,i+1}} E_i(u)) = (u+c)^{s_{i,i+1}} E_i(u+c)$ *for* $1 \le i < n$*;*

(iii) $\eta_c(u^{s_{i+1,i}} F_i(u)) = (u+c)^{s_{i+1,i}} F_i(u+c)$ *for* $1 \le i < n$*.*

Proof. It is immediate from (3.16) that

$$\eta_c(u^{S_{i,j}} T_{i,j;0}(u)) = (u+c)^{S_{i,j}} T_{i,j;0}(u+c).$$

We will deduce the lemma from this formula. To do so, let $\widehat{T}(u)$ denote the $n \times n$ matrix with ij-entry $T_{i,j;0}(u)$. Consider the Gauss factorization $\widehat{T}(u) = \widehat{F}(u)\widehat{D}(u)\widehat{E}(u)$ where $\widehat{D}(u)$ is a diagonal matrix with ii-entry $\widehat{D}_i(u) \in U(\mathfrak{p})[[u^{-1}]]$, $\widehat{E}(u)$ is an upper unitriangular matrix with ij-entry $\widehat{E}_{i,j}(u) \in U(\mathfrak{p})[[u^{-1}]]$ and $\widehat{F}(u)$ is a lower unitriangular matrix with ji-entry $\widehat{F}_{i,j}(u) \in U(\mathfrak{p})[[u^{-1}]]$. Thus,

$$T_{i,j;0}(u) = \sum_{k=1}^{\min(i,j)} \widehat{F}_{k,i}(u)\widehat{D}_k(u)\widehat{E}_{k,j}(u).$$

Since $S_{i,j} = s_{i,k} + p_k + s_{k,j}$, it follows that

$$\eta_c(T_{i,j;0}(u)) = \sum_{k=1}^{\min(i,j)} (1+cu^{-1})^{s_{i,k}} \widehat{F}_{k,i}(u)(1+cu^{-1})^{p_k} \widehat{D}_k(u)(1+cu^{-1})^{s_{k,j}} \widehat{E}_{k,j}(u).$$

From this equation we can read off immediately the Gauss factorization of the matrix $\eta_c(\widehat{T}(u))$, hence the matrices $\eta_c(\widehat{D}(u)), \eta_c(\widehat{E}(u))$ and $\eta_c(\widehat{F}(u))$, to get that

$$\begin{aligned}\eta_c(u^{p_i}\widehat{D}_i(u)) &= (u+c)^{p_i}\widehat{D}_i(u+c),\\ \eta_c(u^{s_{i,j}}\widehat{E}_{i,j}(u)) &= (u+c)^{s_{i,j}}\widehat{E}_{i,j}(u+c),\\ \eta_c(u^{s_{j,i}}\widehat{F}_{i,j}(u)) &= (u+c)^{s_{j,i}}\widehat{F}_{i,j}(u+c).\end{aligned}$$

The first of these equations gives (i), since by [BK5, Corollary 9.4] we have that $\widehat{D}_i(u) = D_i(u)$ in $U(\mathfrak{p})[[u^{-1}]]$. Similarly, (ii) and (iii) follow from the second and third equations with $j = i+1$, looking just at the negative powers of u and using [BK5, Corollary 9.4] again. □

The quotient map $\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi)$ from (3.20) of course depends on the choice of the shift matrix $\sigma = (s_{i,j})_{1 \le i, \le n}$. Hence the particular choice of generators $D_i^{(r)}, E_i^{(r)}$ and $F_i^{(r)}$ in our presentation of the algebra $W(\pi)$ also depends ultimately on the choice of the integers n and k made in §3.1. Using Lemma 3.1, we can explain what happens when we switch to a different choice $\dot\sigma = (\dot s_{i,j})_{1\le i,j\le \dot n}$ of shift matrix coming from different choices $\dot n$ and $\dot k$ of these integers. Assume without loss of generality that $n \ge \dot n$ and set $c := n - \dot n$. When working with $\dot\sigma$, we are numbering the rows of π by $1, 2, \dots, \dot n$ rather than by $1, 2, \dots, n$; in particular, the row lengths $(\dot p_1, \dots, \dot p_{\dot n})$ defined from $\dot\sigma$ are related to the original row lengths $(p_1, \dots, p_n)$ by the formula $\dot p_i = p_{i+c}$.

Lemma 3.2. *With the above notation, the generators $\dot D_i^{(r)}$, $\dot E_{i,j}^{(r)}$, $\dot F_{i,j}^{(r)}$ and $\dot T_{i,j}^{(r)}$ of $W(\pi)$ defined relative to $\dot\sigma$ are related to the original generators by the equations*

$$\begin{aligned}(u+c)^{\dot p_i}\dot D_i(u+c) &= u^{p_{i+c}} D_{i+c}(u),\\ (u+c)^{\dot s_{i,j}}\dot E_{i,j}(u+c) &= u^{s_{i+c,j+c}} E_{i+c,j+c}(u),\\ (u+c)^{\dot s_{j,i}}\dot F_{i,j}(u+c) &= u^{s_{j+c,i+c}} F_{i+c,j+c}(u),\\ (u+c)^{\dot S_{i,j}}\dot T_{i,j}(u+c) &= u^{S_{i+c,j+c}} T_{i+c,j+c}(u).\end{aligned}$$

Moreover, for $1 \le i \le c$, we have that $D_i(u) = 1$ and $E_{i,j}(u) = F_{i,j}(u) = 0$ for all $j > i$.

Proof. By the definition (3.13), the elements $\dot T_{i,j;x}^{(r)}$ of $U(\mathfrak{p})$ defined relative to $\dot\sigma$ are related to the original elements $T_{i,j;x}^{(r)}$ by the formula $\eta_c(\dot T_{i,j;x}^{(r)}) = T_{i+c,j+c;x+c}^{(r)}$; the automorphism η_c appears here because the elements $\tilde e_{i,j}$ from (3.12) involve n. Given this, it follows by Lemma 3.1 that

$$(u+c)^{\dot{p}_i}\dot{D}_i(u+c) = u^{p_{i+c}}D_{i+c}(u),$$
$$(u+c)^{\dot{s}_{i,i+1}}\dot{E}_i(u+c) = u^{s_{i+c,i+1+c}}E_{i+c}(u),$$
$$(u+c)^{\dot{s}_{i+1,i}}\dot{F}_i(u+c) = u^{s_{i+1+c,i+c}}F_{i+c}(u).$$

The remaining equations follow directly from these using (2.21)–(2.22) and (2.33). □

3.6. **Miura transform.** Recall from the definition that $W(\pi)$ is a subalgebra of $U(\mathfrak{p})$, where $\mathfrak{p}$ is the parabolic subalgebra $\bigoplus_{j\geq 0}\mathfrak{g}_j$ of $\mathfrak{g} = \mathfrak{gl}_N$ with Levi factor $\mathfrak{h} = \mathfrak{g}_0 \cong \mathfrak{gl}_{q_1}\oplus\cdots\oplus\mathfrak{gl}_{q_l}$. The elements $\{e_{i,j}^{[c]} \mid 1\leq c\leq l, 1\leq i,j\leq q_c\}$ defined from $e_{i,j}^{[c]} := e_{q_1+\cdots+q_{c-1}+i,q_1+\cdots+q_{c-1}+j}$ form a basis for $\mathfrak{h}$. We will often identify $U(\mathfrak{h})$ with $U(\mathfrak{gl}_{q_1})\otimes\cdots\otimes U(\mathfrak{gl}_{q_l})$ so that $e_{i,j}^{[c]}$ is identified with $1^{\otimes(c-1)}\otimes e_{i,j}\otimes 1^{\otimes(l-c)}$. Following [BK5, (11.5)], we define an algebra automorphism

$$\eta : U(\mathfrak{h}) \to U(\mathfrak{h}), \quad e_{i,j}^{[c]} \mapsto e_{i,j}^{[c]} + \delta_{i,j}(q_{c+1}+\cdots+q_l). \tag{3.29}$$

Also let $\xi : U(\mathfrak{p}) \twoheadrightarrow U(\mathfrak{h})$ be the algebra homomorphism induced by the natural projection $\mathfrak{p}\twoheadrightarrow\mathfrak{h}$. The composite

$$\mu := \eta\circ\xi : W(\pi) \to U(\mathfrak{h}) \tag{3.30}$$

is called the *Miura transform*. By [BK5, Theorem 11.4] or [Ly, Corollary 2.3.2], μ is an injective algebra homomorphism, allowing us to view $W(\pi)$ as a subalgebra of $U(\mathfrak{h})$.

Now suppose that $l = l'+l''$ for non-negative integers l', l'', and let $\pi' := (q_1,\dots,q_{l'})$ and $\pi'' := (q_{l'+1},\dots,q_l)$. We write $\pi = \pi'\otimes\pi''$ whenever a pyramid is cut into two in this way. Letting $\mathfrak{h}' := \mathfrak{gl}_{q_1}\oplus\cdots\oplus\mathfrak{gl}_{q_{l'}}$ and $\mathfrak{h}'' := \mathfrak{gl}_{q_{l'+1}}\oplus\cdots\oplus\mathfrak{gl}_{q_l}$, the Miura transform allows us to view the algebras $W(\pi')$ and $W(\pi'')$ as subalgebras of $U(\mathfrak{h}')$ and $U(\mathfrak{h}'')$, respectively. Moreover, identifying $\mathfrak{h}$ with $\mathfrak{h}'\oplus\mathfrak{h}''$ hence $U(\mathfrak{h})$ with $U(\mathfrak{h}')\otimes U(\mathfrak{h}'')$, it follows from the definition [BK5, (11.2)] and injectivity of the Miura transforms that the subalgebra $W(\pi)$ of $U(\mathfrak{h})$ is contained in the subalgebra $W(\pi')\otimes W(\pi'')$ of $U(\mathfrak{h}')\otimes U(\mathfrak{h}'')$. We denote the resulting inclusion homomorphism by

$$\Delta_{l',l''} : W(\pi) \to W(\pi')\otimes W(\pi''). \tag{3.31}$$

This is precisely the comultiplication from [BK5, §11]. It is coassociative in an obvious sense; see [BK5, Lemma 11.2]. Observing in the case π consists of a single column of height t that $W(\pi)$ is simply equal to $U(\mathfrak{gl}_t)$, the Miura transform μ for general π may be recovered by iterating the comultiplication a total of $(l-1)$ times to split π into its individual columns.

Let us explain the relationship between $\Delta_{l',l''}$ and the comultiplication Δ from (2.68). Let $\dot{\pi}'$ be the right-justified pyramid with the same row lengths as π', and let $\dot{\pi}''$ be the left-justified pyramid with the same row lengths as π''. So $\dot{\pi} := \dot{\pi}'\otimes\dot{\pi}''$ is a pyramid with the same row lengths as π. Fixing an integer n greater than or equal to the height of π, read off a shift matrix $\sigma = (s_{i,j})_{1\leq i,j\leq n}$ from the pyramid $\dot{\pi}$ by choosing the integer k in (3.4) to equal the integer l'. Finally define σ' resp. σ'' to be the strictly lower resp. upper triangular matrices with $\sigma = \sigma' + \sigma''$. Then, $W(\dot\pi)$ is naturally a quotient of the shifted Yangian $Y_n(\sigma)$ and similarly $W(\dot\pi')\otimes W(\dot\pi'')$ is a quotient of $Y_n(\sigma')\otimes Y_n(\sigma'')$. Composing these quotient maps with the canonical isomorphisms $W(\dot\pi) \xrightarrow{\sim} W(\pi)$ and $W(\dot\pi')\otimes W(\dot\pi'') \xrightarrow{\sim} W(\pi')\otimes W(\pi'')$ defined by (3.23), we obtain epimorphisms $Y_n(\sigma) \twoheadrightarrow W(\pi)$ and $Y_n(\sigma')\otimes Y_n(\sigma'') \twoheadrightarrow W(\pi')\otimes W(\pi'')$. Now the second part of [BK5, Theorem 11.9] together with [BK5, Remark 11.10] asserts that the following diagram commutes:

$$\begin{array}{ccc} Y_n(\sigma) & \xrightarrow{\Delta} & Y_n(\sigma')\otimes Y_n(\sigma'') \\ \downarrow & & \downarrow \\ W(\pi) & \xrightarrow{\Delta_{l',l''}} & W(\pi')\otimes W(\pi''). \end{array} \tag{3.32}$$

Using this diagram, the results about Δ obtained in §2.5 easily lead to analogous statements for the maps $\Delta_{l',l''} : W(\pi) \to W(\pi')\otimes W(\pi'')$ in general. For example, (2.61) implies that

$$\Delta_{l'',l'} \circ \tau = P \circ \tau \otimes \tau \circ \Delta_{l',l''}, \tag{3.33}$$

equality of maps from $W(\pi)$ to $W((\pi'')^t)\otimes W((\pi')^t)$. This can also be seen directly from the alternative description of τ as the restriction of the map (3.26).

Note finally that the trivial $Y_n(\sigma)$-module from (2.69) factors through the quotient map κ to induce a one dimensional $W(\pi)$-module on which $D_i^{(r)}, E_i^{(r)}$ and $F_i^{(r)}$ act as zero for all meaningful i and $r > 0$. We call this the *trivial* $W(\pi)$*-module*. Recalling (3.12)–(3.13), the trivial $W(\pi)$-module is precisely the restriction of the $U(\mathfrak{p})$-module $\mathbb{F}_\rho$ from (3.11) to the subalgebra $W(\pi)$.

3.7. **Vanishing of higher $T_{i,j}^{(r)}$'s.** Continue with the notation of §3.3; in particular, we have chosen n, k and the shift matrix $\sigma = (s_{i,j})_{1\le i,j\le n}$ as there. We wish to show that $T_{i,j}^{(r)} \in W(\pi)$, i.e. the image of the element $T_{i,j}^{(r)} \in Y_n(\sigma)$ from (2.33) under the homomorphism $\kappa : Y_n(\sigma) \to W(\pi)$, is zero whenever $r > S_{i,j}$. In order to prove this, we first derive a recursive formula for $T_{i,j}^{(r)}$ as an element of $U(\mathfrak{p})$.

Given $1 \le x \le k+1$ and $k \le y \le l$, let $\pi_{x,y}$ denote the subpyramid $(q_x, q_{x+1}, \dots, q_y)$ of level $(y-x+1)$ of our fixed pyramid $\pi = (q_1,\dots,q_l)$ of level l. Fix the choice $\sigma_{x,y}$ of shift matrix for $\pi_{x,y}$ defined according to the formula (3.4) but replacing k there by $(k-x+1)$. Writing $\mathfrak{p}_{x,y}$ for the parabolic associated to $\pi_{x,y}$, define an algebra embedding

$$J_{x,y} : U(\mathfrak{p}_{x,y}) \to U(\mathfrak{p}) \tag{3.34}$$

mapping $e_{i,j} \in U(\mathfrak{p}_{x,y})$ to $e_{q_1+\cdots+q_{x-1}+i,q_1+\cdots+q_{x-1}+j} - \delta_{i,j}(q_{y+1}+\cdots+q_l) \in U(\mathfrak{p})$. In other words, $\tilde e_{i,j} \in U(\mathfrak{p}_{x,y})$ maps to $\tilde e_{q_1+\cdots+q_{x-1}+i,q_1+\cdots+q_{x-1}+j} \in U(\mathfrak{p})$. In the remainder of the section, we need to consider elements both of $W(\pi) \subseteq U(\mathfrak{p})$ and of $W(\pi_{x,y}) \subseteq U(\mathfrak{p}_{x,y})$, relying on the context to discern which we mean. For instance, recalling the definitions from §2.4, the notation $J_{1,l-1}(\bar E_{a,b;i,j}^{(r)})$ in the following lemma means the image of the element $\bar E_{a,b;i,j}^{(r)}$ of $W(\pi_{1,l-1})$ under the map $J_{1,l-1}$.

Lemma 3.3. *Assume that $l \ge 2$ and work relative to the minimal admissible shape $\nu = (\nu_1,\dots,\nu_m)$ for σ.*

(i) *If $q_1 \geq q_l$ and $k \leq l-1$, then the following equalities hold in $U(\mathfrak{p})$ for all meaningful a,b,i,j,r:*

$$D_{a;i,j}^{(r)} = J_{1,l-1}(D_{a;i,j}^{(r)}) + \delta_{a,m}\sum_{h=1}^{\nu_m} J_{1,l-1}(D_{m;i,h}^{(r-1)})\tilde{e}_{q_1+\cdots+q_{l-1}+h,q_1+\cdots+q_{l-1}+j} + \delta_{a,m}\left[J_{1,l-1}(D_{m;i,j}^{(r-1)}),\tilde{e}_{q_1+\cdots+q_{l-1}+j-q_l,q_1+\cdots+q_{l-1}+j}\right],$$

$$E_{a,b;i,j}^{(r)} = \begin{cases} J_{1,l-1}(E_{a,b;i,j}^{(r)}) & \text{if } b<m,\\ J_{1,l-1}(\bar{E}_{a,m;i,j}^{(r)}) \\ \quad + \sum_{h=1}^{\nu_m} J_{1,l-1}(E_{a,m;i,h}^{(r-1)})\tilde{e}_{q_1+\cdots+q_{l-1}+h,q_1+\cdots+q_{l-1}+j} \\ \quad + \left[J_{1,l-1}(E_{a,m;i,j}^{(r-1)}),\tilde{e}_{q_1+\cdots+q_{l-1}+j-q_l,q_1+\cdots+q_{l-1}+j}\right] & \text{if } b=m,\end{cases}$$

$$F_{a,b;i,j}^{(r)} = J_{1,l-1}(F_{a,b;i,j}^{(r)}).$$

(ii) *If $q_1 \leq q_l$ and $k \geq 1$, then the following equalities hold in $U(\mathfrak{p})$ for all meaningful a,b,i,j,r:*

$$D_{a;i,j}^{(r)} = J_{2,l}(D_{a;i,j}^{(r)}) + \delta_{a,m}\sum_{h=1}^{\nu_m}\tilde{e}_{i,h}J_{2,l}(D_{m;h,j}^{(r-1)}) + \delta_{a,m}\left[\tilde{e}_{i,q_2+i},J_{2,l}(D_{m;i,j}^{(r-1)})\right],$$

$$E_{a,b;i,j}^{(r)} = J_{2,l}(E_{a,b;i,j}^{(r)}),$$

$$F_{a,b;i,j}^{(r)} = \begin{cases} J_{2,l}(F_{a,b;i,j}^{(r)}) & \text{if } b<m,\\ J_{2,l}(\bar{F}_{a,m;i,j}^{(r)}) \\ \quad + \sum_{h=1}^{\nu_m}\tilde{e}_{i,h}J_{2,l}(F_{a,m;h,j}^{(r-1)}) + \left[\tilde{e}_{i,q_2+i},J_{2,l}(F_{a,m;i,j}^{(r-1)})\right] & \text{if } b=m.\end{cases}$$

Proof. (i) The first equation involving $D_{a;i,j}^{(r)}$ and the second two equations in the case $b=a+1$ follow immediately from [BK5, Lemma 10.4]. The second two equations for $b>a+1$ may then be deduced in exactly the same way as [BK5, Lemma 4.2]. In the difficult case when $b=m$, one needs to use Lemma 2.3 and also the observation that

$$\left[J_{1,l-1}(E_{a,m-1;i,h}^{(r-s_{m-1,m}(\nu))}),\tilde{e}_{q_1+\cdots+q_{l-1}+j-q_l,q_1+\cdots+q_{l-1}+j}\right]=0$$

for any $1\leq h\leq \nu_m$ along the way. The latter fact is checked by considering the expansion of $E_{a,m-1;i,h}^{(r-s_{m-1,m}(\nu))}$ using [BK5, Theorem 9.3] and Lemma 2.2.

(ii) Argue similarly using [BK5, Lemma 10.11] instead of [BK5, Lemma 10.4], or alternatively apply the map τ from (3.26) to (i). $\square$

Lemma 3.4. *Assume that $l\geq 2$, $1\leq i,j\leq n$ and $r>0$.*

(i) *If $q_1 \geq q_l$ and $k \leq l-1$ then*

$$T_{i,j}^{(r)} = J_{1,l-1}(T_{i,j}^{(r)}) - \sum_{\substack{1 \leq h \leq n-q_l \\ s_{h,j} \leq r}} J_{1,l-1}(T_{i,h}^{(r-s_{h,j})}) J_{1,l-1}(T_{h,j}^{(s_{h,j})})$$
$$+ \sum_{n-q_l < h \leq n} J_{1,l-1}(T_{i,h}^{(r-1)}) \tilde{e}_{q_1+\cdots+q_l+h-n, q_1+\cdots+q_l+j-n}$$
$$+ \left[J_{1,l-1}(T_{i,j}^{(r-1)}), \tilde{e}_{q_1+\cdots+q_{l-1}+j-n, q_1+\cdots+q_l+j-n} \right],$$

omitting the last three terms on the right hand side if $j \leq n - q_l$.

(ii) *If $q_1 \leq q_l$ and $k \geq 1$ then*

$$T_{i,j}^{(r)} = J_{2,l}(T_{i,j}^{(r)}) - \sum_{\substack{1 \leq h \leq n-q_1 \\ s_{i,h} \leq r}} J_{2,l}(T_{i,h}^{(s_{i,h})}) J_{2,l}(T_{h,j}^{(r-s_{i,h})})$$
$$+ \sum_{n-q_1 < h \leq n} \tilde{e}_{q_1+i-n, q_1+h-n} J_{2,l}(T_{h,j}^{(r-1)}) + \left[\tilde{e}_{q_1+i-n, q_1+q_2+i-n}, J_{2,l}(T_{i,j}^{(r-1)}) \right],$$

omitting the last three terms on the right hand side if $j \leq n - q_1$.

Proof. (i) Let $\nu = (\nu_1, \dots, \nu_m)$ be the minimal admissible shape for σ. Take $1 \leq a, b \leq m, 1 \leq i \leq \nu_a, 1 \leq j \leq \nu_b$ and $r > 0$. By definition,

$$T_{\nu_1+\cdots+\nu_{a-1}+i, \nu_1+\cdots+\nu_{b-1}+j}(u) = \sum_{c=1}^{\min(a,b)} \sum_{s,t=1}^{\nu_c} F_{c,a;i,s}(u) D_{c;s,t}(u) E_{c,b;t,j}(u).$$

Now apply Lemma 3.3(i) to rewrite the terms on the right hand side then simplify using the definition (2.54).

(ii) Similar, or apply τ to (i). □

Theorem 3.5. *The generators $T_{i,j}^{(r)}$ of $W(\pi)$ are zero for all $1 \leq i, j \leq n$ and $r > S_{i,j}$.*

Proof. Proceed by induction on the level l. The base case $l = 1$ is easy to verify directly from the definitions. For $l > 1$, assume that $q_1 \geq q_l$, the argument in the case $q_1 \leq q_l$ being entirely similar. In view of Lemma 3.2 we just need to prove the result for one particular choice of the shift matrix σ, so we may assume moreover that $k \leq l - 1$. Noting that $S_{i,j} - s_{h,j} = S_{i,h}$ for $i, h \leq j$, the induction hypothesis implies that all the terms on the right hand side of Lemma 3.4(i) are zero if $r > S_{i,j}$. Hence $T_{i,j}^{(r)} = 0$ as required. □

Finally we describe some PBW bases for the algebra $W(\pi)$. Recalling the definition of the Kazhdan filtration on $W(\pi)$ from §3.2, [BK5, Theorem 6.2] shows that the associated graded algebra $\operatorname{gr} W(\pi)$ is free commutative on generators

$$\{\operatorname{gr}_r D_i^{(r)} \mid 1 \leq i \leq n, s_{i,i} < r \leq S_{i,i}\}, \tag{3.35}$$

$$\{\operatorname{gr}_r E_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{i,j} < r \leq S_{i,j}\}, \tag{3.36}$$

$$\{\operatorname{gr}_r F_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{j,i} < r \leq S_{j,i}\}. \tag{3.37}$$

Hence, as in [BK5, Corollary 6.3], the monomials in the elements

$$\{D_i^{(r)} \mid 1 \le i \le n, s_{i,i} < r \le S_{i,i}\}, \tag{3.38}$$

$$\{E_{i,j}^{(r)} \mid 1 \le i < j \le n, s_{i,j} < r \le S_{i,j}\}, \tag{3.39}$$

$$\{F_{i,j}^{(r)} \mid 1 \le i < j \le n, s_{j,i} < r \le S_{j,i}\} \tag{3.40}$$

taken in some fixed order give a basis for the algebra $W(\pi)$.

Lemma 3.6. *The associated graded algebra* $\operatorname{gr} W(\pi)$ *is free commutative on generators* $\{\operatorname{gr}_r T_{i,j}^{(r)} \mid 1 \le i, j \le n, s_{i,j} < r \le S_{i,j}\}$. *Hence, the monomials in the elements* $\{T_{i,j}^{(r)} \mid 1 \le i, j \le n, s_{i,j} < r \le S_{i,j}\}$ *taken in some fixed order form a basis for* $W(\pi)$.

Proof. Similar to the proof of Lemma 2.1, but using Theorem 3.5 too. □

3.8. **Harish-Chandra homomorphisms.** Finally in this chapter we review the classical description of the center $Z(U(\mathfrak{gl}_N))$. Recalling the notation (2.71)–(2.72), define a monic polynomial

$$Z_N(u) = \sum_{r=0}^{N} Z_N^{(r)} u^{N-r} \in U(\mathfrak{gl}_N)[u] \tag{3.41}$$

by setting

$$Z_N(u) := \operatorname{rdet} \begin{pmatrix} e_{1,1} + u - N + 1 & \cdots & e_{1,N-1} & e_{1,N} \\ \vdots & \ddots & \vdots & \vdots \\ e_{N-1,1} & \cdots & e_{N-1,N-1} + u - 1 & e_{N-1,N} \\ e_{N,1} & \cdots & e_{N,N-1} & e_{N,N} + u \end{pmatrix} \tag{3.42}$$

$$= \operatorname{cdet} \begin{pmatrix} e_{1,1} + u & e_{1,2} & \cdots & e_{1,N} \\ e_{2,1} & e_{2,2} + u - 1 & \cdots & e_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ e_{N,1} & e_{N,2} & \cdots & e_{N,N} + u - N + 1 \end{pmatrix}. \tag{3.43}$$

Then, the coefficients $Z_N^{(1)}, \dots, Z_N^{(N)}$ of this polynomial are algebraically independent and generate the center $Z(U(\mathfrak{gl}_N))$. For a proof, see [CL, §2.2] where this is deduced from the classical Capelli identity or [MNO, Remark 2.11] where it is deduced from (2.73)–(2.74).

So it is natural to parametrize the central characters of $U(\mathfrak{gl}_N)$ by monic polynomials in $\mathbb{F}[u]$ of degree N, the polynomial $f(u)$ corresponding to the central character $Z(U(\mathfrak{gl}_N)) \to \mathbb{F}, Z_N(u) \mapsto f(u)$. Let P denote the free abelian group

$$P = \bigoplus_{a \in \mathbb{F}} \mathbb{Z}\varepsilon_a. \tag{3.44}$$

Given a monic $f(u) \in \mathbb{F}[u]$ of degree N, we associate the element

$$\theta = \sum_{a \in \mathbb{F}} c_a \varepsilon_a \in P \tag{3.45}$$

whose coefficients $\{c_a \mid a \in \mathbb{F}\}$ are defined from the factorization

$$f(u) = \prod_{a \in \mathbb{F}} (u+a)^{c_a}. \tag{3.46}$$

This defines a bijection between the set of monic polynomials of degree N and the set of elements $\theta \in P$ whose coefficients are non-negative integers summing to N. We will from now on always use this latter set to parametrize central characters.

Let us compute the images of the elements $Z_N^{(1)}, \dots, Z_N^{(N)}$ under the Harish-Chandra homomorphism. Let $\mathfrak{d}_N$ denote the standard Cartan subalgebra of $\mathfrak{gl}_N$ with basis $e_{1,1}, \dots, e_{N,N}$ and let $\varepsilon_1, \dots, \varepsilon_N$ be the dual basis for $\mathfrak{d}_N^*$. We often represent an element $\alpha \in \mathfrak{d}_N^*$ simply as an N-tuple $\alpha = (a_1, \dots, a_N)$ of elements of the field $\mathbb{F}$, defined from $\alpha = \sum_{i=1}^N a_i \varepsilon_i$. Also let $\mathfrak{b}_N$ be the standard Borel subalgebra consisting of upper triangular matrices. We will parametrize highest weight modules already in "ρ-shifted notation": for $\alpha \in \mathfrak{d}_N^*$, let $M(\alpha)$ denote the *Verma module* of highest weight $(\alpha - \rho)$, namely, the module

$$M(\alpha) := U(\mathfrak{gl}_N) \otimes_{U(\mathfrak{b}_N)} \mathbb{F}_{\alpha-\rho} \tag{3.47}$$

induced from the one dimensional $\mathfrak{b}_N$-module of weight $(\alpha - \rho)$, where ρ in this place only means the weight $-\varepsilon_2 - 2\varepsilon_3 - \cdots - (N-1)\varepsilon_N$. Thus, if $\alpha = (a_1, \dots, a_N)$, the diagonal matrix $e_{i,i}$ acts on the highest weight space of $M(\alpha)$ by the scalar $(a_i + i - 1)$. Viewing the symmetric algebra $S(\mathfrak{d}_N)$ as an algebra of functions on $\mathfrak{d}_N^*$, with the symmetric group S_N acting by $we_{i,i} := e_{wi,wi}$ as usual, the Harish-Chandra homomorphism

$$\Psi_N : Z(U(\mathfrak{gl}_N)) \xrightarrow{\sim} S(\mathfrak{d}_N)^{S_N} \tag{3.48}$$

may be defined as the map sending $z \in Z(U(\mathfrak{gl}_N))$ to the unique element of $S(\mathfrak{d}_N)$ with the property that z acts on $M(\alpha)$ by the scalar $(\Psi_N(z))(\alpha)$ for each $\alpha \in \mathfrak{d}_N^*$. Using (3.43) it is easy to see directly from this definition that

$$\Psi_N(Z_N(u)) = (u + e_{1,1})(u + e_{2,2}) \cdots (u + e_{N,N}). \tag{3.49}$$

The coefficients on the right hand side are the elementary symmetric functions. Define the *content* $\theta(\alpha)$ of the weight $\alpha = (a_1, \dots, a_N) \in \mathfrak{d}_N^*$ by setting

$$\theta(\alpha) := \varepsilon_{a_1} + \cdots + \varepsilon_{a_N} \in P. \tag{3.50}$$

By (3.49), the central character of the Verma module $M(\alpha)$ is precisely the central character parametrized by $\theta(\alpha)$.

Now return to the setup of §3.2. The restriction of the projection $\mathrm{pr}_\chi : U(\mathfrak{g}) \to U(\mathfrak{p})$ defined after (3.7) defines an algebra homomorphism

$$\psi : Z(U(\mathfrak{gl}_N)) \to Z(W(\pi)). \tag{3.51}$$

Applying this to the polynomial $Z_N(u)$ we obtain elements $\psi(Z_N^{(1)}), \dots, \psi(Z_N^{(N)})$ of $Z(W(\pi))$. The following lemma explains the relationship between these elements and the elements $C_n^{(1)}, C_n^{(2)}, \dots$ of $Z(W(\pi))$ defined by the formula (2.70).

Lemma 3.7. $\psi(Z_N(u)) = u^{p_1}(u-1)^{p_2} \cdots (u-n+1)^{p_n} C_n(u)$.

Proof. Using (3.42), it is easy to see that the image of $\psi(Z_N(u))$ under the Miura transform $\mu : W(\pi) \to U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l})$ from (3.30) is equal to $Z_{q_1}(u) \otimes \cdots \otimes Z_{q_l}(u)$. So, since μ is injective, we have to check that $\mu(u^{p_1}(u-1)^{p_2} \cdots (u-n+1)^{p_n} C_n(u))$ also equals $Z_{q_1}(u) \otimes \cdots \otimes Z_{q_l}(u)$. By (2.75), we have that $\mu(C_n(u)) = C_n(u) \otimes \cdots \otimes C_n(u)$ (l times). Therefore it just remains to observe in the special case that π consists of a single column of height $t \leq n$, i.e. when $W(\pi) = U(\mathfrak{gl}_t)$, that

$$(u-n+t) \cdots (u-n+2)(u-n+1) C_n(u) = Z_t(u).$$

To see this, one reduces using Lemma 3.2 to the case $t = n$, when it follows as in [MNO, Remark 2.11]. □

We can also consider the Harish-Chandra homomorphism

$$\Psi_{q_1} \otimes \cdots \otimes \Psi_{q_l} : Z(U(\mathfrak{h})) \xrightarrow{\sim} S(\mathfrak{d}_N)^{S_{q_1} \times \cdots \times S_{q_l}} \tag{3.52}$$

for the Levi subalgebra $\mathfrak{h} = \mathfrak{gl}_{q_1} \oplus \cdots \oplus \mathfrak{gl}_{q_l}$, identifying $U(\mathfrak{h})$ with $U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l})$. It follows immediately from (3.49) and the explicit computation of $\mu(\psi(Z_N(u)))$ made in the proof of Lemma 3.7 that the following diagram commutes:

$$\begin{array}{ccc} Z(U(\mathfrak{gl}_N)) & \xrightarrow[\Psi_N]{\sim} & S(\mathfrak{d}_N)^{S_N} \\ {\scriptstyle \mu \circ \psi}\downarrow & & \downarrow \\ Z(U(\mathfrak{h})) & \xrightarrow[\Psi_{q_1} \otimes \cdots \otimes \Psi_{q_l}]{\sim} & S(\mathfrak{d}_N)^{S_{q_1} \times \cdots \times S_{q_l}} \end{array} \tag{3.53}$$

where the right hand map is the obvious inclusion. Hence the Harish-Chandra homomorphism Ψ_N factors through the map ψ, as has been observed in much greater generality than this by Lynch [Ly, Proposition 2.6] and Premet [P1, 6.2]. In particular this shows that ψ is injective, so the elements $\psi(Z_N^{(1)}), \ldots, \psi(Z_N^{(N)})$ of $Z(W(\pi))$ are actually algebraically independent.

4. Dual canonical bases

The appropriate setting for the combinatorics underlying the representation theory of the algebras $W(\pi)$ is provided by certain dual canonical bases for representations of the Lie algebra $\mathfrak{gl}_\infty$. In this chapter we review these matters following [B] closely. Throughout, π denotes a fixed pyramid $(q_1, \ldots, q_l)$ with row lengths $(p_1, \ldots, p_n)$, and $N = p_1 + \cdots + p_n = q_1 + \cdots + q_l$.

4.1. Tableaux.

By a π-*tableau* we mean a filling of the boxes of the diagram of π with arbitrary elements of the ground field $\mathbb{F}$. Let $\mathrm{Tab}(\pi)$ denote the set of all such π-tableaux. If $\pi = \pi' \otimes \pi''$ for pyramids π' and π'' and we are given a π'-tableau A' and a π''-tableau A'', we write $A' \otimes A''$ for the π-tableau obtained by concatenating A' and A''. For example,

$$A = \begin{array}{|c|c|c|} \hline 1 & & \\ \hline 0 & 3 & 2 \\ \hline 4 & 3 & 1 \\ \hline \end{array} = \begin{array}{|c|c|} \hline 1 & \\ \hline 0 & 3 \\ \hline 4 & 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array} = \begin{array}{|c|} \hline 1 \\ \hline 0 \\ \hline 4 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 3 \\ \hline 3 \\ \hline \end{array} \otimes \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}.$$

We always number the rows of $A \in \mathrm{Tab}(\pi)$ by $1, \ldots, n$ from top to bottom and the columns by $1, \ldots, l$ from left to right, like for the diagram of π.

We will use two different ways of reading the entries of a π-tableau A to get an N-tuple $\alpha = (a_1, \dots, a_N) \in \mathbb{F}^N$, i.e. a weight in $\mathfrak{d}_N^*$. The first is the *row reading* denoted $\rho(A)$ defined by reading the entries of A along rows starting with the top row (confusion with (3.11) seems unlikely), the second is the column reading denoted $\gamma(A)$ defined by reading the entries of A down columns starting with the leftmost column. For example, if A is as in the previous paragraph, $\rho(A) = (1, 0, 3, 2, 4, 3, 1)$ and $\gamma(A) = (1, 0, 4, 3, 3, 2, 1)$. Define the *content* $\theta(A)$ of A to be the content of either of the weights $\rho(A)$ or $\gamma(A)$ in the sense of (3.50), an element of the free abelian group $P = \bigoplus_{a \in \mathbb{F}} \mathbb{Z}\varepsilon_a$.

We say that two π-tableaux A and B are *row equivalent*, written $A \sim_{\text{ro}} B$, if one can be obtained from the other by permuting entries within rows. The notion $\sim_{\text{co}}$ of *column equivalence* is defined similarly. Let $\text{Row}(\pi)$ denote the set of all row equivalence classes of π-tableaux. We refer to elements of $\text{Row}(\pi)$ as *row symmetrized π-tableaux*. Also define the set $\text{Col}(\pi)$ of *column strict π-tableaux*, namely, the π-tableaux whose entries are strictly increasing up columns from bottom to top in the partial order $\geq$ on $\mathbb{F}$ defined at the end of the introduction. We stress the deliberate asymmetry of these definitions: $\text{Col}(\pi)$ is a subset of $\text{Tab}(\pi)$ but $\text{Row}(\pi)$ is a quotient.

We need a couple of variations on the usual definition of the *Bruhat ordering* on tableaux. First we have the ordering $\geq$ on the set $\text{Row}(\pi)$, defined as follows. Given π-tableaux A and B, write $A \downarrow B$ if B is obtained from A by swapping an entry x in the ith row of A with an entry y in the jth row of A, and moreover we have that $i < j$ and $x > y$. For example,

$$\begin{array}{|c|c|c}\hline 2&5&\multicolumn{1}{|c}{}\\\hline 7&7&\multicolumn{1}{|c}{}\\\hline 3&3&\multicolumn{1}{|c|}{5}\\\hline\end{array}\ \downarrow\ \begin{array}{|c|c|c}\hline 2&3&\multicolumn{1}{|c}{}\\\hline 7&7&\multicolumn{1}{|c}{}\\\hline 3&5&\multicolumn{1}{|c|}{5}\\\hline\end{array}\ \downarrow\ \begin{array}{|c|c|c}\hline 2&3&\multicolumn{1}{|c}{}\\\hline 7&3&\multicolumn{1}{|c}{}\\\hline 7&5&\multicolumn{1}{|c|}{5}\\\hline\end{array}\ .$$

Now $A \geq B$ means that there exists $r \geq 1$ and π-tableaux $A_1, \dots, A_r$ such that

$$A \sim_{\text{ro}} A_1 \downarrow \cdots \downarrow A_r \sim_{\text{ro}} B. \tag{4.1}$$

It is obvious that if $A \geq B$ then $\theta(A) = \theta(B)$; conversely, if $\theta(A) = \theta(B)$ then there exists $C \in \text{Row}(\pi)$ such that $A \geq C \leq B$. Similarly, we define an ordering $\geq'$ on the set $\text{Col}(\pi)$. Given π-tableaux A and B, write $A \rightarrow B$ if B is obtained from A by swapping an entry x in the ith column with an entry y in the jth column, such that $i < j$ and $x > y$. Then $A \geq' B$ means that there exist $A_1, \dots, A_r \in \text{Tab}(\pi)$ such that

$$A \sim_{\text{co}} A_1 \rightarrow \cdots \rightarrow A_r \sim_{\text{co}} B. \tag{4.2}$$

Also define an equivalence relation $\parallel$ on $\text{Col}(\pi)$ by declaring that $A \parallel B$ if B can be obtained from A by shuffling the columns in such a way that the relative position of all columns belonging to the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$ remains the same. The partial order $\geq'$ on $\text{Col}(\pi)$ induces a partial order also denoted $\geq'$ on $\parallel$-equivalence classes.

It just remains to introduce notions of *dominant* and of *standard* π-tableaux. The first of these is easy: call an element $A \in \text{Row}(\pi)$ *dominant* if it has a representative belonging to $\text{Col}(\pi)$ and let $\text{Dom}(\pi)$ denote the set of all such dominant row symmetrized π-tableaux. The notion of a standard π-tableau is more subtle. Suppose first that π is left-justified, when its diagram is a Young diagram in the usual sense. In that case, a π-tableau $A \in \text{Col}(\pi)$ with entries $a_{i,1}, \dots, a_{i,p_i}$ in its ith row read from left to right is called *standard* if $a_{i,j} \not> a_{i,k}$ for all $1 \leq i \leq n$ and $1 \leq j < k \leq p_i$. If A has integer entries (rather than arbitrary elements of $\mathbb{F}$) this is just saying that the entries of A are strictly increasing up columns from bottom to top and weakly increasing along rows from left to right, i.e. it is the usual notion of standard tableau.

Lemma 4.1. *Assume that π is left-justified. Then any element $A \in \mathrm{Dom}(\pi)$ has a representative that is standard.*

Proof. By definition, we can choose a representative for A that is column strict. Let $a_{i,1}, \dots, a_{i,p_i}$ be the entries on the ith row of this representative read from left to right, for each $i = 1, \dots, n$. We need to show that we can permute entries within rows so that it becomes standard. Proceed by induction on $\#\{(i,j,k) \,|\, 1 \leq i \leq n, 1 \leq j < k \leq p_i \text{ such that } a_{i,j} > a_{i,k}\}$. If this number is zero then our tableau is already standard. Otherwise we can pick $1 \leq i \leq n$ and $1 \leq j < k \leq p_i$ such that $a_{i,j} > a_{i,k}$, none of $a_{i,j+1}, \dots, a_{i,k-1}$ lie in the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$ as $a_{i,j}$, and either $i = n$ or $a_{i+1,j} \not> a_{i+1,k}$. Then define $1 \leq h \leq i$ to be minimal so that $k \leq p_h$ and $a_{r,j} > a_{r,k}$ for all $h \leq r \leq i$. Thus our tableau contains the following entries:

$$\begin{array}{ccc}
a_{h-1,j} & \leq & a_{h-1,k} \\
a_{h,j} & > & a_{h,k} \\
a_{h+1,j} & > & a_{h+1,k} \\
\vdots & \vdots & \vdots \\
a_{i,j} & > & a_{i,k} \\
a_{i+1,j} & \leq & a_{i+1,k},
\end{array}$$

where entries on the $(h-1)$th and/or $(i+1)$th rows should be omitted if they do not exist. Now swap the entries $a_{h,j} \leftrightarrow a_{h,k}, a_{h+1,j} \leftrightarrow a_{h+1,k}, \dots a_{i,j} \leftrightarrow a_{i,k}$ and observe that the resulting tableau is still column strict. Finally by the induction hypothesis we get that the new tableau is row equivalent to a standard tableau. $\square$

To define what it means for $A \in \mathrm{Col}(\pi)$ to be standard for more general pyramids π we need to recall the notion of row insertion; see e.g. [F, §1.1]. Suppose we are given an N-tuple $(a_1, \dots, a_N) \in \mathbb{F}^N$. We decide if it is admissible, and if so construct an element of $\mathrm{Row}(\pi)$, according to the following algorithm. Start from the diagram of π with all boxes empty. Insert a_1 into some box in the bottom (nth) row. Then if $a_2 \not< a_1$ insert a_2 into the bottom row too; else replace the entry a_1 by a_2 and insert a_1 into the next row up instead. Continue in this way: at the ith step the pyramid π has $(i-1)$ boxes filled in and we need to insert the entry a_i into the bottom row. If a_i is $\not<$ all of the entries in this row, simply add it to the row; else find the smallest entry b in the row that is strictly larger than a_i, replace this entry b with a_i, then insert b into the next row up in similar fashion. If at any stage of this process one gets more than p_i entries in the ith row for some i, the algorithm terminates and the tuple $(a_1, \dots, a_N)$ is inadmissible; else, the tuple $(a_1, \dots, a_N)$ is admissible and we have successfully computed a tableau $A \in \mathrm{Row}(\pi)$.

Now, for any pyramid π, we say that $A \in \mathrm{Col}(\pi)$ is *standard* if the tuple $\gamma(A)$ obtained from the column reading of A is admissible. In that case, we define the *rectification* $R(A) \in \mathrm{Row}(\pi)$ to be the row symmetrized π-tableau computed from the tuple $\gamma(A)$ by the algorithm described in the previous paragraph. Actually it is clear from the algorithm that $R(A)$ belongs to $\mathrm{Dom}(\pi)$, i.e. it has a representative that is column strict. Moreover, if $A \parallel B$ for some $B \in \mathrm{Col}(\pi)$, then A is standard if and only if B is standard, and $R(A) = R(B)$. Let $\mathrm{Std}(\pi)$ denote the set of all $\parallel$-equivalence classes of standard π-tableaux. The rectification map R induces a well-defined map

$$R : \mathrm{Std}(\pi) \to \mathrm{Dom}(\pi). \tag{4.3}$$

In the special case that π is left justified, it is straightforward to check that the new definition of standard tableau agrees with the one given before Lemma 4.1, and moreover in this case the map R is simply the map sending a tableau to its row equivalence class. In general, the map R always defines a *bijection* between the sets $\mathrm{Std}(\pi)$ and $\mathrm{Dom}(\pi)$. This follows in the left-justified case using Lemma 4.1, and then in general by a result of Lascoux and Schützenberger [LS]; see [F, §A.5] and [B, §2]. The partial orders $\leq'$ on $\mathrm{Col}(\pi)$ and $\leq$ on $\mathrm{Row}(\pi)$ give partial orders $\leq'$ on $\mathrm{Std}(\pi)$ and $\leq$ on $\mathrm{Dom}(\pi)$ too. In the case that π is left-justified, the map $R : \mathrm{Std}(\pi) \to \mathrm{Dom}(\pi)$ is actually an isomorphism between the partially ordered sets $(\mathrm{Std}(\pi), \leq')$ and $(\mathrm{Dom}(\pi), \leq)$; see [B, Lemma 1]. This is definitely not the case for more general pyramids π.

We have now introduced four sets $\mathrm{Row}(\pi), \mathrm{Col}(\pi), \mathrm{Dom}(\pi)$ and $\mathrm{Std}(\pi)$ of tableaux which will be needed later on to parametrize the various bases/modules that we will meet. One more thing: for any $a \in \mathbb{F}$ we write $\mathrm{Row}_a(\pi), \mathrm{Col}_a(\pi), \mathrm{Dom}_a(\pi)$ and $\mathrm{Std}_a(\pi)$ for the subsets of $\mathrm{Row}(\pi), \mathrm{Col}(\pi), \mathrm{Dom}(\pi)$ and $\mathrm{Std}(\pi)$ consisting just of the tableaux all of whose entries belong to the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$ as a. We will often restrict our attention just to the sets $\mathrm{Row}_0(\pi), \mathrm{Col}_0(\pi), \mathrm{Dom}_0(\pi)$ and $\mathrm{Std}_0(\pi)$, since this covers the most important situation when all the tableaux have *integer* entries. In fact, most of the problems that we will meet are reduced in a straightforward fashion to this special situation. Often, we will identify an element $A \in \mathrm{Row}_0(\pi)$ with its unique representative in $\mathrm{Tab}(\pi)$ whose entries are weakly increasing along rows from left to right.

4.2. **Dual canonical bases.** Now let $\mathfrak{gl}_\infty$ denote the Lie algebra of matrices with rows and columns labelled by $\mathbb{Z}$, all but finitely many entries of which are zero. It is generated by the usual Chevalley generators e_i, f_i, i.e. the matrix units $e_{i,i+1}$ and $e_{i+1,i}$, together with the diagonal matrix units $d_i = e_{i,i}$, for each $i \in \mathbb{Z}$. The associated integral weight lattice P_∞ is the free abelian group on basis $\{\varepsilon_i \mid i \in \mathbb{Z}\}$, with simple roots $\varepsilon_i - \varepsilon_{i+1}$ for $i \in \mathbb{Z}$; we will view P_∞ as a subgroup of the group P from (3.44). We let $U_\mathbb{Z}$ be the Kostant $\mathbb{Z}$-form for the universal enveloping algebra $U(\mathfrak{gl}_\infty)$, generated by the divided powers $e_i^r/r!$, $f_i^r/r!$ and the elements $\binom{d_i}{r} = \frac{d_i(d_i-1)\cdots(d_i-r+1)}{r!}$ for all $i \in \mathbb{Z}, r \geq 0$. Let V be the natural $\mathfrak{gl}_\infty$-module with usual basis v_i $(i \in \mathbb{Z})$. Let $V_\mathbb{Z}$ be the $\mathbb{Z}$-submodule generated by these basis vectors, which is naturally a module for the $\mathbb{Z}$-form $U_\mathbb{Z}$.

Consider to start with the $U_\mathbb{Z}$-module arising as the Nth tensor power $T^N(V_\mathbb{Z})$ of $V_\mathbb{Z}$. It is a free $\mathbb{Z}$-module with the monomial basis $\{M_\alpha \mid \alpha \in \mathbb{Z}^N\}$ defined from $M_\alpha = v_{a_1} \otimes \cdots \otimes v_{a_N}$ for $\alpha = (a_1, \ldots, a_N) \in \mathbb{Z}^N$. We also need the *dual canonical basis* $\{L_\alpha \mid \alpha \in \mathbb{Z}^N\}$. The best way to define this is to first quantize, then define L_α using a natural bar involution on the q-tensor space, then specialize to $q = 1$ at the end. We refer to [B, §4] for the details of this construction (which is due to Lusztig [L, ch.27]); the only significant difference is that in [B] the Lie algebra $\mathfrak{gl}_n$ is used in place of the Lie algebra $\mathfrak{gl}_\infty$ here. We just content ourselves with writing down an explicit formula for the expansion of M_α as a linear combination of L_β's in terms of the usual Kazhdan-Lusztig polynomials $P_{x,y}(q)$ associated to the symmetric group S_N from [KL] evaluated at $q = 1$. To do this, let S_N act on the right on the set $\mathbb{Z}^N$ in the natural way, and given any $\alpha \in \mathbb{Z}^N$ define $d(\alpha) \in S_N$ to be the unique element of minimal length with the property that $\alpha \cdot d(\alpha)^{-1}$ is a weakly increasing sequence. Then, by [B, §4], we have that

$$M_\alpha = \sum_{\beta \in \mathbb{Z}^N} P_{d(\alpha)w_0, d(\beta)w_0}(1) L_\beta, \tag{4.4}$$

writing w_0 for the longest element of S_N.

We also need to consider certain tensor products of symmetric and exterior powers of $V_\mathbb{Z}$. Let $S^N(V_\mathbb{Z})$ denote the Nth symmetric power of $V_\mathbb{Z}$, defined as a quotient of $T^N(V_\mathbb{Z})$ in the usual way. Also let $\bigwedge^N(V_\mathbb{Z})$ denote the Nth exterior power of $V_\mathbb{Z}$, viewed as the subspace of $T^N(V_\mathbb{Z})$ consisting of all skew-symmetric tensors. Recalling the fixed pyramid π, let

$$S^\pi(V_\mathbb{Z}) := S^{p_1}(V_\mathbb{Z}) \otimes \cdots \otimes S^{p_n}(V_\mathbb{Z}), \tag{4.5}$$

$$\textstyle\bigwedge^\pi(V_\mathbb{Z}) := \bigwedge^{q_1}(V_\mathbb{Z}) \otimes \cdots \otimes \bigwedge^{q_l}(V_\mathbb{Z}). \tag{4.6}$$

Thus, $S^\pi(V_\mathbb{Z})$ is a quotient of the space $T^N(V_\mathbb{Z})$, while $\bigwedge^\pi(V_\mathbb{Z})$ is a subspace. Following [B, §5], both of these free $\mathbb{Z}$-modules have two natural bases, a monomial basis and a dual canonical basis, parametrized by the sets $\mathrm{Row}_0(\pi)$ and $\mathrm{Col}_0(\pi)$, respectively.

First we define these two bases for the space $S^\pi(V_\mathbb{Z})$. Take $A \in \mathrm{Row}_0(\pi)$, and identify A with its unique representative in $\mathrm{Tab}(\pi)$ having weakly increasing rows as indicated in the last sentence of the previous section. Define M_A to be the image of $M_{\rho(A)}$ and L_A to be the image of $L_{\rho(A)}$ under the canonical quotient map $T^N(V_\mathbb{Z}) \twoheadrightarrow S^\pi(V_\mathbb{Z})$, where $\rho(A)$ is the row reading as defined in the previous section. Then, the monomial basis for $S^\pi(V_\mathbb{Z})$ is $\{M_A \mid A \in \mathrm{Row}_0(\pi)\}$, and the dual canonical basis is $\{L_A \mid A \in \mathrm{Row}_0(\pi)\}$.

Now we define the two bases for the space $\bigwedge^\pi(V_\mathbb{Z})$. For $A \in \mathrm{Col}_0(\pi)$, let

$$N_A := \sum_{B \sim_{\mathrm{co}} A} (-1)^{\ell(A,B)} M_{\gamma(B)}, \tag{4.7}$$

where $\ell(A, B)$ denotes the minimal number of transpositions of adjacent elements in the same column needed to get from A to B and $\gamma(B)$ is the column reading of B as defined in the previous section. Also let K_A denote the vector $L_{\gamma(A)} \in T^N(V_\mathbb{Z})$. Then, both N_A and K_A belong to the subspace $\bigwedge^\pi(V_\mathbb{Z})$ of $T^N(V_\mathbb{Z})$; see [B, §5]. Moreover, $\{N_A \,|\, A \in \mathrm{Col}_0(\pi)\}$ and $\{K_A \,|\, A \in \mathrm{Col}_0(\pi)\}$ are bases for $\bigwedge^\pi(V_\mathbb{Z})$, giving the monomial basis and the dual canonical basis, respectively.

The following formulae, derived in [B, §5] as consequences of (4.4), express the monomial bases in terms of the dual canonical bases and certain Kazhdan-Lusztig polynomials:

$$M_A = \sum_{B \in \mathrm{Row}_0(\pi)} P_{d(\rho(A))w_0, d(\rho(B))w_0}(1) L_B, \tag{4.8}$$

$$N_A = \sum_{B \in \mathrm{Col}_0(\pi)} \left(\sum_{C \sim_{\mathrm{co}} A} (-1)^{\ell(A,C)} P_{d(\gamma(C))w_0, d(\gamma(B))w_0}(1) \right) K_B, \tag{4.9}$$

for $A \in \mathrm{Row}_0(\pi)$ and $A \in \mathrm{Col}_0(\pi)$, respectively.

Notice that $S^\pi(V_\mathbb{Z})$ is a summand of the commutative algebra $S(V_\mathbb{Z}) \otimes \cdots \otimes S(V_\mathbb{Z})$, that is, the tensor product of n copies of the symmetric algebra $S(V_\mathbb{Z})$. In particular, if $\pi = \pi' \otimes \pi''$, the multiplication in this algebra defines a $U_\mathbb{Z}$-module homomorphism

$$\mu : S^{\pi'}(V_\mathbb{Z}) \otimes S^{\pi''}(V_\mathbb{Z}) \to S^\pi(V_\mathbb{Z}). \tag{4.10}$$

Decompose π into its individual columns as $\pi = \pi_1 \otimes \cdots \otimes \pi_l$ and note that $T^{q_1}(V_\mathbb{Z}) \otimes \cdots \otimes T^{q_l}(V_\mathbb{Z}) = S^{\pi_1}(V_\mathbb{Z}) \otimes \cdots \otimes S^{\pi_l}(V_\mathbb{Z})$. Now we can define the canonical $U_\mathbb{Z}$-module homomorphism

$$F : \textstyle\bigwedge^\pi(V_\mathbb{Z}) \to S^\pi(V_\mathbb{Z}) \tag{4.11}$$

to be the composite first of the natural inclusion $\bigwedge^\pi(V_\mathbb{Z}) \hookrightarrow T^{q_1}(V_\mathbb{Z}) \otimes \cdots \otimes T^{q_l}(V_\mathbb{Z})$ then the multiplication map $S^{\pi_1}(V_\mathbb{Z}) \otimes \cdots \otimes S^{\pi_l}(V_\mathbb{Z}) \to S^\pi(V_\mathbb{Z})$ obtained by iterating the map (4.10) a total of $(l-1)$ times. We define $P^\pi(V_\mathbb{Z})$ to be the image of the map F just constructed. It is a well known $\mathbb{Z}$-form for the irreducible polynomial representation of $\mathfrak{gl}_\infty$ parametrized by the partition $\lambda = (p_1, \dots, p_n)$. For any $A \in \mathrm{Col}_0(\pi)$, define

$$V_A := F(N_A). \tag{4.12}$$

By [B, Theorem 26], $P^\lambda(V_\mathbb{Z})$ is a free $\mathbb{Z}$-module with *standard monomial basis* given by the vectors $\{V_A \mid A \in \mathrm{Std}_0(\pi)\}$. Moreover, for $A \in \mathrm{Col}_0(\pi)$, we have that

$$F(K_A) = \begin{cases} L_{R(A)} & \text{if } A \in \mathrm{Std}_0(\pi), \\ 0 & \text{otherwise,} \end{cases} \tag{4.13}$$

recalling the rectification map R from (4.3). The vectors $\{L_A \mid A \in \mathrm{Dom}_0(\pi)\}$ give another basis for the submodule $P^\pi(V_\mathbb{Z})$, which is the *dual canonical basis* of Lusztig, or Kashiwara's *upper global crystal basis*. Finally, by (4.9) and (4.13), we have for any $A \in \mathrm{Col}_0(\pi)$ that

$$V_A = \sum_{B \in \mathrm{Std}_0(\pi)} \left(\sum_{C \sim_{\mathrm{co}} A} (-1)^{\ell(A,C)} P_{d(\gamma(C))w_0, d(\gamma(B))w_0}(1) \right) L_{R(B)}. \tag{4.14}$$

4.3. **Crystals.** In this section, we introduce the crystals underlying the modules $T^N(V_\mathbb{Z}), \bigwedge^\pi(V_\mathbb{Z}), S^\pi(V_\mathbb{Z})$ and $P^\pi(V_\mathbb{Z})$. First, we define a crystal $(\mathbb{Z}^N, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ in the sense of Kashiwara [K2], as follows. Take $\alpha = (a_1, \dots, a_N) \in \mathbb{Z}^N$ and $i \in \mathbb{Z}$. The *i-signature* of α is the tuple $(\sigma_1, \dots, \sigma_N)$ defined from

$$\sigma_j = \begin{cases} + & \text{if } a_j = i, \\ - & \text{if } a_j = i+1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

From this the *reduced i-signature* is computed by successively replacing subsequences of the form $-+$ (possibly separated by 0's) in the signature with 00 until no $-$ appears to the left of a $+$. Let δ_j denote the N-tuple $(0,\dots,0,1,0,\dots,0)$ where 1 appears in the jth place. Now define

$$\tilde{e}_i(\alpha) := \begin{cases} \varnothing & \text{if there are no } -\text{'s in the reduced } i\text{-signature,} \\ \alpha - \delta_j & \text{if the leftmost } - \text{ is in position } j; \end{cases} \tag{4.16}$$

$$\tilde{f}_i(\alpha) := \begin{cases} \varnothing & \text{if there are no } +\text{'s in the reduced } i\text{-signature,} \\ \alpha + \delta_j & \text{if the rightmost } + \text{ is in position } j; \end{cases} \tag{4.17}$$

$$\varepsilon_i(\alpha) = \text{the total number of } -\text{'s in the reduced } i\text{-signature;} \tag{4.18}$$

$$\varphi_i(\alpha) = \text{the total number of } +\text{'s in the reduced } i\text{-signature.} \tag{4.19}$$

Finally define the weight function $\theta : \mathbb{Z}^N \to P_\infty$ to be the restriction of the map (3.50). This completes the definition of the crystal $(\mathbb{Z}^N, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$. It is the N-fold tensor product of the usual crystal associated to the $\mathfrak{gl}_\infty$-module V, but for the opposite tensor product to the one used in [K2]. This crystal carries information about the action of the Chevalley generators of $U_\mathbb{Z}$ on the dual canonical basis $\{L_\alpha \mid \alpha \in \mathbb{Z}^N\}$ of $T^N(V_\mathbb{Z})$, thanks to the following result of Kashiwara [K1, Proposition 5.3.1]: for $\alpha \in \mathbb{Z}^N$, we have that

$$e_i L_\alpha = \varepsilon_i(\alpha) L_{\tilde{e}_i(\alpha)} + \sum_{\substack{\beta \in \mathbb{Z}^N \\ \varepsilon_i(\beta) < \varepsilon_i(\alpha) - 1}} x^i_{\alpha,\beta} L_\beta \tag{4.20}$$

$$f_i L_\alpha = \varphi_i(\alpha) L_{\tilde{f}_i(\alpha)} + \sum_{\substack{\beta \in \mathbb{Z}^N \\ \varphi_i(\beta) < \varphi_i(\alpha) - 1}} y^i_{\alpha,\beta} L_\beta \tag{4.21}$$

for $x^i_{\alpha,\beta}, y^i_{\alpha,\beta} \in \mathbb{Z}$. The right hand side of (4.20) resp. (4.21) should be interpreted as zero if $\varepsilon_i(\alpha) = 0$ resp. $\varphi_i(\alpha) = 0$.

There are also crystals attached to the modules $S^\pi(V_\mathbb{Z})$ and $\bigwedge^\pi(V_\mathbb{Z})$. To define them, identify $\mathrm{Row}_0(\pi)$ with a subset of $\mathbb{Z}^N$ via the row reading $\rho : \mathrm{Row}_0(\pi) \hookrightarrow \mathbb{Z}^N$ (again we are viewing elements of $\mathrm{Row}_0(\pi)$ as elements of $\mathrm{Tab}(\pi)$ with weakly increasing rows), and identify $\mathrm{Col}_0(\pi)$ with a subset of $\mathbb{Z}^N$ via the column reading $\gamma : \mathrm{Col}_0(\pi) \hookrightarrow \mathbb{Z}^N$. In this way, both $\mathrm{Row}_0(\pi)$ and $\mathrm{Col}_0(\pi)$ become identified with subcrystals of the crystal $(\mathbb{Z}^N, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ via these maps. This defines crystals $(\mathrm{Row}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ and $(\mathrm{Col}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$. These crystals control the action of the Chevalley generators of $U_\mathbb{Z}$ on the dual canonical bases $\{L_A \mid A \in \mathrm{Row}_0(\pi)\}$ and $\{K_A \mid A \in \mathrm{Col}_0(\pi)\}$, just like in (4.20)–(4.21). First, for $A \in \mathrm{Row}_0(\pi)$, we have that

$$e_i L_A = \varepsilon_i(A) L_{\tilde{e}_i(A)} + \sum_{\substack{B \in \mathrm{Row}_0(\pi) \\ \varepsilon_i(B) < \varepsilon_i(A) - 1}} x^i_{\rho(A),\rho(B)} L_B, \tag{4.22}$$

$$f_i L_A = \varphi_i(A) L_{\tilde{f}_i(A)} + \sum_{\substack{B \in \mathrm{Row}_0(\pi) \\ \varphi_i(B) < \varphi_i(A) - 1}} y^i_{\rho(A),\rho(B)} L_B. \tag{4.23}$$

Second, for $A \in \mathrm{Col}_0(\pi)$, we have that

$$e_i K_A = \varepsilon_i(A) K_{\tilde{e}_i(A)} + \sum_{\substack{B \in \mathrm{Col}_0(\pi) \\ \varepsilon_i(B) < \varepsilon_i(A)-1}} x^i_{\gamma(A),\gamma(B)} K_B, \tag{4.24}$$

$$f_i K_A = \varphi_i(A) K_{\tilde{f}_i(A)} + \sum_{\substack{B \in \mathrm{Col}_0(\pi) \\ \varphi_i(B) < \varphi_i(A)-1}} y^i_{\gamma(A),\gamma(B)} K_B. \tag{4.25}$$

Finally, there is a well known crystal attached to the irreducible polynomial representation $P^\pi(V_\mathbb{Z})$. This has various different realizations, in terms of either the set $\mathrm{Dom}_0(\pi)$ or the set $\mathrm{Std}_0(\pi)$; the realization as $\mathrm{Std}_0(\pi)$ when π is left-justified is the usual description from [KN]. In the first case, we note that $\mathrm{Dom}_0(\pi)$ is a subcrystal of the crystal $(\mathrm{Row}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$, indeed it is the connected component of this crystal generated by the row equivalence class of *ground-state tableau* A_π, that is, the tableau having all entries on row i equal to $(1-i)$. In the second case, as explained in [B, §2], $\mathrm{Std}_0(\pi)$ is a subcrystal of the crystal $(\mathrm{Col}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$, indeed again it is the connected component of this crystal generated by the ground-state tableau A_π. In this way, we obtain two new crystals $(\mathrm{Dom}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ and $(\mathrm{Std}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$. The rectification map $R : \mathrm{Std}_0(\pi) \to \mathrm{Dom}_0(\pi)$ is the unique isomorphism between these crystals, and it sends the ground-state tableau A_π to its row equivalence class.

4.4. **Consequences of the Kazhdan-Lusztig conjecture.** In this section, we record a representation theoretic interpretation of the dual canonical basis of the spaces $T^N(V_\mathbb{Z})$ and $\bigwedge^\pi(V_\mathbb{Z})$, which is a well known reformulation of the Kazhdan-Lusztig conjecture for $\mathfrak{gl}_N$ [BB, BrK]. Later on in the article we will formulate analogous interpretations for the dual canonical bases of the spaces $S^\pi(V_\mathbb{Z})$ (conjecturally) and $P^\pi(V_\mathbb{Z})$. Going back to the notation from §3.8, let $\mathcal{O}$ denote the [BGG3] category of all finitely generated $\mathfrak{gl}_N$-modules which are locally finite over $\mathfrak{b}_N$ and semisimple over $\mathfrak{d}_N$. The basic objects in $\mathcal{O}$ are the Verma modules $M(\alpha)$ and their unique irreducible quotients $L(\alpha)$ for $\alpha = (a_1, \dots, a_N) \in \mathbb{F}^N$, using the ρ-shifted notation explained by (3.47). Also recall that we have parametrized the central characters of $U(\mathfrak{gl}_N)$ by the set of elements θ of $P = \bigoplus_{a \in \mathbb{F}} \mathbb{Z}\varepsilon_a$ whose coefficients are non-negative integers summing to N.

For $\theta \in P$, let $\mathcal{O}(\theta)$ denote the full subcategory of $\mathcal{O}$ consisting of the objects all of whose composition factors are of central character θ, setting $\mathcal{O}(\theta) = 0$ by convention if the coefficients of θ are not non-negative integers summing to N. The category $\mathcal{O}$ has the following *block decomposition*:

$$\mathcal{O} = \bigoplus_{\theta \in P} \mathcal{O}(\theta). \tag{4.26}$$

We will write $\mathrm{pr}_\theta : \mathcal{O} \to \mathcal{O}(\theta)$ for the natural projection functor. To be absolutely explicit, if the coefficients of $\theta \in P$ are non-negative integers summing to N, so θ corresponds to the polynomial $f(u) = u^N + f^{(1)} u^{N-1} + \cdots + f^{(N)} \in \mathbb{F}[u]$ according to (3.45)–(3.46), we have that

$$\mathrm{pr}_\theta(M) = \left\{ v \in M \;\middle|\; \begin{array}{l} \text{for each } r = 1, \dots, N \text{ there exists } p \gg 0 \\ \text{such that } (Z_N^{(r)} - f^{(r)})^p v = 0 \end{array} \right\}. \tag{4.27}$$

We have already observed in §3.8 that the Verma module $M(\alpha)$ is of central character $\theta(\alpha)$. Hence, for any $\theta \in P$, the modules $\{L(\alpha) \mid \alpha \in \mathbb{F}^N \text{ with } \theta(\alpha) = \theta\}$ form a complete set of pairwise non-isomorphic irreducibles in the category $\mathcal{O}(\theta)$.

Recall that the integral weight lattice P_∞ of $\mathfrak{gl}_\infty$ is the subgroup $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}\varepsilon_i$ of P. Let us restrict our attention from now on to the full subcategory

$$\mathcal{O}_0 = \bigoplus_{\theta \in P_\infty \subset P} \mathcal{O}(\theta) \tag{4.28}$$

of $\mathcal{O}$ corresponding just to *integral* central characters. The Grothendieck group $[\mathcal{O}_0]$ of this category has the two natural bases $\{[M(\alpha)] \mid \alpha \in \mathbb{Z}^N\}$ and $\{[L(\alpha)] \mid \alpha \in \mathbb{Z}^N\}$. Define a $\mathbb{Z}$-module isomorphism

$$j : T^N(V_\mathbb{Z}) \to [\mathcal{O}_0], \qquad M_\alpha \mapsto [M(\alpha)]. \tag{4.29}$$

Note this isomorphism sends the θ-weight space of $T^N(V_\mathbb{Z})$ isomorphically onto the block component $[\mathcal{O}(\theta)]$ of $[\mathcal{O}_0]$, for each $\theta \in P_\infty$. The Kazhdan-Lusztig conjecture [KL], proved in [BB, BrK], can be formulated as follows for the special case of the Lie algebra $\mathfrak{gl}_N$.

Theorem 4.2. *The map j sends the dual canonical basis element L_α of $T^N(V_\mathbb{Z})$ to the class $[L(\alpha)]$ of the irreducible module $L(\alpha)$.*

Proof. In view of (4.4), it suffices to show for $\alpha, \beta \in \mathbb{Z}^N$ that the composition multiplicity of $L(\beta)$ in the Verma module $M(\alpha)$ is given by the formula

$$[M(\alpha) : L(\beta)] = P_{d(\alpha)w_0, d(\beta)w_0}(1).$$

This is well known consequence of the Kazhdan-Lusztig conjecture combined with the translation principle for singular weights, or see [BGS, Theorem 3.11.4]. □

Using (4.29) we can view the action of $U_\mathbb{Z}$ on $T^N(V_\mathbb{Z})$ instead as an action on the Grothendieck group $[\mathcal{O}_0]$. The resulting actions of the Chevalley generators e_i, f_i of $U_\mathbb{Z}$ on $[\mathcal{O}_0]$ are in fact induced by some exact functors $e_i, f_i : \mathcal{O}_0 \to \mathcal{O}_0$ on the category $\mathcal{O}_0$ itself. Like in [BK1], these functors are certain *translation functors* arising from tensoring with the natural $\mathfrak{gl}_N$ module V_N or its dual V_N^* then projecting onto certain blocks. To be precise, for $i \in \mathbb{Z}$, we have that

$$e_i = \bigoplus_{\theta \in P_\infty} \mathrm{pr}_{\theta + (\varepsilon_i - \varepsilon_{i+1})} \circ (? \otimes V_N^*) \circ \mathrm{pr}_\theta, \tag{4.30}$$

$$f_i = \bigoplus_{\theta \in P_\infty} \mathrm{pr}_{\theta - (\varepsilon_i - \varepsilon_{i+1})} \circ (? \otimes V_N) \circ \mathrm{pr}_\theta. \tag{4.31}$$

Recall that these exact functors are both left and right adjoint to each other in a canonical way. The next lemma is a standard consequence of the tensor identity.

Lemma 4.3. *For $\alpha \in \mathbb{F}^N$, the module $M(\alpha) \otimes V_N$ has a filtration with factors $M(\beta)$ for all tuples $\beta \in \mathbb{F}^N$ obtained from the tuple α by adding 1 to one of its entries.*

Similarly, the module $M(\alpha) \otimes V_N^*$ has a filtration with factors $M(\beta)$ for all tuples $\beta \in \mathbb{F}^N$ obtained from the tuple α by subtracting 1 from one of its entries.

Taking blocks and passing to the Grothendieck group, we deduce for $\alpha \in \mathbb{Z}^N$ and $i \in \mathbb{Z}$ that

$$[e_i M(\alpha)] = \sum_\beta [M(\beta)] \tag{4.32}$$

summing over all tuples $\beta \in \mathbb{Z}^N$ obtained from the tuple α by replacing an entry equal to $(i+1)$ by an i, and

$$[f_i M(\alpha)] = \sum_\beta [M(\beta)] \tag{4.33}$$

summing over all tuples $\beta \in \mathbb{Z}^N$ obtained from the tuple α by replacing an entry equal to i by an $(i+1)$. This verifies that the maps on the Grothendieck group $[\mathcal{O}_0]$ induced by the exact functors e_i, f_i really do coincide with the action of the Chevalley generators of $U_\mathbb{Z}$ from (4.29).

Here is an alternative definition of the functors e_i and f_i, explained in detail in [CR, §7.4]. Let $\Omega = \sum_{i,j=1}^N e_{i,j} \otimes e_{j,i} \in U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_N)$. This element centralizes the image of $U(\mathfrak{gl}_N)$ under the comultiplication $\Delta : U(\mathfrak{gl}_N) \to U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_N)$. For any $M \in \mathcal{O}_0$, $f_i M$ is precisely the generalized i-eigenspace of the operator Ω acting on $M \otimes V_N$, for any $M \in \mathcal{O}_0$. Similarly, $e_i M$ is precisely the generalized $-(N+i)$-eigenspace of Ω acting on $M \otimes V_N^*$.

We need to recall a little more of the setup from [CR]. Define an endomorphism x of the functor $? \otimes V_N$ by letting $x_M : M \otimes V_N \to M \otimes V_N$ be left multiplication by Ω, for all $\mathfrak{g}$-modules M. Also define an endomorphism s of the functor $? \otimes V_N \otimes V_N$ by letting $s_M : M \otimes V_N \otimes V_N \to M \otimes V_N \otimes V_N$ be the permutation $m \otimes v_i \otimes v_j \mapsto m \otimes v_j \otimes v_i$. By [CR, Lemma 7.21], we have that

$$s_M \circ (x_M \otimes \mathrm{id}_{V_N}) = x_{M \otimes V_N} \circ s_M - \mathrm{id}_{M \otimes V_N \otimes V_N} \tag{4.34}$$

for any $\mathfrak{g}$-module M, equality of maps from $M \otimes V_N \otimes V_N$ to itself. It follows that x and s restrict to well-defined endomorphisms of the functors f_i and f_i^2; we denote these restrictions by x and s too. Moreover, we have that

$$(s1_{f_i}) \circ (1_{f_i}s) \circ (s1_{f_i}) = (1_{f_i}s) \circ (s1_{f_i}) \circ (1_{f_i}s), \tag{4.35}$$

$$s^2 = 1_{f_i^2}, \tag{4.36}$$

$$s \circ (1_{f_i}x) = (x1_{f_i}) \circ s - 1_{f_i^2}, \tag{4.37}$$

equality of endomorphisms of f_i^3, f_i^2 and f_i^2, respectively. In the language of [CR, §5.2.1], this shows that the category $\mathcal{O}_0$ equipped with the adjoint pair of functors (f_i, e_i) and the endomorphisms $x \in \mathrm{End}(f_i)$ and $s \in \mathrm{End}(f_i^2)$ is an $\mathfrak{sl}_2$-*categorification* for each $i \in \mathbb{Z}$. This has a number of important consequences, explored in detail in [CR]. We just record one more thing here, our proof of which also depends on Theorem 4.2; see [Ku] for an independent proof. Recall for the statement the definition of the crystal $(\mathbb{Z}^N, \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ from (4.16)–(4.19).

Theorem 4.4. *Let $\alpha \in \mathbb{Z}^N$ and $i \in \mathbb{Z}$.*

(i) *If* $\varepsilon_i(\alpha) = 0$ *then* $e_i L(\alpha) = 0$. *Otherwise,* $e_i L(\alpha)$ *is an indecomposable module with irreducible socle and cosocle isomorphic to* $L(\tilde{e}_i(\alpha))$.

(ii) *If* $\varphi_i(\alpha) = 0$ *then* $f_i L(\alpha) = 0$. *Otherwise,* $f_i L(\alpha)$ *is an indecomposable module with irreducible socle and cosocle isomorphic to* $L(\tilde{f}_i(\alpha))$.

Proof. (i) For $\alpha \in \mathbb{Z}^N$, let $\varepsilon_i'(\alpha)$ be the maximal integer $k \geq 0$ such that $(e_i)^k L(\alpha) \neq 0$. If $\varepsilon_i'(\alpha) > 0$, then [CR, Proposition 5.23] shows that $e_i L(\alpha)$ is an indecomposable module with irreducible socle and cosocle isomorphic to $L(\tilde{e}_i'(\alpha))$ for some $\tilde{e}_i'(\alpha) \in \mathbb{Z}^N$. Moreover, using [CR, Lemma 4.3] too, $\varepsilon_i'(\tilde{e}_i'(\alpha)) = \varepsilon_i'(\alpha) - 1$ and all remaining composition factors of $e_i L(\alpha)$ not isomorphic to $L(\tilde{e}_i'(\alpha))$ are of the form $L(\beta)$ for $\beta \in \mathbb{Z}^N$ with $\varepsilon_i'(\beta) < \varepsilon_i'(\alpha) - 1$.

Observe from (4.20) that $\varepsilon_i(\alpha)$ is the maximal integer $k \geq 0$ such that $(e_i)^k L_\alpha \neq 0$, and assuming $\varepsilon_i(\alpha) > 0$ we know that $e_i L_\alpha = \varepsilon_i(\alpha) L_{\tilde{e}_i(\alpha)}$ plus a linear combination of L_β's with $\varepsilon_i(\beta) < \varepsilon_i(\alpha) - 1$. Applying Theorem 4.2 and comparing the preceeding paragraph, it follows immediately that $\varepsilon_i(\alpha) = \varepsilon_i'(\alpha)$, in which case $\tilde{e}_i(\alpha) = \tilde{e}_i'(\alpha)$. This completes the proof.

(ii) Similar, or follows from (i) using adjointness. □

It just remains to extend all of this to the parabolic case. Continuing with the fixed pyramid $\pi = (q_1, \dots, q_l)$, recall from (3.2) that $\mathfrak{h}$ denotes the standard Levi subalgebra $\mathfrak{gl}_{q_1} \oplus \cdots \oplus \mathfrak{gl}_{q_l}$ of $\mathfrak{g} = \mathfrak{gl}_N$ and $\mathfrak{p}$ is the corresponding standard parabolic subalgebra of $\mathfrak{g}$. Let $\mathcal{O}(\pi)$ denote the *parabolic category* $\mathcal{O}$ consisting of all finitely generated $\mathfrak{g}$-modules that are locally finite dimensional over $\mathfrak{p}$ and semisimple over $\mathfrak{h}$. Note $\mathcal{O}(\pi)$ is a full subcategory of the category $\mathcal{O}$. To define the basic modules in $\mathcal{O}(\pi)$, let $A \in \mathrm{Col}(\pi)$ be a column strict π-tableau and let $\alpha = (a_1, \dots, a_N) \in \mathbb{F}^N$ denote the tuple $\gamma(A)$ obtained from column reading A as in §4.1. Let $Y(A)$ denote the usual finite dimensional irreducible $\mathfrak{h}$-module of highest weight $\alpha - \rho = a_1\varepsilon_1 + (a_2+1)\varepsilon_2 + \cdots + (a_N + N - 1)\varepsilon_N$. View $Y(A)$ as a $\mathfrak{p}$-module through the natural projection $\mathfrak{p} \twoheadrightarrow \mathfrak{h}$, then form the *parabolic Verma module*

$$N(A) := U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} Y(A). \tag{4.38}$$

The unique irreducible quotient of $N(A)$ is denoted $K(A)$; by comparing highest weights we have that $K(A) \cong L(\gamma(A))$. In this way, we obtain two natural bases $\{[N(A)] \mid A \in \mathrm{Col}(\pi)\}$ and $\{[K(A)] \mid A \in \mathrm{Col}(\pi)\}$ for the Grothendieck group $[\mathcal{O}(\pi)]$ of the category $\mathcal{O}(\pi)$. The vectors $\{[N(A)] \mid A \in \mathrm{Col}_0(\pi)\}$ and $\{[K(A)] \mid A \in \mathrm{Col}_0(\pi)\}$ form bases for the Grothendieck group $[\mathcal{O}_0(\pi)]$ of the full subcategory $\mathcal{O}_0(\pi) := \mathcal{O}(\pi) \cap \mathcal{O}_0$. Moreover, the translation functors e_i, f_i from (4.30)–(4.31) send modules in $\mathcal{O}_0(\pi)$ to modules in $\mathcal{O}_0(\pi)$, hence the Grothendieck group $[\mathcal{O}_0(\pi)]$ is a $U_\mathbb{Z}$-submodule of $[\mathcal{O}_0(\pi)]$. Also recall the definition of the crystal structure on $\mathrm{Col}_0(\pi)$ from §4.3.

Theorem 4.5. *There is a unique* $U_\mathbb{Z}$*-module isomorphism* $i : \bigwedge^\pi(V_\mathbb{Z}) \to [\mathcal{O}_0(\pi)]$ *such that* $i(N_A) = [N(A)]$ *and* $i(K_A) = [K(A)]$ *for each* $A \in \mathrm{Col}_0(\pi)$. *Moreover, for* $A \in \mathrm{Col}_0(\pi)$ *and* $i \in \mathbb{Z}$, *the following properties hold:*

(i) *If* $\varepsilon_i(A) = 0$ *then* $e_i K(A) = 0$. *Otherwise,* $e_i K(A)$ *is an indecomposable module with irreducible socle and cosocle isomorphic to* $K(\tilde{e}_i(A))$.

(ii) *If* $\varphi_i(A) = 0$ *then* $f_i K(A) = 0$. *Otherwise,* $f_i K(A)$ *is an indecomposable module with irreducible socle and cosocle isomorphic to* $K(\tilde{f}_i(A))$.

Proof. Define a $\mathbb{Z}$-module isomorphism $i : \bigwedge^\pi(V_\mathbb{Z}) \to [\mathcal{O}_0(\pi)]$ by setting $i(N_A) := [N(A)]$ for each $A \in \mathrm{Col}_0(\pi)$. We observe that the following diagram commutes:

$$\begin{array}{ccc} \bigwedge^\pi(V_\mathbb{Z}) & \longrightarrow & T^N(V_\mathbb{Z}) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ [\mathcal{O}_0(\pi)] & \longrightarrow & [\mathcal{O}_0] \end{array} \tag{4.39}$$

where the horizontal maps are the natural inclusions. This is checked by computing the image either way round the diagram of N_A: one way round one uses the definitions (4.7) and (4.29); the other way round uses the Weyl character formula to express $[Y(A)]$ as a linear combination of Verma modules over $\mathfrak{h}$, then exactness of the functor $U(\mathfrak{g})\otimes_{U(\mathfrak{p})}?$ to express $[N(A)]$ as a linear combination of $[M(\alpha)]$'s. Since we already know that all of the maps apart from i are $U_\mathbb{Z}$-module homomorphisms, it then follows that i is too. To complete the proof of the first statement of the theorem, it just remains to show that $i(K_A) = [K(A)]$. This follows by Theorem 4.2 because $K(A) \cong L(\gamma(A))$ and $K_A = L_{\gamma(A)}$. The remaining statements (i) and (ii) follow from Theorem 4.4. □

5. Highest weight theory

In this chapter, we set up the usual machinery of highest weight theory for the shifted Yangian $Y_n(\sigma)$, exploiting its triangular decomposition. Fix throughout a shift matrix $\sigma = (s_{i,j})_{1\le i,j\le n}$.

5.1. **Admissible modules.** Recall the definition of the subalgebra $\mathfrak{d}_n$ of $Y_n(\sigma)$ from §2.1, and the root decomposition (2.20). Given a $\mathfrak{d}_n$-module M and a weight $\alpha \in \mathfrak{d}_n^*$, the (generalized) *$\alpha$-weight space* of M is the subspace

$$M_\alpha := \left\{ v \in M \;\middle|\; \begin{array}{l} \text{for each } i = 1,\dots,n \text{ there exists } p \gg 0 \\ \text{such that } (D_i^{(1)} - \alpha(D_i^{(1)}))^p v = 0 \end{array} \right\}. \tag{5.1}$$

We say that M is *admissible* if

(a) M is the direct sum of its weight spaces, i.e. $M = \bigoplus_{\alpha\in\mathfrak{d}_n^*} M_\alpha$;
(b) each M_α is finite dimensional;
(c) the set of all $\alpha \in \mathfrak{d}_n^*$ such that M_α is non-zero is contained in a finite union of sets of the form $D(\beta) := \{\alpha \in \mathfrak{d}_n^* \mid \alpha \le \beta\}$ for $\beta \in \mathfrak{d}_n^*$.

For example, all finite dimensional $Y_n(\sigma)$-modules are automatically admissible.

Given an admissible $Y_n(\sigma)$-module M, let M^τ be the $Y_n(\sigma^t)$-module equal as a vector space to

$$M^\tau := \bigoplus_{\alpha\in\mathfrak{d}_n^*} (M_\alpha)^* \subseteq M^* \tag{5.2}$$

with action defined by $(xf)(v) = f(\tau(x)v)$ for each $f \in M^\tau, v \in M$ and $x \in Y_n(\sigma^t)$, where $\tau : Y_n(\sigma) \to Y_n(\sigma^t)$ is the anti-isomorphism from (2.35). It is obvious that M^τ is also admissible. Indeed, making the obvious definition on morphisms, τ can be viewed as a contravariant equivalence between the categories of admissible $Y_n(\sigma)$- and $Y_n(\sigma^t)$-modules.

5.2. **Gelfand-Tsetlin characters.** Next, let $\mathscr{P}_n$ denote the set of all power series $A(u) = A_1(u_1)A_2(u_2)\cdots A_n(u_n)$ in indeterminates $u_1, \dots, u_n$, with the property that each $A_i(u)$ belongs to $1 + u^{-1}\mathbb{F}[[u^{-1}]]$. Note that $\mathscr{P}_n$ is an abelian group under multiplication. For $A(u) \in \mathscr{P}_n$, we always write $A_i(u)$ for the ith power series defined from the equation $A(u) = A_1(u_1)\cdots A_n(u_n)$ and $A_i^{(r)}$ for the u^{-r}-coefficient of $A_i(u)$. The associated weight of $A(u) \in \mathscr{P}_n$ is defined by

$$\operatorname{wt} A(u) := A_1^{(1)}\varepsilon_1 + A_2^{(1)}\varepsilon_2 + \cdots + A_n^{(1)}\varepsilon_n \in \mathfrak{d}_n^*. \tag{5.3}$$

Now we form the *completed group algebra* $\widehat{\mathbb{Z}}[\mathscr{P}_n]$. The elements of $\widehat{\mathbb{Z}}[\mathscr{P}_n]$ consist of formal sums $S = \sum_{A(u)\in\mathscr{P}_n} m_{A(u)}[A(u)]$ for integers $m_{A(u)}$ with the property that

(a) the set $\{\operatorname{wt} A(u) \mid A(u) \in \operatorname{supp} S\}$ is contained in a finite union of sets of the form $D(\beta)$ for $\beta \in \mathfrak{d}_n^*$;
(b) for each $\alpha \in \mathfrak{d}_n^*$ the set $\{A(u) \in \operatorname{supp} S \mid \operatorname{wt} A(u) = \alpha\}$ is finite,

where $\operatorname{supp} S$ denotes $\{A(u) \in \mathscr{P}_n \mid m_{A(u)} \neq 0\}$. There is an obvious multiplication on $\widehat{\mathbb{Z}}[\mathscr{P}_n]$ extending the rule $[A(u)][B(u)] = [A(u)B(u)]$.

Given an admissible $Y_n(\sigma)$-module M and $A(u) \in \mathscr{P}_n$, the corresponding *Gelfand-Tsetlin subspace* of M is defined by

$$M_{A(u)} := \left\{ v \in M \;\middle|\; \begin{array}{l} \text{for each } i = 1,\dots,n \text{ and } r > 0 \text{ there exists} \\ p \gg 0 \text{ such that } (D_i^{(r)} - A_i^{(r)})^p v = 0 \end{array} \right\}. \tag{5.4}$$

Since the weight spaces of M are finite dimensional and the operators $D_i^{(r)}$ commute with each other, we have for each $\alpha \in \mathfrak{d}_n^*$ that

$$M_\alpha = \bigoplus_{\substack{A(u)\in\mathscr{P}_n \\ \operatorname{wt} A(u)=\alpha}} M_{A(u)}. \tag{5.5}$$

Hence, since M is the direct sum of its weight spaces, it is also the direct sum of its Gelfand-Tsetlin subspaces: $M = \bigoplus_{A(u)\in\mathscr{P}_n} M_{A(u)}$. Now we are ready to introduce a notion of *Gelfand-Tsetlin character* of an admissible $Y_n(\sigma)$-module M, which is analogous to the characters of Knight [Kn] for Yangians in general and of Frenkel and Reshetikhin [FR] in the setting of quantum affine algebras: set

$$\operatorname{ch} M := \sum_{A(u)\in\mathscr{P}_n} (\dim M_{A(u)})[A(u)]. \tag{5.6}$$

By the definition of admissibility, $\operatorname{ch} M$ belongs to the completed group algebra $\widehat{\mathbb{Z}}[\mathscr{P}_n]$. For example, the Gelfand-Tsetlin character of the trivial $Y_n(\sigma)$-module is $[1]$.

For the first lemma, recall the comultiplication $\Delta : Y_n(\sigma) \to Y_n(\sigma') \otimes Y_n(\sigma'')$ from (2.68), where σ' resp. σ'' is the strictly lower resp. upper triangular matrix such that $\sigma = \sigma' + \sigma''$. This allows us to view the tensor product of a $Y_n(\sigma')$-module M' and a $Y_n(\sigma'')$-module M'' as a $Y_n(\sigma)$-module. We will always denote this "external" tensor product by $M' \boxtimes M''$, to avoid confusion with the usual "internal" tensor product on $\mathfrak{gl}_N$-modules which we will also exploit later on. We point out that $\Delta(D_i^{(1)}) = D_i^{(1)} \otimes 1 + 1 \otimes D_i^{(1)}$, so the α-weight space of $M \boxtimes N$ is equal to $\sum_{\beta\in\mathfrak{d}_n^*} M_\beta \otimes M_{\alpha-\beta}$.

Lemma 5.1. *Suppose that M' is an admissible $Y_n(\sigma')$-module and M'' is an admissible $Y_n(\sigma'')$-module. Then, $M' \boxtimes M''$ is an admissible $Y_n(\sigma)$-module, and*

$$\mathrm{ch}(M' \boxtimes M'') = (\mathrm{ch}\, M')(\mathrm{ch}\, M'').$$

Proof. The fact that $M' \boxtimes M''$ is admissible is obvious. To compute its character, order the set of weights of M' as $\alpha_1, \alpha_2, \dots$ so that $\alpha_j > \alpha_k \Rightarrow j < k$. Let M'_j denote $\sum_{1 \leq k \leq j} M'_{\alpha_k}$. Then Theorem 2.5(i) implies that the subspace $M'_j \otimes M''$ of $M' \otimes M''$ is invariant under the action of all $D_i^{(r)}$. Moreover in order to compute the character of $M' \boxtimes M''$, we can replace it by

$$\bigoplus_{j \geq 1} (M'_j \otimes M'')/(M'_{j-1} \otimes M'') = \bigoplus_{j \geq 1} (M'_j / M'_{j-1}) \otimes M''$$

with $D_i(u)$ acting as $D_i(u) \otimes D_i(u)$. □

The next lemma is concerned with the duality τ on admissible modules from (5.2).

Lemma 5.2. *For an admissible $Y_n(\sigma)$-module M, we have that* $\mathrm{ch}(M^\tau) = \mathrm{ch}\, M$.

Proof. $\tau(D_i^{(r)}) = D_i^{(r)}$. □

5.3. **Highest weight modules.** A vector v in a $Y_n(\sigma)$-module M is called a *highest weight vector* of *type* $A(u) \in \mathscr{P}_n$ if

(a) $E_i^{(r)} v = 0$ for all $i = 1, \dots, n-1$ and $r > s_{i,i+1}$;
(b) $D_i^{(r)} v = A_i^{(r)} v$ for all $i = 1, \dots, n$ and $r > 0$.

We call M a *highest weight module* of *type* $A(u)$ if it is generated by such a highest weight vector. The following lemma gives an equivalent way to state these definitions in terms of the elements $T_{i,j}^{(r)}$ from (2.33).

Lemma 5.3. *A vector v in a $Y_n(\sigma)$-module is a highest weight vector of type $A(u)$ if and only if $T_{i,j}^{(r)} v = 0$ for all $1 \leq i < j \leq n$ and $r > s_{i,j}$, and $T_{i,i}^{(r)} v = A_i^{(r)} v$ for all $i = 1, \dots, n$ and $r > 0$.*

Proof. By the definition (2.33), the left ideal of $Y_n(\sigma)$ generated by $\{E_i^{(r)} \mid i = 1, \dots, n-1, r > s_{i,i+1}\}$ coincides with the left ideal generated by $\{T_{i,j}^{(r)} \mid 1 \leq i < j \leq n, r > s_{i,j}\}$. Moreover, $T_{i,i}^{(r)} \equiv D_i^{(r)}$ modulo this left ideal. □

In the next lemma, we write $\sigma = \sigma' + \sigma''$ where σ' resp. σ'' is strictly lower resp. upper triangular.

Lemma 5.4. *Suppose v is a highest weight vector in a $Y_n(\sigma')$-module M of type $A(u)$ and w is a highest weight vector in a $Y_n(\sigma'')$-module N of type $B(u)$. Then $v \otimes w$ is a highest weight vector in the $Y_n(\sigma)$-module $M \boxtimes N$ of type $A(u)B(u)$.*

Proof. Apply Theorem 2.5. □

To construct the universal highest weight module of type $A(u)$, let $\mathbb{F}_{A(u)}$ denote the one dimensional $Y_{(1^n)}$-module on which $D_i^{(r)}$ acts as the scalar $A_i^{(r)}$. Inflating through the epimorphism $Y^\sharp_{(1^n)}(\sigma) \twoheadrightarrow Y_{(1^n)}$ from (2.30), we can view $\mathbb{F}_{A(u)}$ instead as a $Y^\sharp_{(1^n)}(\sigma)$-module. Now form the induced module

$$M_n(\sigma, A(u)) := Y_n(\sigma) \otimes_{Y^\sharp_{(1^n)}(\sigma)} \mathbb{F}_{A(u)}. \tag{5.7}$$

This is a highest weight module of type $A(u)$, generated by the highest weight vector $m_+ := 1 \otimes 1$. Clearly it is the universal such module, i.e. all other highest weight modules of this type are quotients of $M_n(\sigma, A(u))$. In the next theorem we record two natural bases for $M_n(\sigma, A(u))$.

Theorem 5.5. *For any $A(u) \in \mathscr{P}_n$, the following sets of vectors give bases for the module $M_n(\sigma, A(u))$:*

(i) *$\{xm_+ \mid x \in X\}$, where X denotes the collection of all monomials in the elements $\{F_{i,j}^{(r)} \mid 1 \le i < j \le n, s_{j,i} < r\}$ taken in some fixed order;*

(ii) *$\{ym_+ \mid y \in Y\}$, where Y denotes the collection of all monomials in the elements $\{T_{j,i}^{(r)} \mid 1 \le i < j \le n, s_{j,i} < r\}$ taken in some fixed order.*

Proof. Let $M := M_n(\sigma, A(u))$.

(i) The isomorphism (2.29) implies that $Y_n(\sigma)$ is a free right $Y^\sharp_{(1^n)}(\sigma)$-module with basis X. Hence M has basis $\{xm_+ \mid x \in X\}$.

(ii) Recall the definition of the canonical filtration $\mathrm{F}_0 Y_n(\sigma) \subseteq \mathrm{F}_1 Y_n(\sigma) \subseteq \cdots$ of $Y_n(\sigma)$ from §2.2. In view of Lemma 2.1, it may also be defined by declaring that all $T_{i,j}^{(r)}$ are of degree r. Also introduce a filtration $\mathrm{F}_0 M \subseteq \mathrm{F}_1 M \subseteq \cdots$ of M by setting $\mathrm{F}_d M := \mathrm{F}_d Y_n(\sigma) m_+$. Let $X^{(d)}$ resp. $Y^{(d)}$ denote the set of all monomials in the elements X resp. Y of total degree at most d in the canonical filtration. Applying (i), one deduces at once that the set of all vectors of the form $\{xm_+ \mid x \in X^{(d)}\}$ form a basis for $\mathrm{F}_d M$. On the other hand using Lemmas 2.1 and 5.3, the vectors $\{ym_+ \mid y \in Y^{(d)}\}$ span $\mathrm{F}_d M$. By dimension they must be linearly independent too. Since $M = \bigcup_{d \ge 0} \mathrm{F}_d M$, this implies that the vectors $\{ym_+ \mid y \in Y\}$ give a basis for M itself. □

This implies in particular that the $\mathrm{wt}\, A(u)$-weight space of $M_n(\sigma, A(u))$ is one dimensional, spanned by the vector m_+, while all other weights are strictly lower in the dominance ordering. Given this, the usual argument shows that $M_n(\sigma, A(u))$ has a unique maximal submodule denoted $\operatorname{rad} M_n(\sigma, A(u))$. Set

$$L_n(\sigma, A(u)) := M_n(\sigma, A(u)) / \operatorname{rad} M_n(\sigma, A(u)). \tag{5.8}$$

This is the unique (up to isomorphism) *irreducible highest weight module* of type $A(u)$ for the algebra $Y_n(\sigma)$. We also note that

$$\dim \operatorname{End}_{Y_n(\sigma)}(L_n(\sigma, A(u))) = 1 \tag{5.9}$$

for any $A(u) \in \mathscr{P}_n$.

5.4. **Classification of admissible irreducible representations.** A natural question arises at this point: the module $M_n(\sigma, A(u))$ is certainly not admissible, since all of its weight spaces other than the highest one are infinite dimensional, but the irreducible quotient $L_n(\sigma, A(u))$ may well be.

Theorem 5.6. *For $A(u) \in \mathscr{P}_n$, the irreducible $Y_n(\sigma)$-module $L_n(\sigma, A(u))$ is admissible if and only if $A_i(u)/A_{i+1}(u)$ is a rational function for each $i = 1, \dots, n-1$.*

Proof. ($\Leftarrow$). Suppose that each $A_i(u)/A_{i+1}(u)$ is a rational function. For $f(u) \in 1 + u^{-1}\mathbb{F}[[u^{-1}]]$, the twist of $L_n(\sigma, A(u))$ by the automorphism μ_f from (2.37) is isomorphic to $L_n(\sigma, f(u_1 \cdots u_n)A(u))$. This allows us to reduce to the case that each $A_i(u)$ is actually a polynomial in u^{-1}. Assuming this, there certainly exists $l \geq s_{n,1} + s_{1,n}$ such that, on setting $p_i := l - s_{n,i} - s_{i,n}$, $u^{p_i} A_i(u)$ is a monic polynomial in u of degree p_i for each $i = 1, \dots, n$. Let $\pi = (q_1, \dots, q_l)$ be the pyramid associated to the shift matrix σ and the level l. For each $i = 1, \dots, n$, factorize $u^{p_i} A_i(u)$ as $(u + a_{i,1}) \cdots (u + a_{i,p_i})$ for $a_{i,j} \in \mathbb{F}$, and write the numbers $a_{i,1}, \dots, a_{i,p_i}$ into the boxes on the ith row of π from left to right. For each $j = 1, \dots, l$, let $b_{j,1}, \dots, b_{j,q_j}$ denote the entries in the jth column of the resulting π-tableau read from top to bottom. Let M_j denote the Verma module for the Lie algebra $\mathfrak{gl}_{q_j}$ of highest weight $b_{j,1}\varepsilon_1 + \cdots + b_{j,q_j}\varepsilon_{q_j}$, as in (3.47). The tensor product $M_1 \boxtimes \cdots \boxtimes M_l$ is naturally a $W(\pi)$-module, hence a $Y_n(\sigma)$-module via the quotient map (3.20). Applying Lemmas 5.1 and 5.4, it is an admissible $Y_n(\sigma)$-module and it contains an obvious highest weight vector of type $A(u)$.

($\Rightarrow$). Assume to start with that the shift matrix σ is the zero matrix, i.e. $Y_n(\sigma)$ is just the usual Yangian Y_n. Writing simply $L_n(A(u))$ instead of $L_n(\sigma, A(u))$ in this case, assume that $L_n(A(u))$ is admissible for some $A(u) \in \mathscr{P}_n$. In particular, for each $i = 1, \dots, n-1$, the $(\mathrm{wt}\, A(u) - \varepsilon_i + \varepsilon_{i+1})$-weight space of $L_n(A(u))$ is finite dimensional. Given this an argument due originally to Tarasov [T1, Theorem 1], see e.g. the proof of [M2, Proposition 3.5], shows that $A_i(u)/A_{i+1}(u)$ is a rational function for each $i = 1, \dots, n-1$.

Assume next that σ is lower triangular, and consider the canonical embedding $Y_n(\sigma) \hookrightarrow Y_n$. Given $A(u) \in \mathscr{P}_n$ such that $L_n(\sigma, A(u))$ is admissible, the PBW theorem implies that the induced module $Y_n \otimes_{Y_n(\sigma)} L_n(\sigma, A(u))$ is also admissible and contains a non-zero highest weight vector of type $A(u)$. Hence by the preceeding paragraph $A_i(u)/A_{i+1}(u)$ is a rational function for each $i = 1, \dots, n-1$.

Finally suppose that σ is arbitrary. Recalling the isomorphism ι from (2.34), the twist of a highest weight module by ι is again a highest weight module of the same type, and the twist of an admissible module is again admissible. So the conclusion in general follows from the lower triangular case. □

In view of this result, let us define

$$\mathscr{Q}_n := \left\{ A(u) \in \mathscr{P}_n \;\middle|\; \begin{array}{l} A_i(u)/A_{i+1}(u) \text{ is a rational function} \\ \text{for each } i = 1, \dots, n-1 \end{array} \right\}. \tag{5.10}$$

Then, Theorem 5.6 implies that the modules $\{L_n(\sigma, A(u)) \,|\, A(u) \in \mathscr{Q}_n\}$ give a full set of pairwise non-isomorphic admissible irreducible $Y_n(\sigma)$-modules. Moreover, the construction explained in the proof shows that every admissible irreducible $Y_n(\sigma)$-module can be obtained from an admissible irreducible $W(\pi)$-module via the homomorphism

$$\kappa \circ \mu_f : Y_n(\sigma) \twoheadrightarrow W(\pi), \tag{5.11}$$

for some pyramid π associated to the shift matrix σ and some $f(u) \in 1 + u^{-1}\mathbb{F}[[u^{-1}]]$.

5.5. **Composition multiplicities.** The final job in this chapter is to make precise the sense in which Gelfand-Tsetlin characters characterize admissible modules. We need to be a little careful here since admissible modules need not possess a composition series. Nevertheless, given admissible $Y_n(\sigma)$-modules M and L with L irreducible, we define the *composition multiplicity* of L in M by

$$[M : L] := \sup \#\{i = 1, \dots, r \mid M_i/M_{i-1} \cong L\} \tag{5.12}$$

where the supremum is taken over all finite filtrations $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$. By general principles, this multiplicity is additive on short exact sequences. Now we repeat some standard arguments from [K, ch. 9].

Lemma 5.7. *Let M be an admissible $Y_n(\sigma)$-module and $\alpha \in \mathfrak{d}_n^*$ be a fixed weight. Then, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ and a subset $I \subseteq \{1, \dots, r\}$ such that*

(i) *for each $i \in I$, $M_i/M_{i-1} \cong L_n(\sigma, A^{(i)}(u))$ for $A^{(i)}(u) \in \mathscr{Q}_n$ with $\mathrm{wt}\, A^{(i)}(u) \geq \alpha$;*

(ii) *for each $i \notin I$, $(M_i/M_{i-1})_\beta = 0$ for all $\beta \geq \alpha$.*

In particular, given $A(u) \in \mathscr{Q}_n$ with $\mathrm{wt}\, A(u) \geq \alpha$, we have that

$$[M : L_n(\sigma, A(u))] = \#\{i \in I \mid A^{(i)}(u) = A(u)\}.$$

Proof. Adapt the proof of [K, Lemma 9.6]. □

Corollary 5.8. *For an admissible $Y_n(\sigma)$-module M, we have that*

$$\mathrm{ch}\, M = \sum_{A(u) \in \mathscr{Q}_n} [M : L_n(\sigma, A(u))]\, \mathrm{ch}\, L_n(\sigma, A(u)).$$

Proof. Argue using the lemma exactly as in [K, Proposition 9.7]. □

Theorem 5.9. *Let M and N be admissible $Y_n(\sigma)$-modules such that $\mathrm{ch}\, M = \mathrm{ch}\, N$. Then M and N have all the same composition multiplicities.*

Proof. This follows from Corollary 5.8 once we check that the $\mathrm{ch}\, L_n(\sigma, A(u))$'s are linearly independent in an appropriate sense. To be precise we need to show, given

$$S = \sum_{A(u) \in \mathscr{Q}_n} m_{A(u)}\, \mathrm{ch}\, L_n(\sigma, A(u)) \in \widehat{\mathbb{Z}}[\mathscr{P}_n]$$

for coefficients $m_{A(u)} \in \mathbb{Z}$ satisfying the conditions from §5.2(a),(b), that $S = 0$ implies each $m_{A(u)} = 0$. Suppose for a contradiction that $m_{A(u)} \neq 0$ for some $A(u)$. Amongst all such $A(u)$'s, pick one with $\mathrm{wt}\, A(u)$ maximal in the dominance ordering. But then, since $\mathrm{ch}\, L_n(\sigma, A(u))$ equals $[A(u)]$ plus a (possibly infinite) linear combination of $[B(u)]$'s for $\mathrm{wt}(B(u)) < \mathrm{wt}(A(u))$, the coefficient of $[A(u)]$ in $\sum_{A(u) \in \mathscr{Q}_n} m_{A(u)}\, \mathrm{ch}\, L_n(\sigma, A(u))$ is non-zero, which is the desired contradiction. □

Corollary 5.10. *For $A(u) \in \mathscr{Q}_n$, we have that $L_n(\sigma, A(u))^\tau \cong L_n(\sigma^t, A(u))$.*

Proof. Using (2.34), it is clear that $L_n(\sigma, A(u))$ and $L_n(\sigma^t, A(u))$ have the same formal characters. Hence by Lemma 5.2 so do $L_n(\sigma, A(u))^\tau$ and $L_n(\sigma^t, A(u))$. □

6. Verma modules

Now we turn our attention to studying highest weight modules over the algebras $W(\pi)$ themselves. Fix throughout the chapter a pyramid $\pi = (q_1, \dots, q_l)$ of height $\leq n$, let $(p_1, \dots, p_n)$ be the tuple of row lengths, and choose a corresponding shift matrix $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ as usual. Notions of weights, highest weight vectors etc... are exactly as in the previous chapter, viewing $W(\pi)$-modules as $Y_n(\sigma)$-modules via the quotient map $\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi)$ from (3.20).

6.1. **Parametrization of highest weights.** Our first task is to understand the universal highest weight module of type $A(u) \in \mathscr{P}_n$ for the algebra $W(\pi)$. This module is obviously the unique largest quotient of the $Y_n(\sigma)$-module $M_n(\sigma, A(u))$ from (5.7) on which the kernel of the homomorphism $\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi)$ from (3.20) acts as zero. In other words, it is the $W(\pi)$-module $W(\pi) \otimes_{Y_n(\sigma)} M_n(\sigma, A(u))$. We will abuse notation and write simply m_+ instead of $1 \otimes m_+$ for the highest weight vector in $W(\pi) \otimes_{Y_n(\sigma)} M_n(\sigma, A(u))$.

Theorem 6.1. *For $A(u) \in \mathscr{P}_n$, $W(\pi) \otimes_{Y_n(\sigma)} M_n(\sigma, A(u))$ is non-zero if and only if $u^{p_i} A_i(u) \in \mathbb{F}[u]$ for each $i = 1, \dots, n$. In that case, the following sets of vectors give bases for $W(\pi) \otimes_{Y_n(\sigma)} M_n(\sigma, A(u))$:*

- (i) *$\{x m_+ \mid x \in X\}$, where X denotes the collection of all monomials in the elements $\{F_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{j,i} < r \leq S_{j,i}\}$ taken in some fixed order;*
- (ii) *$\{y m_+ \mid y \in Y\}$, where Y denotes the collection of all monomials in the elements $\{T_{j,i}^{(r)} \mid 1 \leq i < j \leq n, s_{j,i} < r \leq S_{j,i}\}$ taken in some fixed order.*

Proof. (ii) Let us work with the following reformulation of the definition (5.7): the module $M_n(\sigma, A(u))$ is the quotient of $Y_n(\sigma)$ by the left ideal J generated by the elements $\{E_i^{(r)} \mid i = 1, \dots, n-1, r > s_{i,i+1}\} \cup \{D_i^{(r)} - A_i^{(r)} \mid i = 1, \dots, n, r > 0\}$. Equivalently, by Lemma 5.3, J is the left ideal of $Y_n(\sigma)$ generated by the elements

$$P := \{T_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{i,j} < r\} \cup \{T_{i,i}^{(r)} - A_i^{(r)} \mid 1 \leq i \leq n, s_{i,i} < r\}.$$

Also let $Q := \{T_{j,i}^{(r)} \mid 1 \leq i < j \leq n, s_{j,i} < r\}$. Pick an ordering on $P \cup Q$ so that the elements of Q preceed the elements of P. Obviously all ordered monomials in the elements $P \cup Q$ containing at least one element of P belong to J. Hence by Lemma 2.1 and Theorem 5.5(ii), the ordered monomials in the elements $P \cup Q$ containing at least one element of P in fact form a basis for J.

Now it is clear that $W(\pi) \otimes_{Y_n(\sigma)} M_n(\sigma, A(u))$ is the quotient of $W(\pi)$ by the image $\bar{J}$ of J under the map $\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi)$. If $A_i^{(r)} \neq 0$ for some $1 \leq i \leq n$ and $r > p_i$, i.e. $u^{p_i} A_i(u) \notin \mathbb{F}[u]$, then the image of $T_{i,i}^{(r)} - A_i^{(r)}$ gives us a unit in $\bar{J}$ by Theorem 3.5, hence $W(\pi) \otimes_{Y_n(\sigma)} M_n(\sigma, A(u)) = 0$ in this case. On the other hand, if all $u^{p_i} A_i(u)$ belong to $\mathbb{F}[u]$, we let

$$\bar{P} := \{T_{i,j}^{(r)} \mid 1 \leq i < j \leq n, s_{i,j} < r \leq S_{i,j}\} \cup \{T_{i,i}^{(r)} - A_i^{(r)} \mid 1 \leq i \leq n, s_{i,i} < r \leq S_{i,i}\},$$

$$\bar{Q} := \{T_{j,i}^{(r)} \mid 1 \leq i < j \leq n, s_{j,i} < r \leq S_{j,i}\}.$$

Then Theorem 3.5 implies that $\bar{J}$ is spanned by all ordered monomials in the elements $\bar{P} \cup \bar{Q}$ containing at least one element of $\bar{P}$. By Lemma 3.6, these monomials are also linearly independent, hence form a basis for $\bar{J}$. It follows that the image of Y gives a basis for $W(\pi)/\bar{J}$, proving (ii).

(i) This follows from (ii) by reversing the argument used to deduce (ii) from (i) in the proof of Theorem 5.5. □

Now suppose that m_+ is a non-zero highest weight vector in some $W(\pi)$-module M. By Theorem 6.1, there exist elements $(a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq p_i}$ of $\mathbb{F}$ such that

$$u^{p_1} D_1(u) m_+ = (u + a_{1,1})(u + a_{1,2}) \cdots (u + a_{1,p_1}) m_+, \tag{6.1}$$

$$(u-1)^{p_2} D_2(u-1) m_+ = (u + a_{2,1})(u + a_{2,2}) \cdots (u + a_{2,p_2}) m_+, \tag{6.2}$$

$$\vdots$$

$$(u-n+1)^{p_n} D_n(u-n+1) m_+ = (u + a_{n,1})(u + a_{n,2}) \cdots (u + a_{n,p_n}) m_+. \tag{6.3}$$

In this way, the highest weight vector m_+ defines a row symmetrized π-tableau A in the sense of §4.1, namely, the unique element of $\mathrm{Row}(\pi)$ with entries $a_{i,1}, \dots, a_{i,p_i}$ on its ith row. From now on, we will say simply that the highest weight vector m_+ is of *type* A if these equations hold.

Conversely, suppose that we are given $A \in \mathrm{Row}(\pi)$ with entries $a_{i,1}, \dots, a_{i,p_i}$ on its ith row. Define the corresponding *Verma module* $M(A)$ to be the universal highest weight module of type A, i.e.

$$M(A) := W(\pi) \otimes_{Y_n(\sigma)} M_n(\sigma, A(u)) \tag{6.4}$$

where $A(u) = A_1(u_1) \cdots A_n(u_n)$ is defined from $(u-i+1)^{p_i} A_i(u-i+1) = (u + a_{i,1})(u + a_{i,2}) \cdots (u + a_{i,p_i})$ for each $i = 1, \dots, n$. Theorem 6.1 then shows that the vector $m_+ \in M(A)$ is a non-zero highest weight vector of type A. Moreover, $M(A)$ is admissible and, as in §5.3, it has a unique maximal submodule denoted $\mathrm{rad}\, M(A)$. The quotient

$$L(A) := M(A)/\mathrm{rad}\, M(A) \cong W(\pi) \otimes_{Y_n(\sigma)} L_n(\sigma, A(u)) \tag{6.5}$$

is the unique (up to isomorphism) irreducible highest weight module of type A. The modules $\{L(A) \mid A \in \mathrm{Row}(\pi)\}$ give a complete set of pairwise non-isomorphic irreducible admissible representations of the algebra $W(\pi)$.

An important point to make here is that although all these definitions required us at the outset to make fixed choices of n and k in order to realize $W(\pi)$ as a quotient of $Y_n(\sigma)$, the notion of highest weight vector of type A is actually an intrinsic notion independent of these choices, as follows from Lemma 3.2. Hence the Verma module $M(A)$ and the irreducible module $L(A)$ also only depend (up to isomorphism) on the π-tableau A, not on the particular choices of n or k.

Let us describe in detail the situation when the pyramid π consists of a single column of height n. In this case we have simply that $W(\pi) = U(\mathfrak{gl}_n)$ according to the definition (3.8). Let A be a π-tableau with entries $a_1, \dots, a_n \in \mathbb{F}$ read from top to bottom and let $\alpha = (a_1, \dots, a_n) \in \mathbb{F}^n$. A highest weight vector for $W(\pi)$ of type A means a vector m_+ with the properties

(a) $e_{i,j} m_+ = 0$ for all $1 \leq i < j \leq n$;

(b) $(e_{i,i} - i + 1)m_+ = a_i m_+$ for all $i = 1, \dots, n$.

This is the usual definition of a highest weight vector of weight $(\alpha - \rho) = a_1\varepsilon_1 + (a_2 + 1)\varepsilon_2 + \cdots + (a_n + n - 1)\varepsilon_n$ for the Lie algebra $\mathfrak{gl}_n$. Hence the module $M(A)$ here coincides with the Verma module $M(\alpha)$ from (3.47).

For another example, the trivial $W(\pi)$-module, which we defined earlier as the restriction of the $U(\mathfrak{p})$-module $\mathbb{F}_\rho$ from (3.11), is isomorphic to the module $L(A_\pi)$ where A_π is the ground state tableau from §4.3, i.e. the tableau having all entries on its ith row equal to $(1 - i)$.

6.2. **Characters of Verma modules.** By the character $\mathrm{ch}_n M$ of an admissible $W(\pi)$-module M, we mean its Gelfand-Tsetlin character when viewed as a $Y_n(\sigma)$-module via the homomorphism $\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi)$. Thus $\mathrm{ch}_n M$ is an element of the completed group algebra $\widehat{\mathbb{Z}}[\mathscr{P}_n]$ from §5.2. We include the extra subscript n here to emphasize that this notion definitely does depend on the particular choice of n (though it is independent of the other choice k made when determining the shift matrix σ).

Given a decomposition $\pi = \pi' \otimes \pi''$ with π' of level l' and π'' of level l'', the comultiplication $\Delta_{l',l''}$ from (3.31) allows us to view the tensor product of a $W(\pi')$-module M' and a $W(\pi'')$-module M'' as a $W(\pi)$-module, denoted $M' \boxtimes M''$. Assuming M' and M'' are both admissible, Lemma 5.1 and (3.32) imply that $M' \boxtimes M''$ is also admissible and

$$\mathrm{ch}_n(M' \boxtimes M'') = (\mathrm{ch}_n M')(\mathrm{ch}_n M''). \tag{6.6}$$

Lemma 5.4 also carries over in an obvious way to this setting.

Introduce the following shorthand for some special elements of the completed group algebra $\widehat{\mathbb{Z}}[\mathscr{P}_n]$:

$$x_{i,a} := [1 + (u_i + a + i - 1)^{-1}], \tag{6.7}$$

$$y_{i,a} := [1 + (a + i - 1)u_i^{-1}], \tag{6.8}$$

for $1 \leq i \leq n$ and $a \in \mathbb{F}$. We note that

$$y_{i,a}/y_{i,a-k} = x_{i,a-1}x_{i,a-2}\cdots x_{i,a-k} \tag{6.9}$$

for any $k \in \mathbb{N}$. The following theorem implies in particular that the character of any admissible $W(\pi)$-module actually belongs to the completion of the subalgebra of $\widehat{\mathbb{Z}}[\mathscr{P}_n]$ generated just by the elements $\{y_{i,a}^{\pm 1} \mid i = 1, \dots, n, a \in \mathbb{F}\}$.

Theorem 6.2. *For $A \in \mathrm{Row}(\pi)$ with entries $a_{i,1}, \dots, a_{i,p_i}$ on its ith row for each $i = 1, \dots, n$, we have that*

$$\mathrm{ch}_n M(A) = \sum_c \prod_{i=1}^n \prod_{j=1}^{p_i} \left\{ y_{i,a_{i,j}-(c_{i,j,i+1}+\cdots+c_{i,j,n})} \prod_{k=i+1}^n \frac{y_{k,a_{i,j}-(c_{i,j,k+1}+\cdots+c_{i,j,n})}}{y_{k,a_{i,j}-(c_{i,j,k}+\cdots+c_{i,j,n})}} \right\}$$

where the sum is over all tuples $c = (c_{i,j,k})_{1\leq i<k\leq n, 1\leq j\leq p_i}$ of natural numbers.

The proof of this is more technical than conceptual, so we postpone it to §6.5, preferring to illustrate its importance with some applications first.

Corollary 6.3. *Let $A_1, \dots, A_l$ be the columns of any representative of $A \in \mathrm{Row}(\pi)$, so that $A \sim_{\mathrm{ro}} A_1 \otimes \cdots \otimes A_l$. Then,*

$$\mathrm{ch}_n M(A) = (\mathrm{ch}_n M(A_1)) \times \cdots \times (\mathrm{ch}_n M(A_l)) = \mathrm{ch}_n(M(A_1) \boxtimes \cdots \boxtimes M(A_l)).$$

Proof. This follows from the theorem on interchanging the first two products on the right hand side. □

In order to derive the next corollary we need to explain an alternative way of managing the combinatorics in Theorem 6.2. Continue with $A \in \mathrm{Row}(\pi)$ with entries $a_{i,1}, \dots, a_{i,p_i}$ on its ith row as in the statement of the theorem. By a *tabloid* we mean an array $t = (t_{i,j,a})_{1 \le i \le n, 1 \le j \le p_i, a < a_{i,j}}$ of integers from the set $\{1, \dots, n\}$ such that

(a) $\cdots \le t_{i,j,a_{i,j}-3} \le t_{i,j,a_{i,j}-2} \le t_{i,j,a_{i,j}-1}$;
(b) $t_{i,j,a} = i$ for $a \ll a_{i,j}$;

for each $1 \le i \le n$, $1 \le j \le p_i$. Draw a diagram with rows parametrized by pairs (i,j) for $1 \le i \le n, 1 \le j \le p_i$ such that the (i,j)th row consists of a strip of infinitely many boxes, one in each of the columns parametrized by the numbers $\dots, a_{i,j} - 3, a_{i,j} - 2, a_{i,j} - 1$. Then the tabloid t can be recorded on the diagram by writing the number $t_{i,j,a}$ into the box in the ath column of the (i,j)th row. In this way tabloids can be thought of as fillings of the boxes of the diagram by integers from the set $\{1, \dots, n\}$ so that the entries on each row are weakly increasing and all but finitely many entries on row (i,j) are equal to i.

Given a tabloid $t = (t_{i,j,a})_{1 \le i \le n, 1 \le j \le p_i, a < a_{i,j}}$, define a tuple $c = (c_{i,j,k})_{1 \le i < k \le n, 1 \le j \le p_i}$ by declaring that $c_{i,j,k} = \#\{a < a_{i,j} \,|\, t_{i,j,a} = k\}$, i.e. $c_{i,j,k}$ counts the number of entries equal to k appearing in the (i,j)th row of the tabloid t. In this way we obtain a bijection $t \mapsto c$ from the set of all tabloids to the set of all tuples of natural numbers as in the statement of Theorem 6.2. Moreover, for t corresponding to c via this bijection, the identity (6.9) implies that

$$\prod_{i=1}^{n} \prod_{j=1}^{p_i} \left\{ y_{i, a_{i,j} - (c_{i,j,i+1} + \cdots + c_{i,j,n})} \prod_{k=i+1}^{n} \frac{y_{k, a_{i,j} - (c_{i,j,k+1} + \cdots + c_{i,j,n})}}{y_{k, a_{i,j} - (c_{i,j,k} + \cdots + c_{i,j,n})}} \right\} = \prod_{i=1}^{n} \prod_{j=1}^{p_i} \prod_{a < a_i} x_{t_{i,j,a}, a},$$

where the infinite product on the right hand side is interpreted using the convention that $x_{i,a-1} x_{i,a-2} \cdots = y_{i,a}$ for any $i = 1, \dots, n$ and $a \in \mathbb{F}$. Now we can restate Theorem 6.2:

$$\mathrm{ch}_n M(A) = \sum_t \prod_{i=1}^{n} \prod_{j=1}^{p_i} \prod_{a < a_i} x_{t_{i,j,a}, a} \tag{6.10}$$

where the first summation is over all tabloids $t = (t_{i,j,a})_{1 \le i \le n, 1 \le j \le p_i, a < a_{i,j}}$.

Corollary 6.4. *For any $A \in \mathrm{Row}(\pi)$, all Gelfand-Tsetlin subspaces of $M(A)$ are of dimension less than or equal to $p_1!(p_1 + p_2)! \cdots (p_1 + p_2 + \cdots + p_{n-1})!$.*

Proof. Two different tabloids t and t' contribute the same monomial to the right hand side of (6.10) if and only if they have the same number of entries equal to i appearing in column a for each $i = 1, \dots, n$ and $a \in \mathbb{F}$. So, given non-negative integers $k_{i,a}$ for each $i = 1, \dots, n$ and $a \in \mathbb{F}$, we need to show by (6.10) that there are at most $p_1!(p_1 + p_2)! \cdots (p_1 + \cdots + p_{n-1})!$ different tabloids with $k_{i,a}$ entries equal to i in column a for each $i = 1, \dots, n$ and $a \in \mathbb{F}$. Given such a tabloid, all entries in the rows parametrized by $(n, 1), \dots, (n, p_n)$ must equal to n, while in every other row there are only finitely many entries equal to n and all these entries must form a connected strip at the end of the row. So on removing all the boxes containing the entry n we obtain a smaller diagram with rows indexed by pairs (i, j) for $i = 1, \dots, n-1, j = 1, \dots, p_i$. By induction there are at most $p_1!(p_1+p_2)! \cdots (p_1 + \cdots + p_{n-2})!$ admissible ways of filling the boxes of this smaller diagram with $k_{i,a}$ entries equal to i in column a for each $i = 1, \dots, n-1$ and $a \in \mathbb{F}$. Therefore we just need to show that there are at most $(p_1+\cdots+p_{n-1})!$ admissible ways of inserting $k_{n,a}$ entries equal to n into column a for each $a \in \mathbb{F}$. This follows from the following claim:

Suppose we are given $a_1, \dots, a_N \in \mathbb{F}$ and non-negative integers k_a for each $a \in \mathbb{F}$, all but finitely many of which are zero. Draw a diagram with rows numbered $1, \dots, N$ such that the ith row consists of an infinite strip of boxes, one in each of the columns parametrized by $\dots, a_i - 3, a_i - 2, a_i - 1$. Then there are at most $N!$ different ways of deleting boxes from the ends of each row in such a way that a total of k_a boxes are removed from column a for each $a \in \mathbb{F}$.

This may be proved by reducing first to the case that all a_i belong to the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$, then to the case that all a_i are equal. After these reductions it follows from the obvious fact that there are at most $N!$ different N-part compositions with prescribed transpose partition. $\square$

Remark 6.5. On analyzing the proof of the corollary more carefully, one sees that this upper bound $p_1!(p_1+p_2)! \cdots (p_1+\cdots+p_{n-1})!$ for the dimensions of the Gelfand-Tsetlin subspaces of $M(A)$ is attained if and only if all entries in the first $(n-1)$ rows of the tableau A belong to the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$. At the other extreme, all Gelfand-Tsetlin subspaces of $M(A)$ are one dimensional if and only if all entries in the first $(n-1)$ rows of the tableau A belong to different cosets of $\mathbb{F}$ modulo $\mathbb{Z}$.

6.3. **The linkage principle.** Our next application of Theorem 6.2 is to prove a "linkage principle" showing that the row ordering from (4.1) controls the types of composition factors that can occur in a Verma module. In the special case that π consists of a single column of height n, i.e. $W(\pi) = U(\mathfrak{gl}_n)$, this result is [BGG2, Theorem A1]; even in this case the proof given here is quite different.

Lemma 6.6. *Suppose $A \downarrow B$. Then $\mathrm{ch}_n M(A) = \mathrm{ch}_n M(B) + (*)$ where $(*)$ is the character of some admissible $W(\pi)$-module.*

Proof. In view of Corollary 6.3, it suffices prove this in the special case that π consists of a single column, i.e. $W(\pi) = U(\mathfrak{gl}_t)$ for some $t > 0$. But in that case it is well known that $A \downarrow B$ implies that there is an embedding $M(B) \hookrightarrow M(A)$; see [BGG1] or [Di, Lemma 7.6.13]. $\square$

Theorem 6.7. *Let $A, B \in \mathrm{Row}(\pi)$ with entries $a_{i,1}, \dots, a_{i,p_i}$ and $b_{i,1}, \dots, b_{i,p_i}$ on their ith rows, respectively. The following are equivalent:*

(i) $A \geq B$;
(ii) $[M(A) : L(B)] \neq 0$;

(iii) *there exists a tuple* $c = (c_{i,j,k})_{1\le i<k\le n, 1\le j\le p_i}$ *of natural numbers such that*

$$\prod_{i=1}^n\prod_{j=1}^{p_i} y_{i,b_{i,j}} = \prod_{i=1}^n\prod_{j=1}^{p_i}\left\{ y_{i,a_{i,j}-(c_{i,j,i+1}+\cdots+c_{i,j,n})}\prod_{k=i+1}^n \frac{y_{k,a_{i,j}-(c_{i,j,k+1}+\cdots+c_{i,j,n})}}{y_{k,a_{i,j}-(c_{i,j,k}+\cdots+c_{i,j,n})}}\right\}.$$

Proof. (i)⇒(ii). If $A \ge B$ then Lemma 6.6 implies that $\mathrm{ch}_n M(A) = \mathrm{ch}_n M(B) + (*)$, where $(*)$ is the character of some admissible $W(\pi)$-module. Hence $[M(A) : L(B)] \ge [M(B) : L(B)] = 1$.

(ii)⇒(iii). Suppose that $[M(A) : L(B)] \ne 0$. The highest weight vector m_+ of $L(B)$ contributes $\prod_{i=1}^n\prod_{j=1}^{p_i} y_{i,b_{i,j}}$ to the formal character $\mathrm{ch}_n L(B)$. Hence, by Corollary 5.8, we see that $\mathrm{ch}_n M(A)$ also involves $\prod_{i=1}^n\prod_{j=1}^{p_i} y_{i,b_{i,j}}$ with non-zero coefficient. In view of Theorem 6.2 this implies (iii).

(iii)⇒(i). Suppose that

$$\prod_{i=1}^n\prod_{j=1}^{p_i} y_{i,b_{i,j}} = \prod_{i=1}^n\prod_{j=1}^{p_i}\left\{ y_{i,a_{i,j}-(c_{i,j,i+1}+\cdots+c_{i,j,n})}\prod_{k=i+1}^n \frac{y_{k,a_{i,j}-(c_{i,j,k+1}+\cdots+c_{i,j,n})}}{y_{k,a_{i,j}-(c_{i,j,k}+\cdots+c_{i,j,n})}}\right\}$$

for some tuple $c = (c_{i,j,k})_{1\le i<k\le n, 1\le j\le p_i}$ of natural numbers. We show by induction on $\sum c_{i,j,k}$ that $A \ge B$. If $\sum c_{i,j,k} = 0$ this is trivial since then $A \sim_{\mathrm{ro}} B$. Otherwise, let i_2 be maximal such that $c_{i,j,i_2} \ne 0$ for some $1 \le i < i_2$ and $1 \le j \le p_i$. Considering the $y_{i_2,?}$'s on either side of our equation gives that

$$\prod_{j=1}^{p_{i_2}} y_{i_2,b_{i_2,j}} = \prod_{j=1}^{p_{i_2}} y_{i_2,a_{i_2,j}} \times \prod_{i=1}^{i_2-1}\prod_{j=1}^{p_i}\frac{y_{i_2,a_{i,j}}}{y_{i_2,a_{i,j}-c_{i,j,i_2}}}.$$

Hence there exist $1 \le i_1 < i_2$, $1 \le j_1 \le p_{i_1}$ and $1 \le j_2 \le p_{i_2}$ such that $a_{i_2,j_2} = a_{i_1,j_1} - c_{i_1,j_1,i_2} \ne a_{i_1,j_1}$. Let $\bar A = (\bar a_{i,j})_{1\le i\le n, 1\le j\le p_i}$ be the π-tableau obtained from A by swapping the entries a_{i_1,j_1} and a_{i_2,j_2}. Define a new tuple $(\bar c_{i,j,k})_{1\le i<j\le n, 1\le j\le p_i}$ from

$$\bar c_{i,j,k} = \begin{cases} c_{i,j,k} & \text{if } (i,j,k) \ne (i_1,j_1,i_2),\\ 0 & \text{if } (i,j,k) = (i_1,j_1,i_2).\end{cases}$$

Now using the maximality of the choice of i_2, one checks that

$$\prod_{i=1}^n\prod_{j=1}^{p_i}\left\{ y_{i,\bar a_{i,j}-(\bar c_{i,j,i+1}+\cdots+\bar c_{i,j,n})}\prod_{k=i+1}^n \frac{y_{k,\bar a_{i,j}-(\bar c_{i,j,k+1}+\cdots+\bar c_{i,j,n})}}{y_{k,\bar a_{i,j}-(\bar c_{i,j,k}+\cdots+\bar c_{i,j,n})}}\right\} =$$

$$\prod_{i=1}^n\prod_{j=1}^{p_i}\left\{ y_{i,a_{i,j}-(c_{i,j,i+1}+\cdots+c_{i,j,n})}\prod_{k=i+1}^n \frac{y_{k,a_{i,j}-(c_{i,j,k+1}+\cdots+c_{i,j,n})}}{y_{k,a_{i,j}-(c_{i,j,k}+\cdots+c_{i,j,n})}}\right\} = \prod_{i=1}^n\prod_{j=1}^{p_i} y_{i,b_{i,j}}.$$

Since $\sum \bar c_{i,j,k} < \sum c_{i,j,k}$ we deduce by induction that $\bar A \ge B$. Since $A \downarrow \bar A$ this completes the proof. □

Corollary 6.8. *For $A \in \mathrm{Row}(\pi)$ with entries $a_{i,1},\dots,a_{i,p_i}$ on its ith row, the following are equivalent:*

(i) *$M(A)$ is irreducible;*

(ii) *A is minimal with respect to the ordering $\ge$;*

(iii) *$a_{i_1,j_1} \not> a_{i_2,j_2}$ for every $1 \le i_1 < i_2 \le n$, $1 \le j_1 \le p_{i_1}$ and $1 \le j_2 \le p_{i_2}$.*

Moreover, assuming (i)–(iii) hold, let $A_1, \dots, A_l$ be the columns of any representative of A read from left to right, so that $A \sim_{\mathrm{ro}} A_1 \otimes \cdots \otimes A_l$. Then, we have that

$$M(A) \cong M(A_1) \boxtimes \cdots \boxtimes M(A_l).$$

Proof. The equivalence of (i) and (ii) follows from Theorem 6.7. The equivalence of (ii) and (iii) is clear from the definition of the Bruhat ordering. The final statement follows from Corollary 6.3 and Theorem 5.9. □

6.4. **The center of $W(\pi)$.** Our final application of Theorem 6.2 is to prove that the center $Z(W(\pi))$ is a polynomial algebra on generators $\psi(Z_N^{(1)}), \dots, \psi(Z_N^{(N)})$, notation as in §3.8. In the case that π is an $n \times l$ rectangle, when $W(\pi)$ is the Yangian of level l, this result is due to Cherednik [C1, C2]; see also [M3, Corollary 4.1]. For the first lemma, recall the definition of the Miura transform $\mu : W(\pi) \hookrightarrow U(\mathfrak{h})$ from (3.30).

Lemma 6.9. $\mu(Z(W(\pi))) \subseteq Z(U(\mathfrak{h}))$.

Proof. Take $z \in Z(W(\pi))$ and $u \in U(\mathfrak{h})$. We need to show that $[\mu(z), u] = 0$. This follows by [Di, Theorem 8.4.4] if we can show that $[\mu(z), u]$ annihilates a generic Verma module for $\mathfrak{h}$. Corollary 6.8 shows that a generic Verma module for $\mathfrak{h}$ is irreducible when viewed as a $W(\pi)$-module via μ. Hence $\mu(z)$ acts on it as a scalar by (5.9). So certainly $[\mu(z), u]$ acts as zero. □

Theorem 6.10. *The map $\psi : Z(U(\mathfrak{gl}_N)) \to Z(W(\pi))$ from (3.51) is an isomorphism. Hence, the elements $\psi(Z_N^{(1)}), \dots, \psi(Z_N^{(N)})$ are algebraically independent and generate the center $Z(W(\pi))$.*

Proof. In view of Lemma 6.9 and the commutativity of the diagram (3.53), we just need to show that the image of an element $z \in Z(W(\pi))$ under the map $(\Psi_{q_1} \otimes \cdots \otimes \Psi_{q_l}) \circ \mu$ is invariant under the action of all of S_N. This follows from Theorem 5.9, Corollary 6.3 and the definition of the map $\Psi_{q_1} \otimes \cdots \otimes \Psi_{q_l}$. □

We remark that there is now a quite different proof of this, valid in arbitrary type, due to Ginzburg. For a sketch of the argument, see the footnote to [P2, Question 5.1].

Corollary 6.11. *The elements $C_n^{(1)}, C_n^{(2)}, \dots$ of $Y_n(\sigma)$ are algebraically independent and generate the center $Z(Y_n(\sigma))$. Moreover, the epimorphism $\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi)$ from (3.20) maps $Z(Y_n(\sigma))$ surjectively onto $Z(W(\pi))$.*

Proof. This is immediate from the theorem on recalling that $Y_n(\sigma)$ is a filtered inverse limit of $W(\pi)$'s as explained in [BK5, Remark 6.4]. □

We are grateful to one of the referees of [BK5] for pointing out that we are already in a position to apply [FO] to obtain the following generalization of a theorem of Kostant from [Ko1]. In the case $W(\pi)$ is the Yangian of level l this result is [FO, Theorem 2].

Theorem 6.12. *The algebra $W(\pi)$ is free as a module over its center.*

Proof. Recall by the PBW theorem that the associated graded algebra $\operatorname{gr} W(\pi)$ is free commutative on generators (3.35)–(3.37), in particular $W(\pi)$ is a special filtered algebra in the sense of [FO]. Let A be the quotient of $\operatorname{gr} W(\pi)$ by the ideal generated by the elements (3.36)–(3.37). Let $d_i^{(r)}$ resp. $c_n^{(r)}$ denote the image of $\operatorname{gr}_r D_i^{(r)}$ resp. $\operatorname{gr}_r C_n^{(r)}$ in A. Thus, A is the free polynomial algebra $\mathbb{F}[d_i^{(r)} | i = 1, \dots, n, r = 1, \dots, p_i]$. Moreover by Theorem 3.5 and (2.33) we have that $d_i^{(r)} = 0$ for $r > p_i$. It follows from this and (2.70) that if we set

$$d_i(u) = \sum_{r=0}^{p_i} d_i^{(r)} u^{p_i - r},$$

$$c_n(u) = \sum_{r=0}^{N} c_n^{(r)} u^{N-r}$$

then $c_n(u) = d_1(u) d_2(u) \cdots d_n(u)$. Now applying [FO, Theorem 1] as in the proof of [FO, Theorem 2], it suffices to show that $c_n^{(1)}, c_n^{(2)}, \dots, c_n^{(N)}$ is a regular sequence in A. Hence by [FO, Proposition 1(5)] we need to check that the variety $Z = V(c_n^{(1)}, \dots, c_n^{(N)})$ is equidimensional of dimension 0. Consider the morphism $\varphi : \mathbb{F}^N \to \mathbb{F}^N$ mapping a point $(x_i^{(r)})_{1 \leq i \leq n, 1 \leq r \leq p_i}$ to the coefficients of the following monic polynomial:

$$\prod_{i=1}^{n} (u^{p_i} + d_i^{(1)} u^{p_i - 1} + \cdots + d_i^{(p_i)}).$$

Obviously $Z = \varphi^{-1}(0)$. Since $\mathbb{F}[u]$ is a unique factorization domain, $u^N = u^{p_1} \cdots u^{p_n}$ is the unique decomposition of u^N as a product of monic polynomials of degrees $p_1, \dots, p_n$. Hence $Z = \{0\}$. □

In view of Theorem 6.10, the center of $W(\pi)$ is canonically isomorphic to the center of $U(\mathfrak{gl}_N)$. So we can parametrize the central characters of $W(\pi)$ in exactly the same way as we did for $U(\mathfrak{gl}_N)$ in §3.8, by the set of $\theta \in P = \bigoplus_{a \in \mathbb{F}} \mathbb{Z}\varepsilon_a$ whose coefficients are non-negative integers summing to N. Given such an element θ, define $f(u) = u^N + f^{(1)} u^{N-1} + \cdots + f^{(N)} \in \mathbb{F}[u]$ according to (3.45)–(3.46). Then, for an admissible $W(\pi)$-module M, define

$$\operatorname{pr}_\theta(M) := \left\{ v \in M \;\middle|\; \begin{array}{l} \text{for each } r = 1, \dots, N \text{ there exists } p \gg 0 \\ \text{such that } \psi(Z_N^{(r)} - f^{(r)})^p v = 0 \end{array} \right\}. \tag{6.11}$$

Equivalently, by (2.70) and Lemma 3.7, we have that

$$\operatorname{pr}_\theta(M) = \bigoplus_{A(u)} M_{A(u)} \tag{6.12}$$

where the direct sum is over all $A(u) \in \mathscr{P}_n$ such that

$$u^{p_1} (u-1)^{p_2} \cdots (u-n+1)^{p_n} A_1(u) A_2(u-1) \cdots A_n(u-n+1) = f(u).$$

Since the admissible $W(\pi)$-module M is the direct sum of its Gelfand-Tsetlin subspaces, it follows that

$$M = \bigoplus_{\theta \in P} \mathrm{pr}_\theta(M), \tag{6.13}$$

with the convention that $\mathrm{pr}_\theta(M) = 0$ if the coefficients of θ are not non-negative integers summing to N. This is clearly a decomposition of M as a $W(\pi)$-module.

Lemma 6.13. *All highest weight $W(\pi)$-modules of type $A \in \mathrm{Row}(\pi)$ are of central character $\theta(A)$.*

Proof. Suppose that the entries on the ith row of A are $a_{i,1}, \dots, a_{i,p_i}$. By (2.70), Lemma 3.7 and the definition (6.1)–(6.3), $\psi(Z_N(u))$ acts on any highest weight module of type A as the scalar $\prod_{i=1}^n \prod_{j=1}^{p_i} (u + a_{i,j})$. □

6.5. **Proof of Theorem 6.2.** Let $\bar\pi$ denote the pyramid obtained from π by removing the bottom row. The tuple of row lengths corresponding to the pyramid $\bar\pi$ is $(p_1, \dots, p_{n-1})$ and the submatrix $\bar\sigma = (s_{i,j})_{1 \le i,j \le n-1}$ of the shift matrix $\sigma = (s_{i,j})_{1 \le i,j \le n}$ chosen for π gives a shift matrix for $\bar\pi$. It is elementary to see from the relations that there is a homomorphism $W(\bar\pi) \to W(\pi)$ mapping the generators $D_i^{(r)}$ $(i = 1, \dots, n-1, r > 0)$, $E_i^{(r)}$ $(i = 1, \dots, n-2, r > s_{i,i+1})$ and $F_i^{(r)}$ $(i = 1, \dots, n-2, r > s_{i+1,i})$ of $W(\bar\pi)$ to the elements with the same names in $W(\pi)$. By the PBW theorem this map is in fact injective, allowing us to view $W(\bar\pi)$ as a subalgebra of $W(\pi)$. We will in fact prove the following branching theorem for Verma modules.

Theorem 6.14. *Let $A \in \mathrm{Row}(\pi)$ with entries $a_{i,1}, \dots, a_{i,p_i}$ on its ith row for each $i = 1, \dots, n$. There is a filtration $0 = M_0 \subset M_1 \subset \cdots$ of $M(A)$ as a $W(\bar\pi)$-module with $\bigcup_{i \ge 0} M_i = M(A)$ and subquotients isomorphic to the Verma modules $M(B)$ for $B \in \mathrm{Row}(\bar\pi)$ such that B has the entries $(a_{i,1} - c_{i,1}), \dots, (a_{i,p_i} - c_{i,p_i})$ on its ith row for each $i = 1, \dots, n-1$, one for each tuple $(c_{i,j})_{1 \le i \le n-1, 1 \le j \le p_i}$ of natural numbers.*

Let us first explain how to deduce Theorem 6.2 from this. Proceed by induction on n, the case $n = 1$ being trivial. For the induction step, we have by Theorem 6.14 and the induction hypothesis that the character of $\mathrm{res}^{W(\pi)}_{W(\bar\pi)} M(A)$ equals

$$\sum_c \prod_{i=1}^{n-1} \prod_{j=1}^{p_i} \left\{ y_{i,a_{i,j}-(c_{i,j,i+1}+\cdots+c_{i,j,n})} \prod_{k=i+1}^{n-1} \frac{y_{k,a_{i,j}-(c_{i,j,k+1}+\cdots+c_{i,j,n})}}{y_{k,a_{i,j}-(c_{i,j,k}+\cdots+c_{i,j,n})}} \right\},$$

where the first sum is over all tuples $c = (c_{i,j,k})_{1 \le i < k \le n-1, 1 \le j \le p_i}$ of natural numbers. But just like in the proof of Lemma 6.13, $u^{p_1}(u-1)^{p_2} \cdots (u-n+1)^{p_n} D_1(u) D_2(u-1) \cdots D_n(u-n+1)$ acts on $M(A)$ as the scalar $\prod_{i=1}^n \prod_{j=1}^{p_i} (u + a_{i,j})$. Hence recalling (6.8), each monomial appearing in the expansion of $\mathrm{ch}_n M(A)$ must simplify to $\prod_{i=1}^n \prod_{j=1}^{p_i} (u + a_{i,j})$ on replacing $y_{i,a}$ by $(u+a)$ everywhere. In this way we can recover $\mathrm{ch}_n M(A)$ uniquely from the above expression to complete the proof of Theorem 6.2.

To prove Theorem 6.14, *we will assume from now on that the shift matrix σ is upper triangular*; the result in general then follows easily by twisting with the isomorphism ι from (2.34). Exploiting this assumption, the following lemma can be checked using the formulae in [BK4, §5] and some elementary inductive arguments.

Lemma 6.15. *The following relations hold in $W(\pi)$.*

(i) *For all $i < j$, $[F_{i,j}(u)D_i(u), F_{i,j}(v)D_i(v)] = 0$.*

(ii) *For all $i < j < k$, $(u-v)[F_{j,k}(u), F_{i,j}(v)]$ equals*

$$\sum_{r\geq 0}(-1)^r \sum_{\substack{i<i_1<\cdots<i_r\leq j\\ i_{r+1}=k}} F_{i_r,i_{r+1}}(u)\cdots F_{i_1,i_2}(u)(F_{i,i_1}(v)-F_{i,i_1}(u)).$$

(iii) *For all $i < j$ and $k < i$ or $k > j$, $[D_k(u), F_{i,j}(v)] = 0$.*

(iv) *For all $i < j$, $(u-v)[D_i(u), F_{i,j}(v)] = (F_{i,j}(u) - F_{i,j}(v))D_i(u)$.*

(v) *For all $i < j$, $(u-v)[D_j(u), F_{i,j}(v)]$ equals*

$$\sum_{r\geq 0}(-1)^r \sum_{\substack{i<i_1<\cdots<i_r<j\\ i_{r+1}=j}} F_{i_r,i_{r+1}}(u)\cdots F_{i_1,i_2}(u)(F_{i,i_1}(v)-F_{i,i_1}(u))D_j(u).$$

(vi) *For all $i < j < k$, $(u-v)[D_j(u), F_{i,k}(v)]$ equals*

$$\sum_{r\geq 0}(-1)^r \sum_{i<i_1<\cdots<i_{r+1}=j} F_{i_r,i_{r+1}}(u)\cdots F_{i_1,i_2}(u)(F_{i,i_1}(v)-F_{i,i_1}(u))F_{j,k}(u)D_j(u).$$

(vii) *For all $i < j < k$, $(u-v)[F_{j,k}(u)D_j(u), F_{i,k}(v)]$ equals*

$$\sum_{r\geq 0}(-1)^r \sum_{\substack{i<i_1<\cdots<i_r<j\\ i_{r+1}=k}} F_{i_r,i_{r+1}}(u)\cdots F_{i_1,i_2}(u)(F_{i,i_1}(v)-F_{i,i_1}(u))F_{j,k}(u)D_j(u).$$

Recalling Theorem 3.5, introduce the shorthand

$$L_i(u) = \sum_{r=0}^{p_i} L_i^{(r)} u^{p_i-r} := u^{p_i} T_{n,i}(u) \in W(\pi)[u] \tag{6.14}$$

for each $1 \leq i < n$. Also for $h \geq 0$ set

$$L_{i,h}(u) := \frac{1}{h!}\frac{d^h}{du^h} L_i(u). \tag{6.15}$$

We will apply the following simple observation repeatedly from now on: given a vector m of weight α in a $W(\pi)$-module M with the property that $\alpha+\varepsilon_j-\varepsilon_i$ is not a weight of M for any $1 \leq j < i$, we have by (2.32) that $L_i(u)m = u^{p_i}F_{i,n}(u)D_i(u)m$.

Lemma 6.16. *Suppose we are given $1 \leq i < n$ and a vector m of weight α in a $W(\pi)$-module M such that*

(i) *$\alpha - d(\varepsilon_i - \varepsilon_n) + \varepsilon_j - \varepsilon_i$ is not a weight of M for any $1 \leq j < i$ and $d \geq 0$;*

(ii) *$u^{p_i}D_i(u)m \equiv (u+a_1)\cdots(u+a_{p_i})m \pmod{M'[u]}$ for some scalars $a_1, \dots, a_{p_i} \in \mathbb{F}$ and some subspace M' of M.*

For $j = 1, \dots, p_i$, define $m_j := L_{i,h(j)}(-a_j)m$ where $h(j) = \#\{k = 1, \dots, j-1 \mid a_k = a_j\}$. Then we have that

$$u^{p_i}D_i(u)m_j \equiv (u+a_1)\cdots(u+a_{j-1})(u+a_j-1)(u+a_{j+1})\cdots(u+a_{p_i})m_j$$
$$-\sum_{\substack{k=1,\dots,j-1\\ a_k=a_j}} \frac{(u+a_1)\cdots(u+a_{p_i})}{(u+a_j)^{h(j)-h(k)+1}} m_k \pmod{\sum_{r=1}^{p_i} L_i^{(r)} M'[u]}.$$

Moreover, the subspace of M spanned by the vectors $m_1, \dots, m_{p_i}$ coincides with the subspace spanned by the vectors $L_i^{(1)}m, \dots, L_i^{(p_i)}m$.

Proof. By Lemma 6.15(iv) and the assumptions (i)–(ii), we have that

$$(u-v)[u^{p_i}D_i(u), L_i(v)]m \equiv (v+a_1)\cdots(v+a_{p_i})L_i(u)m \\ -(u+a_1)\cdots(u+a_{p_i})L_i(v)m \pmod{\sum_{r=1}^{p_i} L_i^{(r)}M'[u,v]}.$$

Hence,

$$u^{p_i}D_i(u)L_i(v)m \equiv (u+a_1)\cdots(u+a_{p_i})L_i(v)m - \frac{(u+a_1)\cdots(u+a_{p_i})L_i(v)}{u-v}m \\ + \frac{(v+a_1)\cdots(v+a_{p_i})L_i(u)}{u-v}m.$$

Apply the operator $\frac{1}{h(j)!}\frac{d^{h(j)}}{dv^{h(j)}}$ to both sides using the Leibnitz rule then set $v := -a_j$ to deduce that

$$u^{p_i}D_i(u)L_{i,h(j)}(-a_j)m \equiv (u+a_1)\cdots(u+a_{p_i})L_{i,h(j)}(-a_j)m \\ -\sum_{k=0}^{h(j)} \frac{(u+a_1)\cdots(u+a_{p_i})}{(u+a_j)^{h(j)-k+1}}L_{i,k}(-a_j)m.$$

The left hand side equals $u^{p_i}D_i(u)m_j$ by definition. The right hand side simplifies to give

$$(u+a_1)\cdots(u+a_j-1)\cdots(u+a_{p_i})m_j - \sum_{k=0}^{h(j)-1} \frac{(u+a_1)\cdots(u+a_{p_i})}{(u+a_j)^{h(j)-k+1}}L_{i,k}(-a_j)m$$

which is exactly what we need to prove the first part of the lemma.

For the second part, we observe that the transition matrix between the vectors $L_i(u_1)m, \cdots, L_i(u_{p_i})m$ and $L_i^{(1)}m, \cdots, L_i^{(p_i)}m$ is a Vandermonde matrix with determinant $\prod_{1\le j<k\le p_i}(u_j-u_k)$. Apply $\frac{1}{h(j)!}\frac{d^{h(j)}}{du_j^{h(j)}}$ for $j = 1, \dots, p_i$ to deduce that the determinant of the transition matrix between the vectors $L_{i,h(1)}(u_1)m, \cdots, L_{i,h(p_i)}(u_{p_i})m$ and $L_i^{(1)}m, \cdots, L_i^{(p_i)}m$ is

$$\frac{1}{h(1)!h(2)!\cdots h(p_i)!}\frac{d^{h(1)}}{du_1^{h(1)}}\cdots\frac{d^{h(p_i)}}{du_{p_i}^{h(p_i)}}\prod_{1\le j<k\le p_i}(u_j-u_k).$$

Evaluate this expression at $u_j = -a_j$ for each $j = 1, \dots, p_i$ to get

$$(-1)^{h(1)+\cdots+h(p_i)} \prod_{\substack{1\le j<k\le p_i\\ a_j\neq a_k}} (a_k - a_j) \neq 0.$$

Hence the transition matrix between the vectors $m_1, \dots, m_{p_i}$ and $L_i^{(1)}m, \cdots, L_i^{(p_i)}m$ is invertible, so they span the same space. □

Lemma 6.17. *Under the same assumptions as Lemma 6.16, let C_d denote the set of all p_i-tuples $c = (c_1, \dots, c_{p_i})$ of natural numbers summing to d. Put a total order on C_d so that $c' < c$ if c' is lexicographically greater than c. For $c \in C_d$ let*

$$m_c := \prod_{j=1}^{p_i} \prod_{k=1}^{c_j} L_{i,h_c(j,k)}(-a_j + k - 1)m$$

where $h_c(j,k) = \#\{l = 1, \dots, j-1 \mid a_l - c_l = a_j - k + 1\}$. Then,

$$u^{p_i} D_i(u) m_c \equiv (u + a_1 - c_1) \cdots (u + a_{p_i} - c_{p_i}) m_c \pmod{M'_c[u]}$$

where M'_c is the subspace of M spanned by all $m_{c'}$ for $c' < c$ and by all $L_i^{(r_1)} \cdots L_i^{(r_d)} M'$ for $1 \le r_1, \dots, r_d \le p_i$. Moreover, the vectors $\{m_c \mid c \in C_d\}$ span the same subspace of M as the vectors $L_i^{(r_1)} \cdots L_i^{(r_d)} m$ for all $1 \le r_1, \dots, r_d \le p_i$.

Proof. Note first that the definition of the vectors m_c does not depend on the order taken in the products, thanks to Lemma 6.15(i). Now proceed by induction on d, the case $d = 1$ being precisely the result of the previous lemma. For $d > 1$, define vectors $m_1, \dots, m_{p_i}$ according to the preceeding lemma. For $r = 1, \dots, p_i$, let M'_r be the subspace spanned by $m_1, \dots, m_{r-1}$ and $L_i^{(s)} M'$ for all $s = 1, \dots, p_i$. Then the preceeding lemma shows that

$$u^{p_i} D_i(u) m_r \equiv (u + a_1) \cdots (u + a_r - 1) \cdots (u + a_{p_i}) m_r \pmod{M'_r[u]}$$

and that $m_1, \dots, m_{p_i}$ span the same space as the vectors $L_i^{(1)} m, \dots, L_i^{(p_i)} m$.

For $c \in C_{d-1}$ and $r = 1, \dots, p_i$, let

$$m_{r,c} := \prod_{j=1}^{p_i} \prod_{k=1}^{c_j} L_{i,h_{r,c}(j,k)}(-a_j + \delta_{j,r} + k - 1) m_r$$

where $h_{r,c}(j,k) := \#\{l = 1, \dots, j-1 \mid a_l - \delta_{r,l} - c_l = a_j - \delta_{r,j} - k + 1\}$. Let $M'_{r,c}$ be the subspace of M spanned by all $m_{r,c'}$ for $c' < c$ together with $L_i^{(r_1)} \cdots L_i^{(r_{d-1})} M'_r$ for all $1 \le r_1, \dots, r_{d-1} \le p_i$. Then by the induction hypothesis,

$$u^{p_i} D_i(u) m_{r,c} \equiv (u + a_1 - c_1) \cdots (u + a_r - 1 - c_r) \cdots (u + a_{p_i} - c_{p_i}) m_{r,c} \pmod{M'_{r,c}[u]}.$$

Moreover the vectors $\{m_{r,c} \mid c \in C_{d-1}\}$ span the same subspace of M as the vectors $L_i^{(r_1)} \cdots L_i^{(r_{d-1})} m_r$ for all $1 \le r_1, \dots, r_{d-1} \le p_i$. Now observe that if $c \in C_{d-1}$ satisfies $c_1 = \cdots = c_{r-1} = 0$, then $m_{r,c} = m_{c+\delta_r}$ where $c + \delta_r \in C_d$ is the tuple $(c_1, \dots, c_{r-1}, c_r + 1, c_{r+1}, \dots, c_{p_i})$; otherwise, $m_{r,c}$ lies in the subspace spanned by the $m_{s,c'}$ for $s < r, c' \in C_{d-1}$. The lemma follows. □

At last we can complete the proof of Theorem 6.14. Let C denote the set of all tuples $c = (c_{i,j})_{1 \le i \le n-1, 1 \le j \le p_i}$ of natural numbers. Writing $|c|_i$ for $\sum_{j=1}^{p_i} c_{i,j}$ and $|c|$ for $|c|_1 + |c|_2 + \cdots + |c|_{n-1}$, we put a total order on C so that $c' \le c$ if any of the following hold:

(a) $|c'| < |c|$;
(b) $|c'| = |c|$ but $|c'|_{n-1} = |c|_{n-1}, |c'|_{n-2} = |c|_{n-2}, \dots, |c'|_{i+1} = |c|_{i+1}$ and $|c'|_i > |c|_i$ for some $i \in \{1, \dots, n-1\}$;

(c) $|c'|_i = |c|_i$ but the tuple $(c'_{i,1}, \dots, c'_{i,p_i})$ is lexicographically greater than or equal to the tuple $(c_{i,1}, \dots, c_{i,p_i})$ for every $i = 1, \dots, n-1$.

Now let $M := M(A)$ for short. For each $c \in C$, define a vector $m_c \in M$ by

$$m_c := \prod_{i=1}^{n-1} \left\{ \prod_{j=1}^{p_i} \prod_{k=1}^{c_{i,j}} L_{i,h_c(i,j,k)}(-a_{i,j} + k - 1) \right\} m_+$$

where $h_c(i,j,k) = \#\{l = 1, \dots, j-1 \mid a_{i,l} - c_{i,l} = a_{i,j} - k + 1\}$ and the first product is taken in order of increasing i from left to right. The second part of Lemma 6.17 and Theorem 6.1(ii) imply that the vectors $\{y m_c \mid y \in Y, c \in C\}$ form a basis for M, where Y here denotes the set of all monomials in the elements $\{T_{j,i}^{(r)} \mid 1 \leq i < j \leq n-1, r = 1, \dots, p_i\}$. For each $c \in C$, let M_c resp. M'_c denote the subspace of M spanned by the vectors $\{y m_{c'} \mid y \in Y, c' \leq c\}$ resp. $\{y m_{c'} \mid y \in Y, c' < c\}$. Clearly $M = \bigcup_{c \in C} M_c$. Now we complete the proof of Theorem 6.14 by showing that each M_c is actually a $W(\bar{\pi})$-submodule of M with $M_c / M'_c \cong M(B)$ for $B \in \mathrm{Row}(\bar{\pi})$ such that B has entries $(a_{i,1} - c_{i,1}), \dots, (a_{i,p_i} - c_{i,p_i})$ on its ith row for each $i = 1, \dots, n-1$.

Proceeding by induction on the total ordering on C, the induction hypothesis allows us to assume that M'_c is a $W(\bar{\pi})$-submodule of M. Then the vectors

$$\{y m_{c'} + M'_c \mid y \in Y, c' \geq c\}$$

form a basis for the $W(\bar{\pi})$-module M/M'_c. Hence the vector $\overline{m}_c := m_c + M'_c$ is a vector of maximal weight in M/M'_c, so it is annihilated by all $E_i^{(r)}$ $(i = 1, \dots, n-2, r > s_{i,i+1})$. Moreover, using Lemma 6.15(iii),(vi) and (vii), Lemma 6.17 and the PBW theorem for $Y_{(1^n)}^{\flat}(\sigma)$, one checks that

$$u^{p_i} D_i(u) \overline{m}_c = (u + a_{i,1} - c_{i,1}) \cdots (u + a_{i,p_i} - c_{i,p_i}) \overline{m}_c.$$

Hence, $\overline{m}_c \in M/M'_c$ is a highest weight vector of type B as claimed. Now it follows easily using Theorem 6.1(ii) and the universal property of Verma modules that M_c is a $W(\bar{\pi})$-submodule of M and $M_c/M'_c \cong M(B)$.

7. Standard modules

In this chapter, we begin by classifying the irreducible finite dimensional representations of $W(\pi)$ and of $Y_n(\sigma)$, following the argument in the case of the Yangian Y_n itself due to Tarasov [T2] and Drinfeld [D]. Then we define and study another family of finite dimensional $W(\pi)$-modules which we call standard modules.

7.1. **Two rows.** In this section we assume that $n = 2$ and let π be any pyramid with just two rows of lengths $p_1 \leq p_2$. We will represent the π-tableau with entries $a_1, \dots, a_{p_1}$ on its first row and $b_1, \dots, b_{p_2}$ on its second row by ${}^{a_1 \cdots a_{p_1}}_{b_1 \cdots b_{p_2}}$. The first lemma is well known; see e.g. [CP1]. We reproduce here the detailed argument following [M2, Proposition 3.6] since we need to slightly weaken the hypotheses later on.

Lemma 7.1. *Assume $p_1 = p_2 = l$ and $a_1, \dots, a_l, b_1, \dots, b_l, a, b \in \mathbb{F}$.*

(i) *If $a_i > b$ implies that $a_i \geq a > b$ for each $i = 1, \dots, l$, then all highest weight vectors in the module $L\binom{a_1 \cdots a_l}{b_1 \cdots b_l} \boxtimes L\binom{a}{b}$ are scalar multiples of $m_+ \otimes m_+$.*

(ii) *If $a > b_i$ implies that $a > b \geq b_i$ for each $i = 1, \dots, l$, then all highest weight vectors in the module $L\binom{a}{b} \boxtimes L\binom{a_1 \cdots a_l}{b_1 \cdots b_l}$ are scalar multiples of $m_+ \otimes m_+$.*

Proof. (i) Abbreviate $e := e_{1,2}$, $d_2 := e_{2,2}$ and $f := e_{2,1}$ in the Lie algebra $\mathfrak{gl}_2$. Let $f^{(r)}$ denote $f^r/r!$. Recall that the irreducible $\mathfrak{gl}_2$-module $L\binom{a}{b}$ of highest weight $(a\varepsilon_1 + (b+1)\varepsilon_2)$ has basis $m_+, fm_+, f^{(2)}m_+, \dots$ if $a \not> b$ or $m_+, fm_+, \dots, f^{(a-b-1)}m_+$ if $a > b$. Also $ef^{(r+1)}m_+ = (a-b-r-1)f^{(r)}m_+$.

Suppose that $L\binom{a_1 \cdots a_l}{b_1 \cdots b_l} \boxtimes L\binom{a}{b}$ contains a highest weight vector v that is not a scalar multiple of $m_+ \otimes m_+$. We can write

$$v = \sum_{i=0}^{k} m_i \otimes f^{(k-i)} m_+$$

for vectors $m_0 \neq 0, m_1, \dots, m_k$ and $k \geq 0$ with $k < a - b$ in case $a > b$. The element $T_{1,2}^{(r+1)}$ acts on the tensor product as $T_{1,2}^{(r+1)} \otimes 1 + T_{1,2}^{(r)} \otimes d_2 + T_{1,1}^{(r)} \otimes e \in W(\pi) \otimes U(\mathfrak{gl}_2)$.

Apply $T_{1,2}^{(r+1)}$ to the vector v and compute the $? \otimes y^{(k)} m_+$-coefficient to deduce that

$$T_{1,2}^{(r+1)} m_0 + (b+k+1) T_{1,2}^{(r)} m_0 = 0$$

for all $r \geq 0$. It follows that $T_{1,2}^{(r)} m_0 = 0$ for all $r > 0$, hence m_0 is a scalar multiple of the canonical highest weight vector m_+ of $L\binom{a_1 \cdots a_l}{b_1 \cdots b_l}$. Moreover we must in fact have that $k \geq 1$ since v is not a multiple of $m_+ \otimes m_+$.

Next compute the $? \otimes f^{(k-1)} m_+$-coefficient of $T_{1,2}^{(r+1)} v$ to get that

$$T_{1,2}^{(r+1)} m_1 + (b+k) T_{1,2}^{(r)} m_1 + (a-b-k) T_{1,1}^{(r)} m_0 = 0.$$

Multiply by $(-(b+k))^{l-r}$ and sum over $r = 0, 1, \dots, l$ to deduce that

$$T_{1,2}^{(l+1)} m_1 + (a-b-k) \sum_{r=0}^{l} (-(b+k))^{l-r} T_{1,1}^{(r)} m_0 = 0.$$

But $T_{1,2}^{(l+1)} = 0$ in $W(\pi)$ by a trivial special case of Theorem 3.5. Moreover, by the definition (6.1), we have that $\sum_{r=0}^{l} u^{l-r} T_{1,1}^{(r)} m_0 = (u + a_1) \cdots (u + a_l) m_0$. So we have shown that

$$(a-b-k)(a_1-b-k)(a_2-b-k) \cdots (a_l-b-k) = 0.$$

Since $k \geq 1$ and $k < a - b$ in case $a > b$, we have that $(a-b-k) \neq 0$. Hence we must have that $a_i = b + k$ for some $i = 1, \dots, l$, i.e. $a_i > b$ and either $a \not> b$ or $a_i < a$. This is a contradiction.

(ii) Similar. □

Corollary 7.2. *Assume $p_1 = p_2 = l$ and $a_1, \dots, a_l, b_1, \dots, b_l, a, b \in \mathbb{F}$.*

(i) *If $b < a_i$ implies that $b < a \leq a_i$ for each $i = 1, \dots, l$, then $L\binom{a}{b} \boxtimes L\binom{a_1 \cdots a_l}{b_1 \cdots b_l}$ is a highest weight module generated by the highest weight vector $m_+ \otimes m_+$.*

(ii) *If $b_i < a$ implies that $b_i \leq b < a$ for each $i = 1, \dots, l$, then $L\binom{a_1 \cdots a_l}{b_1 \cdots b_l} \boxtimes L\binom{a}{b}$ is a highest weight module generated by the highest weight vector $m_+ \otimes m_+$.*

Proof. (i) By Lemma 7.1(i), $L\binom{a_1\cdots a_l}{b_1\cdots b_l}\boxtimes L\binom{a}{b}$ has simple socle generated by the highest weight vector $m_+\otimes m_+$. Now apply the duality τ from (5.2) using Corollary 5.10 and (3.33) to deduce that

$$(L\tbinom{a_1\cdots a_l}{b_1\cdots b_l}\boxtimes L\tbinom{a}{b})^\tau \cong L\tbinom{a}{b}\boxtimes L\tbinom{a_1\cdots a_l}{b_1\cdots b_l}$$

has a unique maximal submodule and that the highest weight vector $m_+\otimes m_+$ does not belong to this submodule. Hence it is a highest weight module generated by the vector $m_+\otimes m_+$.

(ii) Similar. □

Remark 7.3. The module $L\binom{a_1\cdots a_l}{b_1\cdots b_l}$ in the statement of Corollary 7.2 can in fact be replaced by any non-zero quotient of the Verma module $M\binom{a_1\cdots a_l}{b_1\cdots b_l}$. This follows because the only property of $L\binom{a_1\cdots a_l}{b_1\cdots b_l}$ needed for the proof of Lemma 7.1 is that all its highest weight vectors are scalar multiples of m_+; any non-zero submodule of the dual Verma module $M\binom{a_1\cdots a_l}{b_1\cdots b_l}^\tau$ also has this property.

Lemma 7.4. *Assume $p_1\leq p_2$ and $a_1,\dots,a_{p_1},b_1,\dots,b_{p_2},b\in\mathbb{F}$.*

(i) *If $a_i>b$ implies that $a_i>b_i\geq b$ for each $i=1,\dots,p_1$ then all highest weight vectors in the module $L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}\boxtimes L\binom{}{b}$ are scalar multiples of $m_+\otimes m_+$.*

(ii) *All highest weight vectors in the module $L\binom{}{b}\boxtimes L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}$ are scalar multiples of $m_+\otimes m_+$.*

Proof. Let $\sigma=(s_{i,j})_{1\leq i,j\leq 2}$ be a shift matrix corresponding to the pyramid π. Also note that $L\binom{}{b}$ is just the one dimensional $\mathfrak{gl}_1$-module with basis m_+ such that $e_{1,1}m_+=bm_+$.

(i) Suppose that $m\otimes m_+$ is a non-zero highest weight vector in $L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}\boxtimes L\binom{}{b}$. So we have that $E_1^{(r+1)}(m\otimes m_+)=0$ for all $r>s_{1,2}$ and

$$u^{p_1}D_1(u)(m\otimes m_+)=(u+c_1)(u+c_2)\cdots(u+c_{p_1})(m\otimes m_+)$$

for some scalars $c_1,\dots,c_{p_1}\in\mathbb{F}$.

By [BK5, Lemma 11.3] and [BK5, Theorem 4.1(i)], we have that $\Delta_{p_2,1}(E_1^{(r+1)})=E_1^{(r+1)}\otimes 1+E_1^{(r)}\otimes(e_{1,1}+1)$ for all $r>s_{1,2}$. Hence $E_1^{(r+1)}m+(b+1)E_1^{(r)}m=0$ for all $r>s_{1,2}$. On setting $m':=E_1^{(s_{1,2}+1)}m$, we deduce that $E_1^{(s_{1,2}+r+1)}m=(-(b+1))^r m'$ for all $r\geq 0$, i.e.

$$E_1(u)m=(1-(b+1)u^{-1}+(b+1)^2u^{-2}-\cdots)u^{-s_{1,2}-1}m'=\frac{u^{-s_{1,2}-1}}{1+(b+1)u^{-1}}m'.$$

If $m'=0$ then we have that $E_1^{(r)}m=0$ for all $r>s_{1,2}$, hence m is a scalar multiple of m_+ as required. So assume from now on that $m'\neq 0$ and aim for a contradiction.

Since $\Delta_{p_2,1}(D_1^{(r)})=D_1^{(r)}\otimes 1$ for all $r>0$ we have that

$$D_1(u)m=(1+c_1u^{-1})(1+c_2u^{-1})\cdots(1+c_{p_1}u^{-1})m.$$

The last two equations combined with the identity $[D_1(u), E_1^{(s_{1,2}+1)}] = u^{s_{1,2}} D_1(u) E_1(u)$ in $W(\pi)[[u^{-1}]]$ show that

$$D_1(u)m' = \frac{(1 + c_1 u^{-1}) \cdots (1 + c_{p_1} u^{-1})(1 + (b + 1)u^{-1})}{1 + bu^{-1}} m'.$$

Since $D_1^{(r)} = 0$ for $r > p_1$ it follows from this that $b = c_i$ for some $1 \leq i \leq p_1$. Without loss of generality we may as well assume that $b = c_1$. Then we have shown that

$$D_1(u)m' = (1 + (c_1 + 1)u^{-1})(1 + c_2 u^{-1}) \cdots (1 + c_{p_1} u^{-1})m'.$$

Now we claim that if we have any non-zero vector in $L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}}$ on which $D_1(u)$ acts as the scalar $(1+d_1u^{-1}) \cdots (1+d_{p_1}u^{-1})$ then there exists a permutation $w \in S_{p_1}$ such that $a_i \geq d_{wi}$ and moreover if $a_i > b_i$ then $d_{wi} > b_i$, for each $i = 1, \dots, p_1$. To prove this, we may as well replace the module $L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}}$ with the tensor product $L\binom{a_1}{b_1} \boxtimes \cdots \boxtimes L\binom{a_{p_1}}{b_{p_1}} \boxtimes L(_{b_{p_1+1}}) \boxtimes \cdots \boxtimes L(_{b_{p_2}})$, since that contains $L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}}$ (possibly twisted by the isomorphism ι) as a subquotient. Now the claim follows from Lemma 5.1 and the familiar fact that if we have a non-zero vector in the irreducible $\mathfrak{gl}_2$-module $L\binom{a}{b}$ on which $D_1(u)$ acts as the scalar $(1 + du^{-1})$ then $a \geq d$ and moreover if $a > b$ then $d > b$.

Applying the claim to the non-zero vectors m and m' of $L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}}$, we deduce (after reordering if necessary) that there exists a permutation $w \in S_{p_1}$ such that

(a) $a_1 \geq c_1 + 1$ and moreover if $a_1 > b_1$ then $c_1 + 1 > b_1$; $a_2 \geq c_2$ and moreover if $a_2 > b_2$ then $c_2 > b_2$; $\dots$; $a_{p_1} \geq c_{p_1}$ and moreover if $a_{p_1} > b_{p_1}$ then $c_{p_1} > b_{p_1}$;

(b) $a_1 \geq c_{w1}$ and moreover if $a_1 > b_1$ then $c_{w1} > b_1$; $a_2 \geq c_{w2}$ and moreover if $a_2 > b_2$ then $c_{w2} > b_2$; $\dots$; $a_{p_1} \geq c_{wp_1}$ and moreover if $a_{p_1} > b_{p_1}$ then $c_{wp_1} > b_{p_1}$.

From this we can derive the required contradiction, as follows. Suppose that we know that $c_i > b$ for some i. Then $a_i \geq c_i > b$, hence by the hypothesis from the statement of the lemma $a_i \geq c_{wi} > b_i \geq b$. Hence $c_{wi} > b$. Now we do know that $c_1 = b$. Hence $a_1 \geq c_1 + 1 > b$, so $a_1 \geq c_{w1} > b_1 \geq b$. Hence $c_{w1} > b$. Combining this with the preceeding observation we deduce that $c_{w^k 1} > b$ for all $k \geq 1$, hence in particular $c_1 > b$.

(ii) We have that $\Delta_{1,p_2}(E_1^{(r)}) = 1 \otimes E_1^{(r)}$ for all $r > s_{1,2}$. So if $m_+ \otimes m$ is a highest weight vector in $L(_b) \boxtimes L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}}$ then $E_1^{(r)} m = 0$ for all $r > s_{1,2}$. Hence m is a scalar multiple of m_+ as required. □

Corollary 7.5. *Assume $p_1 \leq p_2$ and $a_1, \dots, a_{p_1}, b_1, \dots, b_{p_2}, b \in \mathbb{F}$.*

(i) *If $b < a_i$ implies that $b \leq b_i < a_i$ for each $i = 1, \dots, p_1$ then the module $L(_b) \boxtimes L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}}$ is a highest weight module generated by the highest weight vector $m_+ \otimes m_+$.*

(ii) *The module $L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}} \boxtimes L(_b)$ is a highest weight module generated by the highest weight vector $m_+ \otimes m_+$.*

Proof. Argue using the duality τ exactly as in the proof of Corollary 7.2. □

Remark 7.6. As in Remark 7.3, the module $L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}$ in the statement of Corollary 7.5(ii) can be replaced by any non-zero quotient of the Verma module $M\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}$. We can not quite say the same thing for Corollary 7.5(i), but by the proof we can at least replace $L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}$ by any non-zero quotient M of the Verma module $M\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}$ with the property that all of its Gelfand-Tsetlin weights, i.e. the $A(u) \in \mathscr{P}_2$ such that $M_{A(u)} \neq 0$, are also Gelfand-Tsetlin weights of the module $L\binom{a_1}{b_1} \boxtimes \cdots \boxtimes L\binom{a_{p_1}}{b_{p_1}} \boxtimes L(_{b_{p_1+1}}) \boxtimes \cdots \boxtimes L(_{b_{p_2}})$.

Now we can prove the main theorem of the section. This is new only if $p_1 \neq p_2$.

Theorem 7.7. *Assume $p_1 \leq p_2$ and $a_1, \dots, a_{p_1}, b_1, \dots, b_{p_2} \in \mathbb{F}$ are scalars such that the following property holds for each $i = 1, \dots, p_1$:*

> *If the set $\{a_j - b_k \mid i \leq j \leq p_1, i \leq k \leq p_2 \text{ such that } a_j > b_k\}$ is non-empty then $(a_i - b_i)$ is its smallest element.*

Then the irreducible $W(\pi)$-module $L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}}$ is isomorphic to the tensor product of the modules

$$L\binom{a_1}{b_1}, \dots, L\binom{a_{p_1}}{b_{p_1}}, L(_{b_{p_1+1}}), \dots, L(_{b_{p_2}})$$

taken in any order that makes sense.

Proof. Assume to start with that the pyramid π is left-justified. First we show for $p_1 > 0$ that

$$L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_1}} \cong L\binom{a_1}{b_1} \boxtimes L\binom{a_2\cdots a_{p_1}}{b_2\cdots b_{p_1}}.$$

Since $a_1 > b_i$ implies that $a_1 > b_1 \geq b_i$ for all $i = 2, \dots, p_1$, Lemma 7.1(ii) implies that $m_+ \otimes m_+$ is the unique (up to scalars) highest weight vector in the module on the right hand side. Since $b_1 < a_i$ implies $b_1 < a_1 \leq a_i$, Corollary 7.2(i) shows that this vector generates the whole module. Hence it is irreducible, so isomorphic to $L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_1}}$ by Lemma 5.4. Next we show for $p_2 > p_1$ that

$$L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}} \cong L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2-1}} \boxtimes L(_{b_{p_2}}).$$

Since $a_i > b_{p_2}$ implies $a_i > b_i \geq b_{p_2}$ Lemma 7.4(i) implies that $m_+ \otimes m_+$ is the unique (up to scalars) highest weight vector in the module on the right hand side. But by Corollary 7.5(ii) this vector generates the whole module, hence it is irreducible. Using these two facts, it follows by induction on p_2 that

$$L\binom{a_1\cdots a_{p_1}}{b_1\cdots b_{p_2}} \cong L\binom{a_1}{b_1} \boxtimes \cdots \boxtimes L\binom{a_{p_1}}{b_{p_1}} \boxtimes L(_{b_{p_1+1}}) \boxtimes \cdots \boxtimes L(_{b_{p_2}}).$$

This proves the theorem for one particular ordering of the tensor product and for one particular choice of the pyramid π with row lengths (p_1, p_2). The theorem for all other orderings and pyramids follows from this by character considerations. □

Suppose finally that we are given an arbitrary two row tableau A with entries $a_1, \dots, a_{p_1}$ on row one and $b_1, \dots, b_{p_2}$ on row two. We can always reindex the entries in the rows so that the hypothesis of Theorem 7.7 is satisfied: first reindex to ensure if possible that $a_1 - b_1$ is the minimal positive integer difference amongst all the dfferences $a_i - b_j$, then inductively reindex the remaining entries $a_2, \dots, a_{p_1}, b_2, \dots, b_{p_2}$. Hence Theorem 7.7 shows that *every* irreducible admissible $W(\pi)$-module can be realized as a tensor product of irreducible $\mathfrak{gl}_2$- and $\mathfrak{gl}_1$-modules. This remarkable observation was first made by Tarasov [T2] in the case $p_1 = p_2$.

Corollary 7.8. *If $L\binom{a_1 \cdots a_{p_1}}{b_1 \cdots b_{p_2}}$ is finite dimensional for scalars $a_1, \dots, a_{p_1}, b_1, \dots, b_{p_2} \in \mathbb{F}$ then there exists a permutation $w \in S_{p_2}$ such that $a_1 > b_{w1}, a_2 > b_{w2}, \dots, a_{p_1} > b_{wp_1}$.*

Proof. Reindexing if necessary, we may assume that the hypothesis of Theorem 7.7 is satisfied. Then by the theorem we must have that $L\binom{a_i}{b_i}$ is finite dimensional for each $i = 1, \dots, p_1$, i.e $a_i > b_i$ for each such i. □

7.2. **Classification of finite dimensional irreducible representations.** Now assume that $\pi = (q_1, \dots, q_l)$ is an arbitrary pyramid with row lengths $(p_1, \dots, p_n)$. Recall the definitions of the sets $\mathrm{Row}(\pi)$ of row symmetrized π-tableaux, $\mathrm{Col}(\pi)$ of column strict π-tableaux and $\mathrm{Dom}(\pi)$ of dominant row symmetrized π-tableaux from §4.1.

Theorem 7.9. *For $A \in \mathrm{Row}(\pi)$, the irreducible $W(\pi)$-module $L(A)$ is finite dimensional if and only if A is dominant, i.e. it has a representative belonging to* $\mathrm{Col}(\pi)$.

Proof. Suppose first that $L(A)$ is finite dimensional. Let $\sigma = (s_{i,j})_{1 \le i,j \le n}$ be a shift matrix corresponding to π, so that $W(\pi)$ is canonically a quotient of the shifted Yangian $Y_n(\sigma)$. For each $i = 1, \dots, n-1$, let σ_i denote the 2×2 submatrix

$$\begin{pmatrix} s_{i,i} & s_{i,i+1} \\ s_{i+1,i} & s_{i+1,i+1} \end{pmatrix}$$

of the matrix σ. Also let $a_{i,1}, \dots, a_{i,p_i}$ be the entries in the ith row of A for each $i = 1, \dots, n$. The map ψ_{i-1} from (2.66) obviously induces an embedding of the shifted Yangian $Y_2(\sigma_i)$ into $Y_n(\sigma)$. The highest weight vector $m_+ \in L(A)$ is also a highest weight vector in the restriction of $L(A)$ to $Y_2(\sigma_i)$ using this embedding. Hence by Corollary 7.8 there exists $w \in S_{p_{i+1}}$ such that

$$a_{i,1} > a_{i+1,w1}, a_{i,2} > a_{i+1,w2}, \dots, a_{i,p_i} > a_{i+1,wp_i},$$

for each $i = 1, \dots, n-1$. Hence A has a representative belonging to $\mathrm{Col}(\pi)$.

Conversely, suppose that A has a representative belonging to $\mathrm{Col}(\pi)$. Let $A_1, \dots, A_l$ be the columns of this representative, so that $A \sim_{\mathrm{ro}} A_1 \otimes \cdots \otimes A_l$. Since each A_i is column strict, the irreducible $\mathfrak{gl}_{q_i}$-module $L(A_i)$ is finite dimensional. By Lemma 5.4 the tensor product $L(A_1) \boxtimes \cdots \boxtimes L(A_l)$ is then a finite dimensional $W(\pi)$-module containing a highest weight vector of type A. Hence $L(A)$ is finite dimensional. □

Hence, the modules $\{L(A) \,|\, A \in \mathrm{Dom}(\pi)\}$ give a full set of pairwise non-isomorphic finite dimensional irreducible $W(\pi)$-modules. As a corollary, we have the following result classifying the finite dimensional irreducible representations of the shifted Yangians $Y_n(\sigma)$ themselves. Since every finite dimensional $Y_n(\sigma)$-module is admissible, it is enough for this to determine which of the irreducible modules $L_n(\sigma, A(u))$ from (5.8) is finite dimensional.

Corollary 7.10. *Let $\sigma = (s_{i,j})_{1 \le i,j \le n}$ be a shift matrix and set $d_i := s_{i,i+1} + s_{i+1,i}$. For $A(u) \in \mathscr{P}_n$, the irreducible $Y_n(\sigma)$-module $L_n(\sigma, A(u))$ is finite dimensional if* *and only if there exist (necessarily unique) monic polynomials* $P_1(u), \dots, P_{n-1}(u)$, $Q_1(u), \dots, Q_{n-1}(u) \in \mathbb{F}[u]$ *such that* $Q_i(u)$ *is of degree* d_i*,* $(P_i(u), Q_i(u)) = 1$ *and*

$$\frac{A_i(u)}{A_{i+1}(u)} = \frac{P_i(u)}{P_i(u-1)} \times \frac{u^{d_i}}{Q_i(u)}$$

for each $i = 1, \dots, n-1$.

Proof. Recall from the proof of Theorem 5.6 that every admissible irreducible $Y_n(\sigma)$-module may be obtained by inflating an admissible irreducible $W(\pi)$-module through the map (5.11), for some pyramid π with shift matrix σ and some $f(u) \in 1 + u^{-1}\mathbb{F}[[u^{-1}]]$. Given this and Theorem 7.9, we see that $L_n(\sigma, A(u))$ is finite dimensional if and only if there exist $l \geq s_{n,1} + s_{1,n}$, $f(u) \in 1 + u^{-1}\mathbb{F}[[u^{-1}]]$ and scalars $a_{i,j} \in \mathbb{F}$ for $1 \leq i \leq n, 1 \leq j \leq p_i := l - s_{n,i} - s_{i,n}$ such that

(a) $A_i(u) = f(u)(1 + a_{i,1}u^{-1}) \cdots (1 + a_{i,p_i}u^{-1})$ for each $i = 1, \dots, n$;
(b) $a_{i,j} \geq a_{i+1,j}$ for each $i = 1, \dots, n-1$ and $j = 1, \dots, p_i$.

Following the proof of [M2, Theorem 2.8], these conditions are equivalent to the existence of monic polynomials $P_1(u), \dots, P_{n-1}(u), Q_1(u), \dots, Q_{n-1}(u) \in \mathbb{F}[u]$ such that $Q_i(u)$ is of degree d_i and

$$\frac{A_i(u)}{A_{i+1}(u)} = \frac{P_i(u)}{P_i(u-1)} \times \frac{u^{d_i}}{Q_i(u)}$$

for each $i = 1, \dots, n-1$. Finally to get uniqueness of the $P_i(u)$'s and $Q_i(u)$'s we have to insist in addition that $(P_i(u), Q_i(u)) = 1$. □

From Corollary 7.10 and (2.77), it also follows that the isomorphism classes of irreducible $SY_n(\sigma)$-modules are parametrized in the same fashion by monic polynomials $P_1(u), \dots, P_{n-1}(u)$, $Q_1(u), \dots, Q_{n-1}(u) \in \mathbb{F}[u]$ such that $Q_i(u)$ is of degree d_i and $(P_i(u), Q_i(u)) = 1$ for each $i = 1, \dots, n-1$. In the case σ is the zero matrix, each $Q_i(u)$ is of course just equal to 1, so we recover the classification from [D] of finite dimensional irreducible representations of the Yangian of $\mathfrak{sl}_n$ by their *Drinfeld polynomials* $P_1(u), \dots, P_{n-1}(u)$; see also [M2, §2] once more.

7.3. **Tensor products.** Assume throughout the section that $\pi = (q_1, \dots, q_l)$ is a fixed pyramid and make a corresponding choice $\sigma = (s_{i,j})_{1 \leq i,j \leq n}$ of shift matrix. Also set $t := q_l$ for short. For $A \in \mathrm{Col}(\pi)$ with columns $A_1, \dots, A_l$ read from left to right, let

$$V(A) := L(A_1) \boxtimes \cdots \boxtimes L(A_l). \tag{7.1}$$

We will refer to the modules $\{V(A) | A \in \mathrm{Col}(\pi)\}$ as *standard modules*. As we observed already in the proof of Theorem 7.9, each $V(A)$ is a finite dimensional $W(\pi)$-module, and the vector $m_+ \otimes \cdots \otimes m_+ \in V(A)$ is a highest weight vector of type equal to the row equivalence class of A. We wish to give a sufficient condition for $V(A)$ to be a highest weight module generated by this highest weight vector, following an argument due to Chari [C] in the context of quantum affine algebras. The key step is provided by the following lemma; in its statement we work with the usual action of the symmetric group S_t on finite dimensional irreducible $\mathfrak{gl}_t$-modules, and $s_1, \dots, s_{t-1} \in S_t$ denote the basic transpositions.

Lemma 7.11. *Suppose that we are given a π-tableau A with columns $A_1, \dots, A_l$ from left to right, together with $1 \le k < t = q_l$ and $w \in S_t$ such that $k \ge w^{-1}k < w^{-1}(k+1)$. Letting $a_1, \dots, a_p$ resp. $c_1, \dots, c_q, b_1, \dots, b_p$ denote the entries in the $(n-t+k)$th resp. the $(n-t+k+1)$th row of A read from left to right, assume that*

(i) *$a_i > b_i$ for each $i = 1, \dots, p$;*
(ii) *$a_i \not> a_j$ for each $1 \le i < j \le p$;*
(iii) *either $c_i \not< a_j$ or $c_i \le b_j$ for each $i = 1, \dots, q$ and $j = 1, \dots, p$;*
(iv) *none of the elements $c_1, \dots, c_q$ lie in the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$ as a_p;*
(v) *A_l is column strict.*

Then, the vector $m_+ \otimes \cdots \otimes m_+ \otimes s_k w m_+$ lies in the $W(\pi)$-submodule of $L(A_1) \boxtimes \cdots \boxtimes L(A_{l-1}) \boxtimes L(A_l)$ generated by the vector $m_+ \otimes \cdots \otimes m_+ \otimes w m_+$.

Since this is technical, let us postpone the proof until the end of the section and explain the applications. For the first one, recall from §4.1 the definition of the set $\mathrm{Std}(\pi)$ of standard π-tableaux in the case that π is left-justified.

Theorem 7.12. *Assume that the pyramid π is left-justified and let $A \in \mathrm{Std}(\pi)$. Then the $W(\pi)$-module $V(A)$ is a highest weight module generated by the highest weight vector $m_+ \otimes \cdots \otimes m_+$.*

Proof. Let $A_1, \dots, A_l$ denote the columns of A from left to right, and set $M := L(A_1) \boxtimes \cdots \boxtimes L(A_{l-1}), L := L(A_l)$ for short. By induction on l, M is a highest weight module generated by the vector $m_+ \otimes \cdots \otimes m_+$. Fix the following choice of reduced expression for the longest element w_0 of the symmetric group S_t:

$$w_0 = s_{i_h} \cdots s_{i_1} \text{ where } (i_1, \dots, i_h) = (t-1; t-2, t-1; \dots; 2, \dots, t-1; 1, \dots, t-1).$$

For $r = 0, \dots, h$ let $m_r := s_{i_r} \cdots s_{i_1} m_+ \in L$. Note by the choice of reduced expression that $i_{r+1} \ge s_{i_1} \cdots s_{i_r}(i_{r+1}) < s_{i_1} \cdots s_{i_r}(i_{r+1}+1)$. So, taking $w = s_{i_r} \cdots s_{i_1}$ and $k = i_{r+1}$ for some $r = 0, \dots, h-1$, the hypotheses of Lemma 7.11 are satisfied. Hence the lemma implies that the vector $m_+ \otimes \cdots \otimes m_+ \otimes m_{r+1}$ lies in the $W(\pi)$-submodule of $M \boxtimes L$ generated by the vector $m_+ \otimes \cdots \otimes m_+ \otimes m_r$. Since this is true for all $r = 0, \dots, h-1$ and $m_h = w_0 m_+$, this shows that the vector $m_+ \otimes \cdots \otimes m_+ \otimes w_0 m_+$ lies in the $W(\pi)$-submodule of $M \boxtimes L$ generated by the highest weight vector $m_+ \otimes \cdots \otimes m_+ \otimes m_+$.

Now to complete the proof we show that $M \boxtimes L$ is generated as a $W(\pi)$-module by the vector $m_+ \otimes \cdots \otimes m_+ \otimes w_0 m_+$. Let M_d denote the span of all weight spaces of M of weight $\lambda - (\varepsilon_{j_1} - \varepsilon_{j_1+1}) - \cdots - (\varepsilon_{j_d} - \varepsilon_{j_d+1})$ for $1 \le j_1, \dots, j_d < n$, where λ is the weight of the highest weight vector $m_+ \otimes \cdots \otimes m_+$ of M. We will prove by induction on $d \ge 0$ that $M_d \otimes L$ is contained in the $W(\pi)$-submodule of $M \boxtimes L$ generated by the vector $(m_+ \otimes \cdots \otimes m_+) \otimes w_0 m_+$. Note to start with for any vector $y \in L$ and $1 \le i < t$ that

$$E_{n-t+i}^{(1)}((m_+ \otimes \cdots \otimes m_+) \otimes y) = (m_+ \otimes \cdots \otimes m_+) \otimes (e_{i,i+1} y).$$

Since L is generated as a $\mathfrak{gl}_t$-module by the lowest weight vector $w_0 m_+$ this is enough to verify the base case. Now for the induction step we know already that M is a highest weight module, hence it suffices to show that every vector of the form

$(F_i^{(r)}x) \otimes y$ for $1 \leq i < n, r > 0, x \in M_{d-1}$ and $y \in L$ lies in the $W(\pi)$-submodule of $M \boxtimes L$ generated by $M_{d-1} \otimes L$. But for this we have that

$$F_i^{(r)}(x \otimes y) \equiv (F_i^{(r)}x) \otimes y \pmod{M_{d-1} \otimes L}$$

by Theorem 2.5(iii). □

For the second application, we return to an arbitrary pyramid $\pi = (q_1, \dots, q_l)$. The following theorem reduces the problem of computing the characters of all finite dimensional irreducible $W(\pi)$-modules to that of computing the characters just of the modules $L(A)$ where all entries of A lie in the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$. Twisting moreover with the automorphism η_a from (3.28) using Lemma 3.1 one can reduce further to the case that all entries of A actually lie in $\mathbb{Z}$ itself, i.e. $A \in \mathrm{Dom}_0(\pi)$.

Theorem 7.13. *Suppose $\pi = \pi' \otimes \pi''$ for pyramids π' and π''. Pick $A' \in \mathrm{Dom}(\pi')$ and $A'' \in \mathrm{Dom}(\pi'')$ such that no entry of A' lies in the same coset of $\mathbb{F}$ modulo $\mathbb{Z}$ as an entry of A''. Then the $W(\pi)$-module $L(A') \boxtimes L(A'')$ is irreducible with highest weight vector $m_+ \otimes m_+$.*

Proof. By character considerations, we may assume for the proof that the pyramid π' is right-justified of level l' and the pyramid π'' is left-justified of level l''. Pick a standard π''-tableau representing A'' and let $A_{l'+1}, A_{l'+2}, \dots, A_l$ be its columns read from left to right. We claim that $L(A') \boxtimes L(A_{l'+1}) \boxtimes \cdots \boxtimes L(A_l)$ is a highest weight module generated by the highest weight vector $m_+ \otimes m_+ \otimes \cdots \otimes m_+$. The theorem follows from this claim as follows. By Theorem 7.12, $L(A'')$ is a quotient of $L(A_{l'+1}) \boxtimes \cdots \boxtimes L(A_l)$. Hence we get from the claim that $L(A') \boxtimes L(A'')$ is a highest weight module generated by the highest weight vector $m_+ \otimes m_+$. Similarly so is $L(A'')^\tau \boxtimes L(A')^\tau$, hence on twisting with τ we see that $m_+ \otimes m_+$ is actually the unique (up to scalars) highest weight vector in $L(A') \boxtimes L(A'')$. Thus $L(A') \boxtimes L(A'')$ is irreducible.

To prove the claim, fix the same reduced expression $w_0 = s_{i_h} \cdots s_{i_1}$ for the longest element of S_t as in the proof of Theorem 7.12. Let $m_r := s_{i_r} \cdots s_{i_1} m_+ \in L(A_l)$. We will actually show that m_{r+1} lies in the $W(\pi)$-submodule of $L(A') \boxtimes L(A_{l'+1}) \boxtimes \cdots \boxtimes L(A_l)$ generated by the vector $m_+ \otimes m_+ \otimes \cdots \otimes m_+ \otimes m_r$ for each $r = 0, \dots, h-1$. Given this, it follows that $m_+ \otimes m_+ \otimes \cdots \otimes m_+ \otimes w_0 m_+$ lies in the $W(\pi)$-submodule generated by the highest weight vector. Since we already know by induction that $L(A') \boxtimes L(A_{l'+1}) \boxtimes \cdots \boxtimes L(A_{l-1})$ is highest weight, the argument can then be completed in the same way as in last paragraph of the proof of Theorem 7.12.

So finally fix a choice of $r = 0, \dots, h-1$. Let $w := s_{i_r} \cdots s_{i_1}$ and $k := i_{r+1}$. Pick a representative for A' so that, letting $a_1, \dots, a_p$ resp. $c_1, \dots, c_q, b_1, \dots, b_p$ denote the entries in its $(n-t+k)$th resp. $(n-t+k+1)$th row read from left to right, we have that

(i) $a_i > b_i$ for each $i = 1, \dots, p$;
(ii) $a_i \not> a_j$ for each $1 \leq i < j \leq p$;
(iii) either $c_i \not< a_j$ or $c_i \leq b_j$ for each $i = 1, \dots, q$ and $j = 1, \dots, p$.

To see that this is possible, it is easy to arrange things so that (i) and (ii) are satisfied. Then if $p > 0$ we can rearrange the $(n-t+i+1)$th row so that the difference $a_p - b_p$ is the smallest of all the positive integers in the set $\{a_p - b_1, \dots, a_p - b_p, a_p - c_1, \dots, a_p -$

$c_q\}$. The condition (iii) is then automatic for $j = p$, and the remaining entries $c_1, \dots, c_q, b_1, \dots, b_{p-1}$ can then be rearranged inductively to get (iii) in general. Let $A_1, \dots, A_{l'}$ denote the columns of this representative from left to right. It then follows by Lemma 7.11 that the vector $m_+ \otimes \cdots \otimes m_+ \otimes m_{r+1}$ lies in the $W(\pi)$-submodule of $L(A_1) \boxtimes \cdots \boxtimes L(A_{l-1}) \otimes L(A_l)$ generated by the vector $m_+ \otimes \cdots \otimes m_+ \otimes m_r$. Since $L(A')$ is a quotient of the submodule of $L(A_1) \boxtimes \cdots \boxtimes L(A_{l'})$ generated by the highest weight vector $m_+ \otimes \cdots \otimes m_+$, this completes the proof. □

We still need to explain the proof of Lemma 7.11. Let the notation be as in the statement of the lemma and abbreviate $(n - t + k)$ by i. Let π' be the pyramid consisting just of the ith and $(i+1)$th rows of π. The 2×2 submatrix σ' consisting just of the ith and $(i+1)$th rows and columns of σ gives a choice of shift matrix for π'. As in the proof of Theorem 7.9, the map ψ_{i-1} from (2.66) induces an embedding $\varphi : Y_2(\sigma') \hookrightarrow Y_n(\sigma)$. For $j = 1, \dots, l$, let

$$q'_j := \begin{cases} 2 & \text{if } n - q_j < i, \\ 1 & \text{if } n - q_j = i, \\ 0 & \text{if } n - q_j > i. \end{cases}$$

So, numbering the columns of the pyramid π' by $1, \dots, l$ in the same way as in the pyramid π, its columns are of heights $q'_1, q'_2, \dots, q'_l$ from left to right (including possibly some empty columns at the left hand edge). Recall the quotient map $\kappa : Y_n(\sigma) \twoheadrightarrow W(\pi)$ from (3.20) and also the Miura transform $\mu : W(\pi) \hookrightarrow U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l})$ from (3.30). Similarly we have the quotient map $\kappa' : Y_2(\sigma') \twoheadrightarrow W(\pi')$ and the Miura transform $\mu' : W(\pi') \hookrightarrow U(\mathfrak{gl}_{q'_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q'_l})$. For each $j = 1, \dots, l$, define an algebra embedding $\varphi_j : U(\mathfrak{gl}_{q'_j}) \hookrightarrow U(\mathfrak{gl}_{q_j})$ so that if $q'_j = 2$ then

$$e_{1,1} \mapsto e_{q_j-n+i,q_j-n+i} + n - q_j, \qquad e_{1,2} \mapsto e_{q_j-n+i,q_j-n+i+1},$$
$$e_{2,1} \mapsto e_{q_j-n+i+1,q_j-n+i}, \qquad e_{2,2} \mapsto e_{q_j-n+i+1,q_j-n+i+1} + n - q_j,$$

and if $q'_j = 1$ then $e_{1,1} \mapsto e_{1,1} + n - q_j - 1$. We have now defined all the maps in the following diagram:

$$\begin{array}{ccccc} Y_2(\sigma') & \xrightarrow{\kappa'} & W(\pi') & \xrightarrow{\mu'} & U(\mathfrak{gl}_{q'_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q'_l}) \\ \varphi\big\downarrow & & & & \big\downarrow\varphi_1 \otimes \cdots \otimes \varphi_l \\ Y_n(\sigma) & \xrightarrow{\kappa} & W(\pi) & \xrightarrow{\mu} & U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l}) \end{array} \tag{7.2}$$

This diagram definitely does *not* commute. So the actions of $Y_2(\sigma')$ on the $U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l})$-module $L(A_1) \boxtimes \cdots \boxtimes L(A_l)$ defined either using the homomorphism $\mu \circ \kappa \circ \varphi$ or using the homomorphism $\varphi_1 \otimes \cdots \otimes \varphi_l \circ \mu \circ \kappa$ are in general different. In the proof of the following lemma we will see that in fact the two actions coincide on special vectors.

Lemma 7.14. *The subspaces* $(\mu \circ \kappa \circ \varphi)(Y_2(\sigma'))(m_+ \otimes \cdots \otimes m_+ \otimes wm_+)$ *and* $(\varphi_1 \otimes \cdots \otimes \varphi_l \circ \mu' \circ \kappa')(Y_2(\sigma'))(m_+ \otimes \cdots \otimes m_+ \otimes wm_+)$ *of* $L(A_1) \boxtimes \cdots \boxtimes L(A_{l-1}) \boxtimes L(A_l)$ *are equal.*

Proof. For $j = 1, \dots, l-1$, let m_j be an element of $L(A_j)$ whose weight is equal to the weight of the highest weight vector m_+ of $L(A_j)$ minus some multiple of the ith simple root $\varepsilon_i - \varepsilon_{i+1} \in \mathfrak{d}_n^*$. Also let m_l be any element of $L(A_l)$. We claim for any element x of $Y_2(\sigma')$ that

$$(\mu \circ \kappa \circ \varphi)(x)(m_1 \otimes \cdots \otimes m_l) = (\varphi_1 \otimes \cdots \otimes \varphi_l \circ \mu' \circ \kappa')(x)(m_1 \otimes \cdots \otimes m_l).$$

Clearly the lemma follows from this claim. The advantage of the claim is that it suffices to prove it for x running over a set of generators for the algebra $Y_2(\sigma')$, since the vector on the right hand side of the equation can obviously be expressed as a linear combination of vectors of the form $m_1' \otimes \cdots \otimes m_l'$ where again the weight of m_j' is equal to the weight of m_+ minus some multiple of $\varepsilon_i - \varepsilon_{i+1}$ for each $j = 1, \dots, l-1$,

So now we proceed to prove the claim just for $x = D_1^{(r)}, D_2^{(r)}, E_1^{(r)}$ and $F_1^{(r)}$ and all meaningful r. For each of these choices for x, explicit formulae for each of $\kappa'(x) \in W(\pi')$ and of $\kappa \circ \varphi(x) \in W(\pi)$ are recorded in (3.13) and (3.20). On applying the Miura transforms one obtains explicit formulae for $(\mu \circ \kappa \circ \varphi)(x)$ and for $(\varphi_1 \otimes \cdots \otimes \varphi_l \circ \mu' \circ \kappa')(x)$ as elements of $U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l})$. By considering these formulae directly, one observes finally that $(\mu \circ \kappa \circ \varphi)(x) - (\varphi_1 \otimes \cdots \otimes \varphi_l \circ \mu' \circ \kappa')(x)$ is a linear combination of terms of the form $x_1 \otimes \cdots \otimes x_l$ such that some x_j $(j = 1, \dots, l-1)$ annihilates m_j by weight considerations, which proves the claim. Let us explain this last step in detail just in the case $x = D_2^{(r)}$, all the other cases being entirely similar. In this case, we have that

$$(\mu \circ \kappa \circ \varphi - \varphi_1 \otimes \cdots \otimes \varphi_l \circ \mu' \circ \kappa')(x) = \sum_{\substack{i_1, \dots, i_r \\ j_1, \dots, j_r}} (-1)^{\#\{t=1,\dots,r-1 \,|\, \mathrm{row}(j_t) \leq i\}} \bar{e}_{i_1,j_1} \cdots \bar{e}_{i_r,j_r}$$

where for $1 \leq s, t \leq q_c$ the notation $\bar{e}_{q_1+\cdots+q_{c-1}+s, q_1+\cdots+q_{c-1}+t}$ here denotes $1^{\otimes(c-1)} \otimes e_{s,t} \otimes 1^{\otimes(l-c-1)} + \delta_{s,t}(n - q_c) \in U(\mathfrak{gl}_{q_1}) \otimes \cdots \otimes U(\mathfrak{gl}_{q_l})$, and the sum is over all $1 \leq i_r, \dots, i_r, j_1, \dots, j_r \leq n$ with

(a) $\mathrm{row}(i_1) = \mathrm{row}(j_r) = i + 1$;
(b) $\mathrm{col}(i_t) = \mathrm{col}(j_t)$ for all $t = 1, \dots, r$;
(c) $\mathrm{row}(j_t) = \mathrm{row}(i_{t+1})$ for all $t = 1, \dots, r-1$;
(d) if $\mathrm{row}(j_t) \geq i + 1$ then $\mathrm{col}(j_t) < \mathrm{col}(i_{t+1})$ for all $t = 1, \dots, r-1$;
(e) if $\mathrm{row}(j_t) \leq i$ then $\mathrm{col}(j_t) \geq \mathrm{col}(i_{t+1})$ for all $t = 1, \dots, r-1$;
(f) $\mathrm{row}(j_t) \notin \{i, i+1\}$ for at least one $t = 1, \dots, r-1$.

Take such a monomial $\bar{e}_{i_1,j_1} \cdots \bar{e}_{i_r,j_r}$. Let c be minimal such that there exists j_t with $\mathrm{col}(j_t) = c$ and $\mathrm{row}(j_t) \notin \{i, i+1\}$, then take the maximal such t. Consider the component of $\bar{e}_{i_1,j_1} \cdots \bar{e}_{i_r,j_r}$ in the cth tensor position. If $\mathrm{row}(j_t) > i+1$, then by the choices of c and t, this component is of the form $u\bar{e}_{i_t,j_t}v$ where $\mathrm{row}(i_t) \leq i{+}1 < \mathrm{row}(j_t)$ and the weight of v is some multiple of $\varepsilon_i - \varepsilon_{i+1}$. Similarly if $\mathrm{row}(j_t) < i$ then this component is of the form $u\bar{e}_{i_{t+1},j_{t+1}}v$ where $\mathrm{row}(j_{t+1}) \geq i > \mathrm{row}(i_{t+1})$ and the weight of v is some multiple of $\varepsilon_i - \varepsilon_{i+1}$. In either case, this component annihilates the vector $m_c \in L(A_c)$ by weight considerations. □

Now let L_j be the irreducible $U(\mathfrak{gl}_{q_j'})$-submodule of $L(A_j)$ generated by the highest weight vector m_+ for each $j = 1, \dots, l-1$, embedding $U(\mathfrak{gl}_{q_j'})$ into $U(\mathfrak{gl}_{q_j})$ via φ_j. Similarly, let L_l be the $U(\mathfrak{gl}_{q_l'})$-submodule of $L(A_l)$ generated by the vector wm_+. Recall by the hypotheses in Lemma 7.11 that the tableau A_l is column strict and

$k \geq w^{-1}(k) < w^{-1}(k+1)$. It follows that the vector $wm_+ \in L_l$ is a highest weight vector for the action of $U(\mathfrak{gl}_{q_l'})$ with $e_{1,1}$ acting as $(a+i-1)$ and $e_{2,2}-1$ acting as $(b+i-1)$, for some $b < a \geq a_p$. In particular L_l is also irreducible. So in our usual notation the $W(\pi')$-module $L_1 \boxtimes \cdots \boxtimes L_l$ is isomorphic to the tensor product

$$L({\scriptstyle c_1+i-1}) \boxtimes \cdots \boxtimes L({\scriptstyle c_q+i-1}) \boxtimes L\left({\scriptstyle a_1+i-1 \atop b_1+i-1}\right) \boxtimes \cdots \boxtimes L\left({\scriptstyle a_{p-1}+i-1 \atop b_{p-1}+i-1}\right) \boxtimes L\left({\scriptstyle a+i-1 \atop b+i-1}\right)$$

for some $b > a \leq a_p$. Using the remaining hypotheses (i)–(iv) from Lemma 7.11, we apply Corollaries 7.2(i) and 7.5(i), or rather the slightly stronger versions of these corollaries described in Remarks 7.3 and 7.6, repeatedly to this tensor product working from right to left to deduce that $L_1 \boxtimes \cdots \boxtimes L_l$ is actually a highest weight $W(\pi')$-module generated by the highest weight vector $m_+ \otimes \cdots \otimes m_+ \otimes wm_+$. Hence in particular, since $s_k wm_+ \in L_l$, we get that

$$m_+ \otimes \cdots \otimes m_+ \otimes s_k wm_+ \in W(\pi')(m_+ \otimes \cdots \otimes wm_+).$$

In view of Lemma 7.14, this completes the proof of Lemma 7.11.

7.4. **Characters of standard modules.** We wish to explain how to compute the Gelfand-Tsetlin characters of the standard modules $\{V(A) \mid A \in \mathrm{Col}(\pi)\}$ from (7.1). In view of (6.6) it suffices just to explain how to compute the character of $V(A)$ when A consists just of a single column with entries $a_1 > \cdots > a_n$ read from top to bottom, i.e. the character of the finite dimensional irreducible $\mathfrak{gl}_n$-module of highest weight $a_1\varepsilon_1 + (a_2+1)\varepsilon_2 + \cdots (a_n+n-1)\varepsilon_n$. Choose an arbitrary scalar $c \in \mathbb{F}$ so that $a_n + n - 1 \geq c$. Then, $(a_1 - c, a_2 + 1 - c, \dots, a_n + n - 1 - c)$ is a partition. Draw its Young diagram in the usual English way and define the *residue* of the box in the ith row and jth column to be $(c + j - i)$. For example, if $n = 3$, $c = 0$ and $(a_1, a_2 + 1, a_3 + 2) = (5, 3, 2)$ then the Young diagram with boxes labelled by their residues is as follows

0	1	2	3	4
-1	0	1		
-2	-1			

Given a filling t of the boxes of this diagram with the integers $\{1, \dots, n\}$ we associate the monomial

$$x(t) := \prod_{i=1}^{n} \prod_{j=1}^{a_i+i-1-c} x_{t_{i,j}, c+j-i} \in \widehat{\mathbb{Z}}[\mathscr{P}_n] \tag{7.3}$$

where $t_{i,j}$ denotes the entry of t in the ith row and jth column and $x_{i,a}$ and $y_{i,a}$ are as in (6.7)–(6.8). Then we have that

$$\mathrm{ch}_n\, V(A) = y_{1,c} y_{2,c-1} \cdots y_{n,c-n+1} \times \sum_t x(t) \tag{7.4}$$

summing over all fillings t of the boxes of the diagram with integers $\{1, \dots, n\}$ such that the entries are weakly increasing along rows from left to right and strictly increasing down columns from top to bottom. (Slightly more generally, for any $m \geq n$, $\mathrm{ch}_m\, V(A)$ can be deduced from (7.4) by simply replacing each $x_{i,a}$ by $x_{m-n+i,a}$ and each $y_{i,a}$ by $y_{m-n+i,a}$.) For example, suppose that the entries of A

are $1, -1, -2, \dots, 1-n$ from top to bottom. Then $V(A)$ is the n-dimensional *natural representation* V_n of $\mathfrak{gl}_n$ of highest weight ε_1. The possible fillings of the Young diagram $\square$ are $\boxed{1}, \boxed{2}, \dots, \boxed{n}$. Hence

$$\mathrm{ch}_n V_n = x_{1,0} + x_{2,0} + \cdots + x_{n,0}. \tag{7.5}$$

The proof of the formula (7.4) is standard, based like the proof of Theorem 6.1 on branching $V(A)$ from $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-1}$. This time however the restriction is completely understood by the classical branching theorem for finite dimensional representations of $\mathfrak{gl}_n$, so everything is easy. The closest reference that we could find in the literature is [NT, Lemma 2.1]; see also [GT, C1] and [FM, Lemma 4.7] (the last of these references greatly influenced our choice of notation here).

Let us make a few further comments. By (6.9), we have that

$$y_{i,a} = \begin{cases} x_{i,-i+1}x_{i,-i+2}\cdots x_{i,a-1} & \text{if } a > 1-i, \\ 1 & \text{if } a = 1-i, \\ x_{i,a}^{-1}x_{i,a+1}^{-1}\cdots x_{i,-i}^{-1} & \text{if } a < 1-i \end{cases} \tag{7.6}$$

for each $i = 1, \dots, n$. Hence if the scalar c in (7.4) is an integer, i.e. if the representation $V(A)$ is a *rational representation* of $\mathfrak{gl}_n$, then $\mathrm{ch}_n V(A)$ belongs to the subalgebra $\mathbb{Z}[x_{i,a}^{\pm 1} \mid i = 1, \dots, n, a \in \mathbb{Z}]$ of $\widehat{\mathbb{Z}}[\mathscr{P}_n]$. Moreover, the character of a rational representation of $\mathfrak{gl}_n$ in the usual sense can be deduced from its Gelfand-Tsetlin character by applying the algebra homomorphism

$$\mathbb{Z}[x_{i,a}^{\pm 1} \mid i = 1, \dots, n, a \in \mathbb{Z}] \to \mathbb{Z}[x_i^{\pm 1} \mid i = 1, \dots, n], \quad x_{i,a} \mapsto x_i. \tag{7.7}$$

Finally, if one can choose the scalar c in (7.4) to be 0, i.e. if the representation $V(A)$ is actually a *polynomial representation* of $\mathfrak{gl}_n$, then the formula (7.4) is especially simple since the leading monomial $y_{1,c}y_{2,c-1}\cdots y_{n,c-n+1}$ is equal to 1. So the Gelfand-Tsetlin character of any polynomial representation of $\mathfrak{gl}_n$ belongs to the subalgebra $\mathbb{Z}[x_{i,a} \mid i = 1, \dots, n, a \in \mathbb{Z}]$ of $\widehat{\mathbb{Z}}[\mathscr{P}_n]$.

7.5. **Grothendieck groups.** Let us at long last introduce some categories of $W(\pi)$-modules. First, let $\mathcal{M}(\pi)$ denote the category of all finitely generated, admissible $W(\pi)$-modules. Obviously $\mathcal{M}(\pi)$ is an abelian category closed under taking finite direct sums. Note that the duality τ from (5.2) defines a contravariant equivalence $\mathcal{M}(\pi) \to \mathcal{M}(\pi^t)$. Also, for any other pyramid $\dot{\pi}$ with the same row lengths as π, the isomorphism ι from (3.23) induces an isomorphism $\mathcal{M}(\pi) \to \mathcal{M}(\dot{\pi})$.

Lemma 7.15. *Every module in the category $\mathcal{M}(\pi)$ has a composition series.*

Proof. Copying the standard proof that modules in the usual category $\mathcal{O}$ have composition series, it suffices to prove the lemma for the Verma module $M(A)$, $A \in \mathrm{Row}(\pi)$. In that case it follows because all the weight spaces of $M(A)$ are finite dimensional, and moreover there are only finitely many irreducibles $L(B)$ with the same central character as $M(A)$ by Lemma 6.13. □

Hence, the Grothendieck group $[\mathcal{M}(\pi)]$ of the category $\mathcal{M}(\pi)$ is the free abelian group on basis $\{[L(A)] \mid A \in \mathrm{Row}(\pi)\}$. By Theorem 6.7, we have that $[M(A)] = [L(A)]$ plus an $\mathbb{N}$-linear combination of $[L(B)]$'s for $B < A$. It follows that the Verma modules

$\{[M(A)] \mid A \in \mathrm{Row}(\pi)\}$ also form a basis for $[\mathcal{M}(\pi)]$. By Theorem 5.9, the character map ch_n defines an injective map

$$\mathrm{ch}_n : [\mathcal{M}(\pi)] \hookrightarrow \widehat{\mathbb{Z}}[\mathscr{P}_n]. \tag{7.8}$$

Now suppose $\pi = \pi' \otimes \pi''$ for pyramids π' and π''. Then the tensor product $\boxtimes$ induces a multiplication

$$\mu : [\mathcal{M}(\pi')] \otimes [\mathcal{M}(\pi'')] \to [\mathcal{M}(\pi)]. \tag{7.9}$$

To see that this makes sense, the only difficulty is to see that the tensor product $M' \boxtimes M''$ of $M' \in \mathcal{M}(\pi')$ and $M'' \in \mathcal{M}(\pi'')$ is finitely generated. It suffices to check this for Verma modules. So take $A' \in \mathrm{Row}(\pi')$ and $A'' \in \mathrm{Row}(\pi'')$. Then, by Corollary 6.3, we have that $\mathrm{ch}_n(M(A') \boxtimes M(A'')) = \mathrm{ch}_n M(A)$ where $A \sim_{\mathrm{ro}} A' \otimes A''$. In view of Theorem 5.9 and Lemma 7.15, this shows that $M(A') \boxtimes M(A'')$ has a composition series, hence it is finitely generated as required. Moreover,

$$\mu([M(A')] \otimes [M(A'')]) = [M(A)]. \tag{7.10}$$

Recalling the decomposition (6.13), the category $\mathcal{M}(\pi)$ has the following *block decomposition*

$$\mathcal{M}(\pi) = \bigoplus_{\theta \in P} \mathcal{M}(\pi, \theta) \tag{7.11}$$

where $\mathcal{M}(\pi, \theta)$ is the full subcategory of $\mathcal{M}(\pi)$ consisting of objects all of whose composition factors are of central character θ; by convention, we set $\mathcal{M}(\pi, \theta) = 0$ if the coefficients of θ are not non-negative integers summing to N. Like in (4.28), we now restrict our attention just to modules with integral central characters: let

$$\mathcal{M}_0(\pi) := \bigoplus_{\theta \in P_\infty \subset P} \mathcal{M}(\pi, \theta). \tag{7.12}$$

The Grothendieck group $[\mathcal{M}_0(\pi)]$ has the two natural basis $\{[M(A)] \mid A \in \mathrm{Row}_0(\pi)\}$ and $\{[L(A)] \mid A \in \mathrm{Row}_0(\pi)\}$.

Next recall the definition of the $U_{\mathbb{Z}}$-module $S^\pi(V_{\mathbb{Z}})$ from (4.5). This is also a free abelian group, with two natural bases $\{M_A \mid A \in \mathrm{Row}_0(\pi)\}$ and $\{L_A \mid A \in \mathrm{Row}_0(\pi)\}$. Define an isomorphism of abelian groups

$$k : S^\pi(V_{\mathbb{Z}}) \to [\mathcal{M}_0(\pi)], \qquad M_A \mapsto [M(A)] \tag{7.13}$$

for each $A \in \mathrm{Row}_0(\pi)$. Under this isomorphism, the θ-weight space of $S^\pi(V_{\mathbb{Z}})$ corresponds to the block component $[\mathcal{M}(\pi, \theta)]$ of $[\mathcal{M}_0(\pi)]$, for each $\theta \in P_\infty$. Moreover, the isomorphism is compatible with the multiplications μ arising from (4.10) and (7.9) in the sense that for every decomposition $\pi = \pi' \otimes \pi''$ the following diagram commutes:

$$\begin{array}{ccc} S^{\pi'}(V_{\mathbb{Z}}) \otimes S^{\pi''}(V_{\mathbb{Z}}) & \xrightarrow{\mu} & S^\pi(V_{\mathbb{Z}}) \\ {\scriptstyle k\otimes k}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ [\mathcal{M}_0(\pi')] \otimes [\mathcal{M}_0(\pi'')] & \xrightarrow[\mu]{} & [\mathcal{M}_0(\pi)] \end{array} \tag{7.14}$$

Now we can formulate the following conjecture, which may be viewed as a more precise formulation in type A of [VD]. Note this conjecture is true if π consists of a single column; see Theorem 4.2.

Conjecture 7.16. *For each $A \in \mathrm{Row}_0(\pi)$, the isomorphism $k : S^\pi(V_\mathbb{Z}) \to [\mathcal{M}_0(\pi)]$ maps the dual canonical basis element L_A to the class $[L(A)]$ of the irreducible module $L(A)$. In other words, for every $A, B \in \mathrm{Row}_0(\pi)$, we have that*

$$[M(A) : L(B)] = P_{d(\rho(A))w_0, d(\rho(B))w_0}(1),$$

notation as in (4.8).

Let us turn our attention to finite dimensional $W(\pi)$-modules. Let $\mathcal{F}(\pi)$ denote the category of all finite dimensional $W(\pi)$-modules, a full subcategory of the category $\mathcal{M}(\pi)$. Let $\mathcal{F}_0(\pi) = \mathcal{F}(\pi) \cap \mathcal{M}_0(\pi)$. Like in (7.11)–(7.12), we have the block decompositions

$$\mathcal{F}(\pi) = \bigoplus_{\theta \in P} \mathcal{F}(\pi, \theta), \tag{7.15}$$

$$\mathcal{F}_0(\pi) = \bigoplus_{\theta \in P_\infty \subset P} \mathcal{F}(\pi, \theta). \tag{7.16}$$

By Theorem 7.9, the Grothendieck group $[\mathcal{F}(\pi)]$ has basis $\{[L(A)] \mid A \in \mathrm{Dom}(\pi)\}$ coming from the simple modules. Hence $[\mathcal{F}_0(\pi)]$ has basis $\{[L(A)] \mid A \in \mathrm{Dom}_0(\pi)\}$; we will refer to these $L(A) \in \mathcal{F}_0(\pi)$ as the *rational* irreducible representations of $W(\pi)$.

Recall the subspace $P^\pi(V_\mathbb{Z})$ of $S^\pi(V_\mathbb{Z})$ from §4.3. Comparing (4.12) and (7.1) and using (7.14), it follows that the map $k : S^\pi(V_\mathbb{Z}) \to [\mathcal{M}_0(\pi)]$ maps V_A to $[V(A)]$. Hence there is a well-defined map $j : P^\pi(V_\mathbb{Z}) \to [\mathcal{F}_0(\pi)]$ such that $V_A \mapsto [V(A)]$ for each $A \in \mathrm{Col}_0(\pi)$. Moreover, the following diagram commutes:

$$\begin{array}{ccc} P^\pi(V_\mathbb{Z}) & \longrightarrow & S^\pi(V_\mathbb{Z}) \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle k} \\ [\mathcal{F}_0(\pi)] & \longrightarrow & [\mathcal{M}_0(\pi)] \end{array} \tag{7.17}$$

where the horizontal maps are the natural inclusions.

Lemma 7.17. *The map $j : P^\pi(V_\mathbb{Z}) \to [\mathcal{F}_0(\pi)], V_A \mapsto [V(A)]$ is an isomorphism of abelian groups.*

Proof. Arguing with the isomorphism ι, it suffices to prove this in the special case that π is left-justified. In this case, recall from (4.3) that $R(A)$ denotes the row equivalence class of $A \in \mathrm{Std}_0(\pi)$. By Theorem 7.12, for each $A \in \mathrm{Std}_0(\pi)$ the standard module $V(A)$ is a quotient of $M(R(A))$, hence we have that $V(A) = L(R(A))$ plus an $\mathbb{N}$-linear combination of $L(B)$'s for $B < A$. It follows that $\{[V(A)] \mid A \in \mathrm{Std}_0(\pi)\}$ is a basis for $[\mathcal{F}_0(\pi)]$. Since the map $j : P^\pi(V_\mathbb{Z}) \to [\mathcal{F}_0(\pi)]$ maps the basis $\{V_A \mid A \in \mathrm{Std}_0(\pi)\}$ of $P^\pi(V_\mathbb{Z})$ onto this basis of $[\mathcal{F}_0(\pi)]$, it follows that j is indeed an isomorphism. □

This lemma implies that $\{[V(A)] \mid A \in \mathrm{Std}_0(\pi)\}$ gives a basis for the Grothendieck group $[\mathcal{F}_0(\pi)]$. In particular, this means that the Gelfand-Tsetlin character of any module in $\mathcal{F}_0(\pi)$ belongs to the subalgebra $\mathbb{Z}[x_{i,a}^{\pm 1} \mid i = 1, \dots, n, a \in \mathbb{Z}]$ of $\widehat{Z}[\mathscr{P}_n]$, since we know already that this is true for the standard modules. In the next lemma we extend this "standard basis" from $[\mathcal{F}_0(\pi)]$ to all of the Grothendieck group $[\mathcal{F}(\pi)]$.

Recall for the statement the definition of the set $\mathrm{Std}(\pi)$ from §4.1; its elements are $\|$-equivalence classes rather than elements of $\mathrm{Col}(\pi)$. For any $A \in \mathrm{Std}(\pi)$, we define $[V(A)] \in [\mathcal{F}(\pi)]$ to be the class of the standard module parametrized by any representative of A. This definition is independent of the choice of representative, since if $A, B \in \mathrm{Col}(\pi)$ satisfy $A \parallel B$ then we obviously have that $[V(A)] = [V(B)]$ in the Grothendieck group. (Actually we believe here that $A \parallel B$ implies $V(A) \cong V(B)$ but have been unable to prove this stronger statement except in special cases, e.g. it is true if π is left-justified by Theorem 7.13.)

Lemma 7.18. *The elements* $\{[V(A)] \mid A \in \mathrm{Std}(\pi)\}$ *form a basis for* $[\mathcal{F}(\pi)]$*. In particular, the elements* $\{[V(A)] \mid A \in \mathrm{Std}_0(\pi)\}$ *form a basis for* $[\mathcal{F}_0(\pi)]$*.*

Proof. We have already proved the second statement about $\mathcal{F}_0(\pi)$. The first statement follows from this and Theorem 7.13. □

8. Finite dimensional representations

Throughout the chapter, we fix a pyramid $\pi = (q_1, \dots, q_l)$ of height $\leq n$. Conjecture 7.16 immediately implies that the isomorphism $j : P^\pi(V_\mathbb{Z}) \to [\mathcal{F}_0(\pi)]$ from (7.17) maps the dual canonical basis of the polynomial representation $P^\pi(V_\mathbb{Z})$ to the basis of the Grothendieck group $[\mathcal{F}_0(\pi)]$ arising from irreducible modules. In this chapter, we will give an independent proof of this statement. Hence we can in principle compute the Gelfand-Tsetlin characters of all finite dimensional irreducible $W(\pi)$-modules.

8.1. **Skryabin's theorem.** We begin by recalling the relationship between the algebra $W(\pi)$ and the representation theory of $\mathfrak{g}$. Let Q_χ denote the *generalized Gelfand-Graev representation*

$$Q_\chi := U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{F}_\chi \tag{8.1}$$

induced from the one dimensional $\mathfrak{m}$-module $\mathbb{F}_\chi$ associated to χ. To avoid confusion, we will denote the element $1 \otimes 1 \in Q_\chi$ by 1_χ. Obviously the map $U(\mathfrak{p}) \to Q_\chi, u \mapsto u1_\chi$ is a vector space isomorphism. As explained in the introduction of [BK5], $W(\pi)$ is naturally identified with the endomorphism algebra $\mathrm{End}_{U(\mathfrak{g})}(Q_\chi)^{\mathrm{op}}$, so that $w \in W(\pi)$ corresponds to the endomorphism of Q_χ mapping $u1_\chi \mapsto uw1_\chi$ for any $u \in U(\mathfrak{g})$. So Q_χ is a $(U(\mathfrak{g}), W(\pi))$-bimodule.

Let $\mathcal{W}(\pi)$ denote the category of *generalized Whittaker modules* of type π, that is, the category of all $\mathfrak{g}$-modules on which $(x - \chi(x))$ acts locally nilpotently for all $x \in \mathfrak{m}$. If $V \in \mathcal{W}(\pi)$ then the subspace

$$V^\mathfrak{m} := \{v \in V \mid (x - \chi(x))v = 0 \text{ for all } x \in \mathfrak{m}\} \cong \mathrm{Hom}_{U(\mathfrak{g})}(Q_\chi, V) \tag{8.2}$$

is invariant under the action of $W(\pi)$, hence $?^\mathfrak{m}$ is a functor from $\mathcal{W}(\pi)$ to the category $W(\pi)$-mod of all left $W(\pi)$-modules. Conversely, $Q_\chi \otimes_{W(\pi)} ?$ is a functor from $W(\pi)$-mod to $\mathcal{W}(\pi)$. *Skryabin's theorem* [Sk] asserts that the functors $?^\mathfrak{m}$ and $Q_\chi \otimes_{W(\pi)} ?$ are quasi-inverse equivalences between the categories $\mathcal{W}(\pi)$ and $W(\pi)$-mod.

Skryabin also proved that Q_χ is a free right $W(\pi)$-module, and explained how to write down an explicit basis. Since we want to keep track of certain additional weight information, we must refine this basis slightly. Let $\mathfrak{c}$ denote the subalgebra of $\mathfrak{g} = \mathfrak{gl}_N$

consisting of all diagonal matrices that centralize the nilpotent matrix e from (3.6). So $\mathfrak{c}$ is spanned by the matrices

$$\sum_{\substack{1\leq j\leq N\\ \mathrm{row}(j)=i}} e_{j,j} \tag{8.3}$$

for each $i = 1,\dots,n$. Clearly, $\mathfrak{c} \subset W(\pi)$, so we can view any $W(\pi)$-module (for instance, the restriction of a $U(\mathfrak{p})$- or $U(\mathfrak{g})$-module) as a $\mathfrak{c}$-module too. Since

$$D_i^{(1)} = \sum_{\substack{1\leq j\leq N\\ \mathrm{row}(j)=i}} \tilde{e}_{j,j} = \sum_{\substack{1\leq j\leq N\\ \mathrm{row}(j)=i}} (e_{j,j} + (n - q_{\mathrm{col}(j)} - q_{\mathrm{col}(j)+1} - \cdots - q_l)), \tag{8.4}$$

the weight space decomposition of a $W(\pi)$-module with respect to $\mathfrak{c}$ coincides with its usual weight space decomposition with respect to the subalgebra $\mathfrak{d}_n$ of $W(\pi)$ spanned by $D_1^{(1)},\dots,D_n^{(1)}$. Now let $b_1,\dots,b_k$ be a homogeneous basis for $\mathfrak{m}$ such that b_i of degree $-d_i$ and of $\mathrm{ad}\,\mathfrak{c}$-weight $-\gamma_i \in \mathfrak{c}^*$. The elements $[b_1,e],\dots,[b_k,e]$ are again linearly independent, and $[b_i,e]$ is of degree $(1-d_i)$ and of $\mathrm{ad}\,\mathfrak{c}$-weight $-\gamma_i$. Hence there exist elements $a_1,\dots,a_k \in \mathfrak{p}$ such that a_i is of degree $(d_i - 1)$, of $\mathrm{ad}\,\mathfrak{c}$-weight γ_i, and

$$([a_i,b_j],e) = (a_i,[b_j,e]) = \delta_{i,j}. \tag{8.5}$$

We will refer to an ordered set of elements $a_1,\dots,a_k$ chosen in this way as a *Skryabin basis*. Given such a choice, it follows from [Sk] that Q_χ is a free right $W(\pi)$-module on basis $\big\{(a_1-\rho(a_1))^{i_1}\cdots(a_k-\rho(a_k))^{i_k}1_\chi \mid i_1,\dots,i_k \geq 0\big\}$, where ρ is the map from (3.11). Let

$$p : Q_\chi \twoheadrightarrow W(\pi) \tag{8.6}$$

denote the unique right $W(\pi)$-module homomorphism with

$$(a_1-\rho(a_1))^{i_1}\cdots(a_k-\rho(a_k))^{i_k}1_\chi \mapsto \begin{cases} 1 & \text{if } i_1=\cdots=i_k=0,\\ 0 & \text{otherwise.}\end{cases}$$

Applying the functor $? \otimes_{W(\pi)} M$ to p, we get induced a surjective linear map

$$p_M : Q_\chi \otimes_{W(\pi)} M \twoheadrightarrow M \tag{8.7}$$

for any $W(\pi)$-module M. Note that p, hence each p_M, is actually a $\mathfrak{c}$-module homomorphism. In the first lemma, for any vector space M, $\overline{\mathrm{Hom}}(U(\mathfrak{m}),M)$ denotes the space of all linear maps from $U(\mathfrak{m})$ to M which annihilate $(I_\chi)^r$ for $r \gg 0$, recalling that I_χ is the two-sided ideal of $U(\mathfrak{m})$ generated by the elements $\{x-\chi(x)\mid x\in\mathfrak{m}\}$. We view $\overline{\mathrm{Hom}}(U(\mathfrak{m}),M)$ as an $\mathfrak{m}$-module with action defined by $(xf)(u) = f(ux)$ for $x\in\mathfrak{m}$, $u \in U(\mathfrak{m})$.

Lemma 8.1. *For $M \in W(\pi)\text{-mod}$, there is a natural $\mathfrak{m}$-module isomorphism*

$$\varphi_M : Q_\chi \otimes_{W(\pi)} M \to \overline{\mathrm{Hom}}(U(\mathfrak{m}),M)$$

defined by $\varphi_M(x)(u) = p_M(ux)$ for $x \in Q_\chi\otimes_{W(\pi)} M$ and $u\in U(\mathfrak{m})$. Moreover φ_M is $\mathfrak{c}$-equivariant for the action of $\mathfrak{c}$ on $\overline{\mathrm{Hom}}(U(\mathfrak{m}),M)$ defined by $(hf)(u) = h(f(u)) - f([h,u])$ for $h\in\mathfrak{c}$, $u\in U(\mathfrak{m})$ and $f : U(\mathfrak{m})\to M$.

Proof. The fact that φ_M is an $\mathfrak{m}$-module isomorphism follows from Skryabin's proof [Sk]. Using the additional fact that p_M is $\mathfrak{c}$-equivariant, one checks easily that φ_M is $\mathfrak{c}$-equivariant too. □

8.2. **Tensor identity.** Let V be a finite dimensional $\mathfrak{g}$-module. Given any $M \in \mathcal{W}(\pi)$, it is clear that $M \otimes V$ (the usual tensor product of $\mathfrak{g}$-modules) also belongs to the category $\mathcal{W}(\pi)$. Thus $? \otimes V$ gives an exact functor from $\mathcal{W}(\pi)$ to $\mathcal{W}(\pi)$. Using Skryabin's equivalence of categories, we can transport this functor directly to the category $W(\pi)$-mod: for a $W(\pi)$-module M, let

$$M \circledast V := ((Q_\chi \otimes_{W(\pi)} M) \otimes V)^{\mathfrak{m}}. \tag{8.8}$$

This defines an exact functor $? \circledast V : W(\pi)\text{-mod} \to W(\pi)\text{-mod}$.

Writing V^* for the dual $\mathfrak{g}$-module, the functor $? \circledast V^*$ is right adjoint to $? \circledast V$. To write down a canonical adjunction, let $v_1, \dots, v_r$ be a basis for V and let $f_1, \dots, f_r$ be the dual basis for V^*. Then, the unit of the canonical adjunction is the map $\eta : \mathrm{Id} \to (? \circledast V) \circledast V^*$ defined on a $W(\pi)$-module M by $m \mapsto \sum_{i=1}^r (1_\chi \otimes m) \otimes v_i \otimes f_i$, and the counit $\varepsilon : (? \circledast V^*) \circledast V \to \mathrm{Id}$ is the restriction of the map $(u1_\chi \otimes m) \otimes f \otimes v \mapsto f(v)u1_\chi \otimes m$.

The following lemma is a more precise formulation of [Ly, Theorem 4.2].

Lemma 8.2. *Let V be a finite dimensional $U(\mathfrak{g})$-module on basis $v_1, \dots, v_r$. Define $c_{i,j} \in U(\mathfrak{g})^*$ from $uv_j = \sum_{i=1}^r c_{i,j}(u) v_i$ for $i, j = 1, \dots, r$. Also fix a choice of Skryabin basis, determining projections p and p_M as in as in (8.6)–(8.7). For any $W(\pi)$-module M, the restriction of the map $p_M \otimes \mathrm{id}_V : (Q_\chi \otimes_{W(\pi)} M) \otimes V \to M \otimes V$ defines a natural $\mathfrak{c}$-module isomorphism*

$$\chi_{M,V} : M \circledast V \xrightarrow{\sim} M \otimes V.$$

The inverse isomorphism maps $m \otimes v_j$ to $\sum_{i=1}^r (x_{i,j} 1_\chi \otimes m) \otimes v_i$, where $(x_{i,j})_{1 \le i,j \le r}$ is the invertible matrix with entries in $U(\mathfrak{p})$ determined uniquely by the properties

(i) $p(x_{i,j} 1_\chi) = \delta_{i,j} 1$;

(ii) $\mathrm{pr}_\chi([x, x_{i,j}]) + \sum_{k=1}^r c_{i,k}(x) x_{k,j} = 0$ *for all* $x \in \mathfrak{m}$.

Proof. Let $\overline{\mathrm{Hom}}_{\mathfrak{m}}(U(\mathfrak{m}), V)$ denote the space of all $\mathfrak{m}$-module homomorphisms from $U(\mathfrak{m})$ to V which annihilate some power of I_χ, where the left action of $\mathfrak{m}$ on $U(\mathfrak{m})$ here is by $(xf)(u) = f(u(\chi(x) - x))$ for $x \in \mathfrak{m}$, $u \in U(\mathfrak{m})$ and $f : U(\mathfrak{m}) \to V$. Evaluation at $1_{U(\mathfrak{m})}$ defines an isomorphism $\overline{\mathrm{Hom}}_{\mathfrak{m}}(U(\mathfrak{m}), V) \to V$. Combining this with Lemma 8.1, we get natural isomorphisms of $\mathfrak{c}$-modules

$$\begin{aligned} M \circledast V = ((Q_\chi \otimes_{W(\pi)} M) \otimes V)^{\mathfrak{m}} &\xrightarrow{\sim} (\overline{\mathrm{Hom}}(U(\mathfrak{m}), M) \otimes V)^{\mathfrak{m}} \\ &\xrightarrow{\sim} M \otimes (\overline{\mathrm{Hom}}(U(\mathfrak{m}), \mathbb{F}) \otimes V)^{\mathfrak{m}} \xrightarrow{\sim} M \otimes \overline{\mathrm{Hom}}_{\mathfrak{m}}(U(\mathfrak{m}), V) \xrightarrow{\sim} M \otimes V. \end{aligned}$$

Let $\chi_{M,V} : M \circledast V \to M \otimes V$ denote the composite isomorphism.

Now assume that $M = W(\pi)$, the regular $W(\pi)$-module. In this case, the inverse image of $1 \otimes v_j$ under the isomorphism $\chi_{M,V}$ must be of the form $\sum_{i=1}^r (x_{i,j} 1_\chi \otimes 1) \otimes v_i$

for unique elements $x_{i,j} \in U(\mathfrak{p})$. Moreover, computing explicitly from the definition of the map $\chi_{M,V}$, each $x_{i,j}$ is determined by the property that

$$p(ux_{i,j}1_\chi) = c_{i,j}(u^*)1 \tag{8.9}$$

for all $u \in U(\mathfrak{m})$, where $* : U(\mathfrak{m}) \to U(\mathfrak{m})$ here is the antiautomorphism with $x^* = \chi(x) - x$ for each $x \in \mathfrak{m}$. Taking $u = 1$ in (8.9), we see that $p(x_{i,j}1_\chi) = \delta_{i,j}1$, as in property (i). Moreover, $\sum_{i=1}^r (x_{i,j}1_\chi \otimes 1) \otimes v_i$ is $\mathfrak{m}$-invariant, which is equivalent to property (ii). Conversely, one checks that properties (i) and (ii) imply (8.9), hence they also determine the $x_{i,j}$'s uniquely. To see that the matrix $(x_{i,j})_{1\le i,j\le r}$ is invertible, we may assume without loss of generality that the basis $v_1, \dots, v_r$ has the property that $xv_i \in \mathbb{F}v_1 + \cdots + \mathbb{F}v_{i-1}$ for each $i = 1, \dots, r$ and $x \in \mathfrak{m}$, i.e. $c_{i,j}(x) = 0$ for $i \ge j$. But then, if one replaces $x_{i,j}$ by $\delta_{i,j}$ for all $i \ge j$, the new elements still satisfy (8.9). Hence by uniqueness we must already have that $x_{i,j} = \delta_{i,j}$ for $i \ge j$, i.e. the matrix $(x_{i,j})_{1\le i,j\le r}$ is unitriangular, so it is invertible.

Finally return to general M. Property (ii) implies that $\sum_{i=1}^r (x_{i,j}1_\chi \otimes m) \otimes v_i$ belongs to $M \circledast V$ for any $m \in M$. By functoriality, the image of this element under the isomorphism $\chi_{M,V}$ constructed in the first paragraph must equal $m \otimes v_j$. By property (i) this is also its image under the restriction of the map $p_M \otimes \mathrm{id}_V$. This shows that the isomorphism $\chi_{M,V}$ coincides with the restriction of $p_M \otimes \mathrm{id}_V$ in general, completing the proof. □

Corollary 8.3. *For any finite dimensional $\mathfrak{g}$-module V, the functor $? \circledast V$ maps admissible $W(\pi)$-modules to admissible $W(\pi)$-modules.*

Now we can prove the following important "tensor identity".

Theorem 8.4. *Let M be any $\mathfrak{p}$-module and V be a finite dimensional $\mathfrak{g}$-module on basis $v_1, \dots, v_r$. The restriction of the map $(Q_\chi \otimes_{W(\pi)} M) \otimes V \to M \otimes V$ sending $(u1_\chi \otimes m) \otimes v$ to $um \otimes v$ for each $u \in U(\mathfrak{p}), m \in M, v \in V$ defines a natural isomorphism*

$$\mu_{M,V} : M \circledast V \xrightarrow{\sim} M \otimes V$$

of $W(\pi)$-modules. The inverse map sends $m \otimes v_k$ to $\sum_{i,j=1}^r (x_{i,j}1_\chi \otimes y_{j,k}m) \otimes v_i$, where $(x_{i,j})_{1\le i,j\le r}$ is the matrix defined in Lemma 8.2 and $(y_{i,j})_{1\le i,j\le r}$ is the inverse matrix.

Proof. The map $(Q_\chi \otimes_{W(\pi)} M) \otimes V \to M \otimes V, (x1_\chi \otimes m) \otimes v \mapsto xm \otimes v$ is a $\mathfrak{p}$-module homomorphism, hence its restriction $\mu_{M,V}$ is a $W(\pi)$-module homomorphism. To prove that $\mu_{M,V}$ is an isomorphism, just note by Lemma 8.2 that there is a well-defined map $M \otimes V \to M \circledast V, m \otimes v_k \mapsto \sum_{i,j=1}^r (x_{i,j}1_\chi \otimes y_{j,k}m) \otimes v_i$. This is a two-sided inverse to $\chi_{M,V}$. □

We should also make some comments about associativity of $\circledast$. Suppose that we are given another finite dimensional $\mathfrak{g}$-module V'. For any $W(\pi)$-module M, the restriction of the linear map

$$(Q_\chi \otimes_{W(\pi)} ((Q_\chi \otimes_{W(\pi)} M) \otimes V)) \otimes V' \to (Q_\chi \otimes_{W(\pi)} M) \otimes V \otimes V',$$
$$(u'1_\chi \otimes ((u1_\chi \otimes m) \otimes v)) \otimes v' \mapsto (u'((u1_\chi \otimes m) \otimes v)) \otimes v'$$

defines a natural isomorphism

$$a_M : (M \circledast V) \circledast V' \to M \circledast (V \otimes V') \tag{8.10}$$

of $W(\pi)$-modules. If M is actually a $\mathfrak{p}$-module, it is straightforward to check that the following diagram commutes:

$$\begin{array}{ccc} (M \circledast V) \circledast V' & \xrightarrow{a_M} & M \circledast (V \otimes V') \\ \Big\downarrow{\scriptstyle \mathrm{id}_{Q_\chi} \otimes \mu_{M,V} \otimes \mathrm{id}_{V'}} & & \Big\downarrow{\scriptstyle \mu_{M,V\otimes V'}} \\ (M \otimes V) \circledast V' & \xrightarrow[\mu_{M\otimes V, V'}]{} & M \otimes V \otimes V'. \end{array} \tag{8.11}$$

Finally, given a third finite dimensional module V''', the following diagram commutes:

$$\begin{array}{ccc} (((M \circledast V) \circledast V') \circledast V'') & \xrightarrow{a_{M\circledast V}} & (M \circledast V) \circledast (V' \otimes V'') \\ \Big\downarrow{\scriptstyle \mathrm{id}_{Q_\chi} \otimes a_M \otimes \mathrm{id}_{V''}} & & \Big\downarrow{\scriptstyle a_M} \\ (M \circledast (V \otimes V')) \circledast V'' & \xrightarrow[a_M]{} & M \circledast (V \otimes V' \otimes V''). \end{array} \tag{8.12}$$

8.3. **Translation functors.** In this section we extend the definition of the translation functors e_i, f_i on the category $\mathcal{O}_0(\pi)$ from §4.4 to the category $\mathcal{M}_0(\pi)$ from §7.5. We begin by defining an endomorphism

$$x : ? \circledast V_N \to ? \circledast V_N \tag{8.13}$$

of the functor $? \circledast V_N : W(\pi)\text{-mod} \to W(\pi)\text{-mod}$. On a $W(\pi)$-module M, x_M is the endomorphism of $M \circledast V_N = ((Q_\chi \otimes_{W(\pi)} M) \otimes V_N)^{\mathfrak{m}}$ defined by left multiplication by $\Omega = \sum_{i,j=1}^N e_{i,j} \otimes e_{j,i} \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$. Here, we treat the $\mathfrak{g}$-module $Q_\chi \otimes_{W(\pi)} M$ as the first tensor position and V_N as the second, so $\Omega((u1_\chi \otimes m) \otimes v)$ means $\sum_{i,j=1}^N (e_{i,j} u1_\chi \otimes m) \otimes e_{j,i} v$. Next, we define an endomorphism

$$s : (? \circledast V_N) \circledast V_N \to (? \circledast V_N) \circledast V_N \tag{8.14}$$

of the functor $(? \circledast V_N) \circledast V_N : W(\pi)\text{-mod} \to W(\pi)\text{-mod}$. For the definition, we use the isomorphism $a : (? \circledast V_N) \circledast V_N \to ? \circledast (V_N \otimes V_N)$ from (8.10). Then, s is the composite $a^{-1} \circ \widehat{s} \circ a$, where $\widehat{s}$ is the endomorphism of the functor $? \circledast (V_N \otimes V_N)$ defined by letting $\widehat{s}_M$ be the endomorphism of $M \circledast (V_N \otimes V_N) = ((Q_\chi \otimes_{W(\pi)} M) \otimes V_N \otimes V_N)^{\mathfrak{m}}$ defined by left multiplication by $\Omega^{[2,3]}$, i.e. Ω acting on the second and third tensor positions so $\Omega^{[2,3]}((u1_\chi \otimes m) \otimes v \otimes v')$ means $\sum_{i,j=1}^N (u1_\chi \otimes m) \otimes e_{i,j} v \otimes e_{j,i} v' = (u1_\chi \otimes m) \otimes v' \otimes v$. Actually these definitions are just the natural translations through Skryabin's equivalence of categories of the endomorphisms x and s from §4.4 of the functors $? \otimes V_N$ and $? \otimes V_N \otimes V_N$ on the category $\mathcal{W}(\pi)$.

More generally, suppose that we are given $d \geq 1$, and introduce the following endomorphisms of the dth power $(? \circledast V_N)^d$: for $1 \leq i \leq d$ and $1 \leq j < d$, let

$$x_i := (1_{? \circledast V_N})^{d-i} x (1_{? \circledast V_N})^{i-1}, \qquad s_j := (1_{? \circledast V_N})^{d-j-1} s (1_{? \circledast V_N})^{j-1}. \tag{8.15}$$

There is an easier description of these endomorphisms. To formulate this, we exploit the natural isomorphism $a^{(d-1)}$ between the functor $(? \circledast V_N)^d$ and the functor $? \circledast V_N^{\otimes d}$

defined by iterating the map a from (8.10). To be precise, set $a^{(0)} := 1_{?\circledast V_N}$ and then inductively define $a^{(d-1)}$ for $d > 1$ by picking $1 \leq k \leq d-1$ and letting

$$a^{(d-1)} := a \circ (a^{(d-k-1)} a^{(k-1)}), \tag{8.16}$$

where a here is the isomorphism $a : (?\circledast V_N^{\otimes(d-k)}) \circ (?\circledast V_N^{\otimes k}) \to ?\circledast(V_N^{\otimes k} \otimes V_N^{\otimes(d-k)})$ from (8.10). By the commutativity of the diagram (8.12), this definition is independent of the particular choice of k. Now for $1 \leq i \leq d$ and $1 \leq j < d$, let $\widehat{x}_i$ and $\widehat{s}_j$ denote the endomorphisms of the functor $? \circledast V_N^{\otimes d}$ defined by left multiplication by the elements $\sum_{h=1}^{i} \Omega^{[h,i+1]}$ and $\Omega^{[j+1,j+2]}$, respectively, notation as above. Then, we have that

$$x_i = \left(a^{(d-1)}\right)^{-1} \circ \widehat{x}_i \circ a^{(d-1)}, \qquad s_j = \left(a^{(d-1)}\right)^{-1} \circ \widehat{s}_j \circ a^{(d-1)}. \tag{8.17}$$

Using this alternate description, the following identities are straightforward to check:

$$x_i x_j = x_j x_i, \tag{8.18}$$
$$s_i x_i = x_{i+1} s_i - 1, \tag{8.19}$$
$$s_i^2 = 1, \tag{8.20}$$
$$s_i s_j = s_j s_i \qquad \text{if } |i-j| > 1, \tag{8.21}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{8.22}$$

These commutation relations, which are the defining relations of the degenerate affine Hecke algebra H_d as in [CR], can also be deduced directly from (4.35)–(4.37) using Skryabin's equivalence of categories.

Let us next bring the adjoint functor $? \circledast V_N^*$ into the picture. Actually, we want at the same time to restrict our attention just to the subcategory $\mathcal{M}_0(\pi)$ of $W(\pi)$-mod, but we do not yet know that the functors $? \circledast V_N$ and $? \circledast V_N^*$ leave this subcategory invariant.

Lemma 8.5. *For $A \in \mathrm{Row}(\pi)$, we have that*

(i) $\mathrm{ch}_n(M(A) \circledast V_N) = \sum_B \mathrm{ch}_n M(B)$ *summing over all B obtained from A by adding 1 to one of its entries;*
(ii) $\mathrm{ch}_n(M(A) \circledast V_N^*) = \sum_B \mathrm{ch}_n M(B)$ *summing over all B obtained from A by subtracting 1 from one of its entries.*

Proof. (i) It follows by Corollary 8.3 that $M(A) \circledast V_N$ is admissible, hence it makes sense to consider its Gelfand-Tsetlin character. In view of Corollary 6.3, Theorem 5.9 and exactness of the functor $? \circledast V_N$, we can replace $M(A)$ by $M(A_1) \boxtimes \cdots \boxtimes M(A_l)$, where $A_1 \otimes \cdots \otimes A_l$ is some representative for A. But this is just the restriction from $U(\mathfrak{p})$ to $W(\pi)$ of the inflation, M say, of a Verma module over the Levi subalgebra $\mathfrak{h}$. By Theorem 8.4, we have that $M \circledast V_N \cong M \otimes V_N$ as $W(\pi)$-modules. Now observe that V_N has a filtration as a $\mathfrak{p}$-module with factors $V_{q_1}, \dots, V_{q_l}$ being the natural modules of the blocks $\mathfrak{gl}_{q_1}, \dots, \mathfrak{gl}_{q_l}$ of $\mathfrak{h}$. Hence $M \otimes V_N$ has a filtration with factors $M \otimes V_{q_i}$. Now apply Lemma 4.3 to each of these factors in turn, to deduce that the Gelfand-Tsetlin character of $M(A) \circledast V_N$ is equal to

$$\sum_{i=1}^{l} \sum_{B_i} \mathrm{ch}_n(M(A_1) \boxtimes \cdots \boxtimes M(A_{i-1}) \boxtimes M(B_i) \boxtimes M(A_{i+1}) \boxtimes \cdots \boxtimes M(A_l)),$$

where the second sum is over all B_i obtained from the column tableau A_i by adding 1 to one of its entries. Now we are done on applying Corollary 6.3 once more.

(ii) Similar. □

Corollary 8.6. *The functors* $? \circledast V_N$ *and* $? \circledast V_N^*$ *map objects in* $\mathcal{M}_0(\pi)$ *to objects in* $\mathcal{M}_0(\pi)$.

Proof. It suffices to check that $M(A) \circledast V_N$ and $M(A) \circledast V_N^*$ belong to $\mathcal{M}_0(\pi)$, for each $A \in \mathrm{Row}_0(\pi)$. We have already observed that $M(A) \circledast V_N$ is admissible. By Lemma 8.5(i) and Theorem 5.9, it has a composition series and all of its composition factors are of integral central character. Hence it belongs to $\mathcal{M}_0(\pi)$. Similarly so does $M(A) \circledast V_N^*$. □

For $\theta \in P_\infty$, let $\mathrm{pr}_\theta : \mathcal{M}_0(\pi) \to \mathcal{M}(\pi, \theta)$ be the projection functor along the decomposition (7.12). Explicitly, for a module $M \in \mathcal{M}_0(\pi)$, we have that $\mathrm{pr}_\theta(M)$ is the summand of M defined by (6.11), or $\mathrm{pr}_\theta(M) = 0$ if the coefficients of θ are not non-negative integers summing to N. In view of Corollary 8.6, it makes sense to define exact functors $e_i, f_i : \mathcal{M}_0(\pi) \to \mathcal{M}_0(\pi)$ by setting

$$e_i := \bigoplus_{\theta \in P_\infty} \mathrm{pr}_{\theta + (\varepsilon_i - \varepsilon_{i+1})} \circ (? \circledast V_N^*) \circ \mathrm{pr}_\theta, \tag{8.23}$$

$$f_i := \bigoplus_{\theta \in P_\infty} \mathrm{pr}_{\theta - (\varepsilon_i - \varepsilon_{i+1})} \circ (? \circledast V_N) \circ \mathrm{pr}_\theta. \tag{8.24}$$

Note e_i is right adjoint to f_i, indeed, the canonical adjunction fixed earlier between $? \circledast V_N$ and $? \circledast V_N^*$ induces a canonical adjunction between f_i and e_i. Similarly, f_i is also right adjoint to e_i. Moreover, applying Lemma 8.5 and taking blocks, we see for $A \in \mathrm{Row}_0(\pi)$ and $i \in \mathbb{Z}$ that

$$[e_i M(A)] = \sum_B [M(B)] \tag{8.25}$$

summing over all B obtained from A by replacing an entry equal to $(i+1)$ by an i, and

$$[f_i M(A)] = \sum_B [M(B)] \tag{8.26}$$

summing over all B obtained from A by replacing an entry equal to i by an $(i+1)$; cf. (4.32)–(4.33). Hence if we identify the Grothendieck group $[\mathcal{M}_0(\pi)]$ with the $U_{\mathbb{Z}}$-module $S^\pi(V_{\mathbb{Z}})$ via the isomorphism (7.13), the maps on the Grothendieck group induced by the exact functors e_i, f_i coincide with the action of $e_i, f_i \in U_{\mathbb{Z}}$.

Lemma 8.7. *For* $M \in \mathcal{M}_0(\pi)$, $f_i M$ *coincides with the generalized* i*-eigenspace of* $x_M \in \mathrm{End}_{W(\pi)}(M \circledast V_N)$.

Proof. It suffices to check this on a Verma module $M(A)$ for $A \in \mathrm{Row}_0(\pi)$. Say the entries of A in some order are $a_1, \dots, a_N$ and let B be obtained from A by replacing the entry a_t by a_t+1, for some $1 \leq t \leq N$. Recall the elements

$$Z_N^{(1)} = \sum_{i=1}^N (e_{i,i} - N + i),$$
$$Z_N^{(2)} = \sum_{i<j} ((e_{i,i} - N + i)(e_{j,j} - N + j) - e_{i,j}e_{j,i})$$

of $Z(U(\mathfrak{g}))$ from (3.41). For any $\mathfrak{g}$-module M, the operator Ω acts on $M \otimes V_N$ in the same way as $\Delta(Z_N^{(2)}) - Z_N^{(2)} \otimes 1 - Z_N^{(1)} \otimes 1$. Also by Lemma 6.13, $\psi(Z_N^{(1)})$ acts on $M(A)$ as $\sum_{r=1}^N a_r$ and $\psi(Z_N^{(2)})$ acts as $\sum_{r<s} a_r a_s$. It follows that $x_{M(A)}$ stabilizes any $W(\pi)$-submodule of $M(A) \circledast V_N$, and it acts on any irreducible subquotient having the same central character as $L(B)$ by scalar multiplication by

$$a_t = \sum_{r<s} (a_r + \delta_{r,t})(a_s + \delta_{s,t}) - \sum_{r<s} a_r a_s - \sum_{r=1}^N a_r.$$

Since $M(A) \circledast V_N = \bigoplus_{i \in \mathbb{Z}} f_i M(A)$ and all irreducible subquotients of $f_i M(A)$ have the same central character as $L(B)$ for some B obtained from A by replacing an entry i by an $(i+1)$, this identifies $f_i M(A)$ as the generalized i-eigenspace of $x_{M(A)}$. □

As in [CR, §7.4], this lemma together with the relations (8.18)–(8.22) imply that the endomorphisms x and s restrict to well-defined endomorphisms also denoted x and s of the functors f_i and f_i^2, respectively. Moreover, the identities (4.35)–(4.37) also hold in this setting. This means that the category $\mathcal{M}_0(\pi)$ equipped with the adjoint pair of functors (f_i, e_i) and the endomorphisms $x \in \mathrm{End}(f_i)$ and $s \in \mathrm{End}(f_i^2)$ is an $\mathfrak{sl}_2$-categorification in the sense of [CR], for all $i \in \mathbb{Z}$. So we can appeal to all the general results developed in [CR] in our study of the category $\mathcal{M}_0(\pi)$.

Theorem 8.8. *Let $A \in \mathrm{Row}_0(\pi)$ and $i \in \mathbb{Z}$.*

(i) *Define $\varepsilon_i'(A)$ to be the maximal integer $k \geq 0$ such that $(e_i)^k L(A) \neq 0$. Assuming $\varepsilon_i'(A) > 0$, $e_i L(A)$ has irreducible socle and cosocle isomorphic to $L(\tilde{e}_i'(A))$ for some $\tilde{e}_i'(A) \in \mathrm{Row}_0(\pi)$ with $\varepsilon_i'(\tilde{e}_i'(A)) = \varepsilon_i'(A) - 1$. The multiplicity of $L(\tilde{e}_i'(A))$ as a composition factor of $e_i L(A)$ is equal to $\varepsilon_i'(A)$, and all other composition factors are of the form $L(B)$ for $B \in \mathrm{Row}_0(\pi)$ with $\varepsilon_i'(B) < \varepsilon_i'(A) - 1$.*

(ii) *Define $\varphi_i'(A)$ to be the maximal integer $k \geq 0$ such that $(f_i)^k L(A) \neq 0$. Assuming $\varphi_i'(A) > 0$, $f_i L(A)$ has irreducible socle and cosocle isomorphic to $L(\tilde{f}_i'(A))$ for some $\tilde{f}_i'(A) \in \mathrm{Row}_0(\pi)$ with $\varphi_i'(\tilde{f}_i'(A)) = \varphi_i'(A) - 1$. The multiplicity of $L(\tilde{f}_i'(A))$ as a composition factor of $f_i L(A)$ is equal to $\varphi_i'(A)$, and all other composition factors are of the form $L(B)$ for $B \in \mathrm{Row}_0(\pi)$ with $\varphi_i'(B) < \varphi_i'(A) - 1$.*

Proof. This follows from [CR, Lemma 4.3] and [CR, Proposition 5.23], as in the first paragraph of the proof of Theorem 4.4. □

Remark 8.9. This theorem gives a representation theoretic definition of a crystal structure $(\mathrm{Row}_0(\pi), \tilde{e}'_i, \tilde{f}'_i, \varepsilon'_i, \varphi'_i, \theta)$ on the set $\mathrm{Row}_0(\pi)$. In §4.3, we gave a combinatorial definition of another crystal structure $(\mathrm{Row}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ on the same underlying set. If Conjecture 7.16 is true, then it follows by (4.22)–(4.23) (as in the proof of Theorem 4.4) that these two crystal structures are in fact *equal*, that is, $\varepsilon'_i(A) = \varepsilon_i(A), \varphi'_i(A) = \varphi_i(A), \tilde{e}'_i(A) = \tilde{e}_i(A)$ and $\tilde{f}'_i(A) = \tilde{f}_i(A)$ for all $A \in \mathrm{Row}_0(\pi)$. Even without Conjecture 7.16, one can show using [CR, Lemma 4.3] and [BK, §4] that the two crystals $(\mathrm{Row}_0(\pi), \tilde{e}'_i, \tilde{f}'_i, \varepsilon'_i, \varphi'_i, \theta)$ and $(\mathrm{Row}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ are at least *isomorphic*. However, there is a labelling problem here: without invoking Conjecture 7.16 we do not know how to prove that the identity map on the underlying set $\mathrm{Row}_0(\pi)$ is an isomorphism between the two crystals. An analogous labelling problem arises in a number of other situations; compare for example [BK2] and [BK3].

8.4. **Characters of finite dimensional irreducible representations.** Following Kostant [Ko2] and Lynch [Ly], we now consider the following important functor from $\mathfrak{g}$-modules to $W(\pi)$-modules: for a $\mathfrak{g}$-module M, let

$$D(M) := (M^*)^{\mathfrak{m}}, \tag{8.27}$$

the space of all generalized Whittaker vectors in the full dual space M^* (viewed as a $\mathfrak{g}$-module in the standard way). Making the obvious definition on morphisms, this defines a contravariant functor $D : \mathfrak{g}\text{-mod} \to W(\pi)\text{-mod}$. The first lemma can be found already in Lynch's thesis [Ly, Lemma 4.6]; Lynch attributes it to N. Wallach.

Lemma 8.10. *The functor D sends short exact sequences of $\mathfrak{g}$-modules that are finitely generated over $\mathfrak{m}$ to short exact sequences of finite dimensional $W(\pi)$-modules.*

Proof. Note first for any $\mathfrak{g}$-module M that is finitely generated over $\mathfrak{m}$ that

$$D(M) \cong \mathrm{Hom}_{\mathfrak{m}}(\mathbb{F}_\chi, M^*) \cong \mathrm{Hom}_{\mathfrak{m}}(M, \mathbb{F}_{-\chi}),$$

which is finite dimensional. Now consider instead the functor $\widehat{D} : \mathfrak{g}\text{-mod} \to \mathcal{W}(\pi)$ mapping an object M to the largest submodule of M^* that lies in the category $\mathcal{W}(\pi)$. The functor D is the composite of $\widehat{D}$ and Skryabin's equivalence of categories $?^{\mathfrak{m}}$. So to complete the proof of the lemma, it is enough to show that the functor $\widehat{D}$ is exact on short exact sequences of $\mathfrak{g}$-modules that are finitely generated over $\mathfrak{m}$. Well, for a $\mathfrak{g}$-module M that is finitely generated over $\mathfrak{m}$, there are natural isomorphisms of vector spaces

$$\begin{aligned}\widehat{D}(M) &\cong \varinjlim \mathrm{Hom}_{\mathfrak{m}}(U(\mathfrak{m})/I_\chi^r, M^*)\\ &\cong \varinjlim \mathrm{Hom}_{\mathfrak{m}}(M, (U(\mathfrak{m})/I_\chi^r)^*) \cong \mathrm{Hom}_{\mathfrak{m}}(M, \varinjlim((U(\mathfrak{m})/I_\chi^r)^*)),\end{aligned}$$

where the last isomorphism here depends on the finite generation of M to see that the direct limit stabilizes after finitely many terms. Now we observe that $\varinjlim((U(\mathfrak{m})/I_\chi^r)^*)$ is the set of all $f \in U(\mathfrak{m})^*$ such that $f(I_\chi^r) = 0$ for $r \gg 0$, with $\mathfrak{m}$-action given by $(xf)(u) = -f(xu)$ for $x \in \mathfrak{m}$, $u \in U(\mathfrak{m})$ and $f : U(\mathfrak{m}) \to \mathbb{F}$. This is isomorphic to the set of all $f \in U(\mathfrak{m})^*$ such that $f(I_{-\chi}^r) = 0$ for $r \gg 0$, with $\mathfrak{m}$-action given instead by $(xf)(u) = f(ux)$. Hence by [Sk, Assertion 2], we get that $\varinjlim((U(\mathfrak{m})/I_\chi^r)^*)$ is an injective $\mathfrak{m}$-module. The desired exactness follows. □

Recall the definition of the parabolic category $\mathcal{O}(\pi)$ from the end of §4.4. By the PBW theorem, the parabolic Verma modules in $\mathcal{O}(\pi)$ are finitely generated over $\mathfrak{m}$, hence so are the simple modules. Since every object in $\mathcal{O}(\pi)$ is of finite length, it follows that all $M \in \mathcal{O}(\pi)$ are finitely generated over $\mathfrak{m}$. Hence the functor D restricts to a well-defined exact contravariant functor

$$D : \mathcal{O}(\pi) \to \mathcal{F}(\pi), \tag{8.28}$$

where $\mathcal{F}(\pi)$ is the category of all finite dimensional $W(\pi)$-modules as in §7.5. The next two lemmas give some further properties of this functor D.

Lemma 8.11. *For any $\mathfrak{p}$-module M, there is a natural isomorphism of $W(\pi)$-modules between $D(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M)$ and the restriction of the dual $\mathfrak{p}$-module M^* from $U(\mathfrak{p})$ to $W(\pi)$.*

Proof. We have that

$$\begin{aligned} D(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M) &\cong \operatorname{Hom}_{\mathfrak{m}}(\mathbb{F}_\chi, (U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M)^*) \cong \operatorname{Hom}_{\mathfrak{m}}(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M, \mathbb{F}_{-\chi}) \\ &= \operatorname{Hom}_{\mathfrak{m}}(U(\mathfrak{m}) \otimes M, \mathbb{F}_{-\chi}) \cong \operatorname{Hom}(M, \mathbb{F}) = M^*. \end{aligned}$$

It just remains to check that the action of $W(\pi)$ corresponds under these isomorphisms to the restriction of the usual $U(\mathfrak{p})$-action on the dual space M^*. □

Lemma 8.12. *For a $\mathfrak{g}$-module M and a finite dimensional $\mathfrak{g}$-module V, there is a natural isomorphism of $W(\pi)$-modules between $D(M \otimes V^*)$ and $D(M) \circledast V$*

Proof. We obviously have that $(M \otimes V^*)^* \cong M^* \otimes V$. Hence, $D(M \otimes V^*) \cong (M^* \otimes V)^{\mathfrak{m}}$. We can replace M^* here by its largest submodule belonging to the category $\mathcal{W}(\pi)$, which by Skryabin's equivalence of categories is isomorphic to $Q_\chi \otimes_{W(\pi)} ((M^*)^{\mathfrak{m}})$. Hence, $D(M \otimes V^*) \cong ((Q_\chi \otimes_{W(\pi)} D(M)) \otimes V)^{\mathfrak{m}} = D(M) \circledast V$. □

We are ready to categorify the linear map $F : \bigwedge^\pi(V_\mathbb{Z}) \to S^\pi(V_\mathbb{Z})$ from (4.11). Letting π^t denote the transpose pyramid $(q_l, \dots, q_1)$, define an exact functor

$$\gamma : \mathcal{O}(\pi) \to \mathcal{O}(\pi^t) \tag{8.29}$$

mapping a module M to the same vector space but with new action defined by the twist $x \cdot m := -(w_\pi^{-1} x w_\pi)^t m$, for $x \in \mathfrak{g}$ and $m \in M$. Here, w_π is the permutation matrix defined in §3.5. Also recall the duality τ from (5.2), which gives us an exact contravariant functor $\tau : \mathcal{F}(\pi^t) \to \mathcal{F}(\pi)$. Let $D : \mathcal{O}(\pi^t) \to \mathcal{F}(\pi^t)$ be the exact contravariant functor from (8.28) with π replaced by π^t, and consider the composite functor

$$F := \tau \circ D \circ \gamma : \mathcal{O}(\pi) \to \mathcal{F}(\pi). \tag{8.30}$$

This is an exact covariant functor. In the next lemma, we adapt Lemma 8.11 to this functor F. For the statement, recall the definitions of the 1-dimensional $\mathfrak{p}$-modules $\mathbb{F}_\rho$ from (3.11) and $\mathbb{F}_{\bar\rho}$ from (3.25).

Lemma 8.13. *For any finite dimensional $\mathfrak{p}$-module M, there is a natural isomorphism of $W(\pi)$-modules between $F(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\mathbb{F}_{\bar\rho} \otimes M))$ and the restriction of the $\mathfrak{p}$-module $\mathbb{F}_\rho \otimes M$ from $U(\mathfrak{p})$ to $W(\pi)$.*

Proof. Let $\gamma : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the automorphism sending $x \mapsto -(w_\pi^{-1} x w_\pi)^t$ and let $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ be the antiautomorphism sending $x \mapsto -x$, for each $x \in \mathfrak{g}$. Recall that $\tau : W(\pi) \to W(\pi^t)$ is the restriction of the map $\tau : U(\mathfrak{p}) \to U(\mathfrak{p}^\tau)$ from (3.26). The composite $S \circ \gamma \circ \tau : U(\mathfrak{p}) \to U(\mathfrak{p})$ maps $x \in \mathfrak{p}$ to $x + (\rho - \bar{\rho})(x)$. By Lemma 8.11 and the definition of F, $F(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} (\mathbb{F}_{\bar{\rho}} \otimes M))$ is naturally isomorphic to $\mathbb{F}_{\bar{\rho}} \otimes M$ viewed as a $W(\pi)$-module with action $x \cdot m = (S \circ \gamma \circ \tau)(x) m$ for each $x \in W(\pi), m \in \mathbb{F}_{\bar{\rho}} \otimes M$. This is the restriction of the $U(\mathfrak{p})$-module $\mathbb{F}_\rho \otimes M$. □

Corollary 8.14. *For $A \in \mathrm{Col}(\pi)$, we have that $F(N(A)) \cong V(A)$.*

Proof. This is immediate from Lemma 8.13 and the definitions of $N(A)$ and $V(A)$ from (4.38) and (7.1). □

For a while now, we will restrict our attention to integral central characters. Note Corollary 8.14 implies that the functor F restricts to a well-defined exact functor

$$F : \mathcal{O}_0(\pi) \to \mathcal{F}_0(\pi). \tag{8.31}$$

We also write $F : [\mathcal{O}_0(\pi)] \to [\mathcal{F}_0(\pi)]$ for the induced map at the level of Grothendieck groups. Recall also the isomorphism $i : \bigwedge^\pi(V_\mathbb{Z}) \to [\mathcal{O}_0(\pi)], N_A \mapsto [N(A)]$ from the proof of Theorem 4.5, and the isomorphisms $j : P^\pi(V_\mathbb{Z}) \to [\mathcal{F}_0(\pi)], V_A \mapsto [V(A)]$ and $k : S^\pi(V_\mathbb{Z}) \to [\mathcal{M}_0(\pi)], M_A \mapsto [M(A)]$ from (7.17). We observe that the following diagram commutes:

$$\begin{array}{ccccc} \bigwedge^\pi(V_\mathbb{Z}) & \xrightarrow{F} & P^\pi(V_\mathbb{Z}) & \longrightarrow & S^\pi(V_\mathbb{Z}) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle k} \\ [\mathcal{O}_0(\pi)] & \xrightarrow[F]{} & [\mathcal{F}_0(\pi)] & \longrightarrow & [\mathcal{M}_0(\pi)], \end{array} \tag{8.32}$$

where the top F is the map from (4.11) and the two unnamed maps are the natural inclusions. To see this, we already checked in (7.17) that the right hand square commutes, and Corollary 8.14 is exactly what is needed to check that the left hand square does too.

Lemma 8.15. *There are isomorphisms of functors $F \circ e_i \cong e_i^\tau \circ F$ and $F \circ f_i \cong f_i^\tau \circ F$, where $e_i^\tau := \tau \circ e_i \circ \tau$ and $f_i^\tau := \tau \circ f_i \circ \tau$.*

Proof. For $M \in \mathcal{O}_0(\pi)$, there are natural isomorphisms $\gamma(M \otimes V_N) \cong \gamma(M) \otimes \gamma(V_N) \cong \gamma(M) \otimes V_N^*$. Hence applying Lemma 8.12, there is an isomorphism of functors $D \circ \gamma \circ (? \otimes V_N) \cong (? \circledast V_N) \circ D \circ \gamma$. Equivalently, $F \circ (? \otimes V_N) \cong (? \circledast V_N)^\tau \circ F$. The lemma for f_i follows because in view of Corollary 8.14 we already know that both functors map $\mathcal{O}(\pi, \theta)$ to $\mathcal{F}(\pi, \theta)$ for each $\theta \in P_\infty$. The proof for e_i is similar. □

Remark 8.16. With substantially more work, one can show that the functors e_i and f_i commute with τ, i.e. $e_i^\tau \cong e_i$ and $f_i^\tau \cong f_i$ in the notation of this lemma. In fact, the functor F is a morphism of $\mathfrak{sl}_2$-categorifications in the sense of [CR]. We will not need these stronger statements here.

Now we are ready to invoke Theorem 4.5, or rather, to invoke the Kazhdan-Lusztig conjecture, since Theorem 4.5 was a direct consequence of it. For the statement of the following theorem, recall the definition of the bijection $R : \mathrm{Std}_0(\pi) \to \mathrm{Dom}_0(\pi)$ from §4.3; in the case that π is left-justified the rectification $R(A)$ of a standard π-tableau A simply means its row equivalence class.

Theorem 8.17. *For $A \in \mathrm{Col}_0(\pi)$, we have that*

$$F(K(A)) \cong \begin{cases} L(R(A)) & \text{if } A \text{ is standard,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Note that it suffices to prove the theorem in the special case that π is left-justified. Indeed, in view of Theorem 4.5, the properties of the map $F : \bigwedge^\pi(V_\mathbb{Z}) \to P^\pi(V_\mathbb{Z})$ and the commutativity of the left hand square in (8.32), the theorem follows if we can show that $j(L_A) = [L(A)]$ for every $A \in \mathrm{Dom}_0(\pi)$. This last statement is independent of the particular choice of π, thanks to the existence of the isomorphism ι. So assume from now on that π is left-justified.

Using Theorem 4.5 and the commutativity of the left hand square in (8.32) again, we know already for $A \in \mathrm{Col}_0(\pi)$ that $F(K(A)) \neq 0$ if and only if $A \in \mathrm{Std}_0(\pi)$. Let $A_\pi \in \mathrm{Col}_0(\pi)$ be the ground state, with all entries on row i equal to $(1-i)$. Since the crystal $(\mathrm{Std}_0(\pi), \tilde{e}_i, \tilde{f}_i, \varepsilon_i, \varphi_i, \theta)$ is connected, it makes sense to define the *height* of $A \in \mathrm{Std}_0(\pi)$ to be the minimal number of applications of the operators $\tilde{e}_i$, $\tilde{f}_i$ $(i \in \mathbb{Z})$ needed to map A to A_π. We proceed to prove that $F(K(A)) \cong L(R(A))$ for $A \in \mathrm{Std}_0(\pi)$ by induction on height. For the base case, observe that no other elements of $\mathrm{Col}_0(\pi)$ have the same content as A_π, hence $N(A_\pi) = K(A_\pi)$. Similarly, $V(R(A_\pi)) = L(R(A_\pi))$. Hence by Corollary 8.14, we have that $F(K(A_\pi)) \cong L(R(A_\pi))$.

Now for the induction step, take $B \in \mathrm{Std}_0(\pi)$ of height > 0. We can write B as either $\tilde{e}_i(A)$ or as $\tilde{f}_i(A)$, where $A \in \mathrm{Std}_0(\pi)$ is of strictly smaller height. We will assume that the first case holds, i.e. that $B = \tilde{e}_i(A)$, since the argument in the second case is entirely similar. By the induction hypothesis, we know already that $F(K(A)) \cong L(R(A))$. We need to show that $F(K(B)) \cong L(R(B))$.

Note by Corollary 8.14 that $F(N(B)) \cong V(B)$, and by exactness of the functor F, we know that $F(K(B))$ is a non-zero quotient of $V(B)$. Since $B \in \mathrm{Std}_0(\pi)$, Theorem 7.12 shows that $V(B)$ is a highest weight module of type $R(B)$. Hence $F(K(B))$ is a highest weight module of type $R(B)$ too. Also $K(B)$ is both a quotient and a submodule of $e_i K(A)$ by Theorem 4.5. Hence by Lemma 8.15, $F(K(B))$ is both a quotient and a submodule of $F(e_i K(A)) \cong e_i^\tau L(R(A))$. In particular, $L(R(B))$ is a quotient of $e_i^\tau L(R(A))$ and $F(K(B))$ is a non-zero submodule of it.

Finally, we know by Theorem 8.8 that the socle and cosocle of $e_i^\tau L(R(A))$ are irreducible and isomorphic to each other. Since we know already that $L(R(B))$ is a quotient of $e_i^\tau L(R(A))$, it follows that the socle of $e_i^\tau L(R(A))$ is isomorphic to $L(R(B))$. Since $F(K(B))$ embeds into $e_i^\tau L(R(A))$, this means that $F(K(B))$ has irreducible socle isomorphic to $L(R(B))$ too. But $F(K(B))$ is a highest weight module of type $R(B)$. These too statements together imply that $F(K(B))$ is indeed irreducible. □

Corollary 8.18. *The isomorphism $j : P^\pi(V_\mathbb{Z}) \to [\mathcal{F}_0(\pi)]$ maps L_A to $[L(A)]$ for each $A \in \mathrm{Dom}_0(\pi)$. Hence, for $A \in \mathrm{Col}_0(\pi)$ and $B \in \mathrm{Std}_0(\pi)$*

$$[V(A) : L(R(B))] = \sum_{C \sim_{\mathrm{co}} A} (-1)^{\ell(A,C)} P_{d(\gamma(C))w_0, d(\gamma(B))w_0}(1),$$

notation as in (4.14).

Proof. The first statement follows from the theorem and the commutativity of the diagram (8.32). The second two statement then follows by (4.14). □

Corollary 8.19. *For $A \in \mathrm{Dom}_0(\pi)$ and $i \in \mathbb{Z}$, the following properties hold:*

(i) *If $\varepsilon_i(A) = 0$ then $e_i L(A) = 0$. Otherwise, $e_i L(A)$ is an indecomposable module with irreducible socle and cosocle isomorphic to $L(\tilde{e}_i(A))$.*

(ii) *If $\varphi_i(A) = 0$ then $f_i L(A) = 0$. Otherwise, $f_i L(A)$ is an indecomposable module with irreducible socle and cosocle isomorphic to $L(\tilde{f}_i(A))$.*

Proof. Argue using (4.22)–(4.23), Theorem 8.8 and Corollary 8.18, like in the proof of Theorem 4.4. □

Since the Gelfand-Tsetlin characters of standard modules are known, one can now in principle compute the characters of the finite dimensional irreducible $W(\pi)$-modules with integral central character, by inverting the unitriangular square submatrix $([V(A) : L(R(B))])_{A,B\in \mathrm{Std}_0(\pi)}$ of the decomposition matrix from Corollary 8.18. Using Theorem 7.13 too, one can deduce from this the characters of arbitrary finite dimensional irreducible $W(\pi)$-modules. All the other combinatorial results just formulated can also be extended to arbitrary central characters in similar fashion. We just record here the extension of Theorem 8.17 itself to arbitrary central characters.

Corollary 8.20. *For $A \in \mathrm{Col}(\pi)$, we have that*

$$F(K(A)) \cong \begin{cases} L(R(A)) & \textit{if } A \textit{ is standard,} \\ 0 & \textit{otherwise.} \end{cases}$$

Thus, the functor $F : \mathcal{O}(\pi) \to \mathcal{F}(\pi)$ sends irreducible modules to irreducible modules or to zero.

Proof. This is a consequence of Corollary 8.18, Corollary 8.14, Theorem 7.13 and [BG, Proposition 5.12]. □

The final result gives a criterion for the irreducibility of the standard module $V(A)$, in the spirit of [LNT]. Note as a special case of this corollary, we recover the main result of [M4] concerning Yangians. Following [LZ, Lemma 3.8], we say that two sets $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_s\}$ of numbers from $\mathbb{F}$ are *separated* if

(a) $r < s$ and there do not exist $a, c \in A - B$ and $b \in B - A$ such that $a < b < c$;

(b) $r = s$ and there do not exist $a, c \in A - B$ and $b, d \in B - A$ such that $a < b < c < d$ or $a > b > c > d$;

(c) $r > s$ and there do not exist $c \in A - B$ and $b, d \in B - A$ such that $b > c > d$.

Say that a π-tableau $A \in \mathrm{Col}(\pi)$ is *separated* if the sets A_i and A_j of entries in the ith and jth columns of A, respectively, are separated for each $1 \leq i < j \leq l$.

Theorem 8.21. *For $A \in \mathrm{Col}(\pi)$, the standard module $V(A)$ is irreducible if and only if A is separated, in which case it is isomorphic to $L(B)$ where $B \in \mathrm{Dom}(\pi)$ is the row equivalence class of A.*

Proof. Using Theorem 7.13, the proof reduces to the special case that $A \in \mathrm{Col}_0(\pi)$. In that case, we apply [LZ, Theorem 1.1] and the main result of Leclerc, Nazarov and

Thibon [LNT, Theorem 31]; see also [Ca]. These references imply that V_A is equal to L_B for some $B \in \mathrm{Dom}_0(\pi)$ if and only if A is separated. Actually, the references cited only prove the q-analog of this statement, but it follows at $q = 1$ too by the positivity of the structure constants from [B, Remark 24]; see the argument from the proof of [LNT, Proposition 15]. By Theorem 8.17, this shows that $V(A)$ is irreducible if and only if A is separated. Finally, when this happens, we must have that $V(A) \cong L(B)$ where B is the row equivalence class of A, since $V(A)$ always contains a highest weight vector of that type. □

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