JAMES' REGULARIZATION THEOREM FOR DOUBLE COVERS OF SYMMETRIC GROUPS

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Dedicated to Gordon James

1. Introduction

In [Ja$_1$], Gordon James described leading terms in decomposition matrices and branching rules for representations of symmetric groups. To be more precise, let F be a field of characteristic p and S_n be the symmetric group. As in [Ja$_2$], $\leq$ denotes the dominance order on partitions, S^λ is the Specht module corresponding to a partition $\lambda \vdash n$, and D^μ is the irreducible FS_n-module corresponding to a p-regular partition $\mu \vdash n$. For every partition $\lambda \vdash n$, James defines its regularization λ^R, which is a p-regular partition of n. Also, the *shadow* $\mathrm{sh}(\mu) \vdash (n-1)$ of a p-regular partition $\mu \vdash n$ is the p-regular partition obtained by removing the *shadow node* from the Young diagram of μ, that is, the leftmost node of the *outer ladder* of μ. The main result of [Ja$_1$] is now as follows.

Theorem 1.1. (James)

(i) *For $\lambda \vdash n$, D^{λ^R} appears as a composition factor of S^λ with multiplicity* 1*, and for any other composition factor D^μ of S^λ we have that $\mu > \lambda^R$.*

(ii) *For a p-regular $\mu \vdash n$, let the outer ladder of μ be of size m. Then $D^{\mathrm{sh}(\mu)}$ appears as a composition factor of the restriction $D^\mu{\downarrow}_{S_{n-1}}$ with multiplicity m, and for any other composition factor D^ν of $D^\mu{\downarrow}_{S_{n-1}}$ we have that $\nu > \mathrm{sh}(\mu)$.*

The goal of this paper is to obtain a similar result for *projective* or *spin* representations of symmetric groups. Let $\mathcal{T}_n$ be the (non-trivially) twisted group algebra of S_n. Then spin representations of S_n are the same as representations of $\mathcal{T}_n$. In fact it is more convenient to work with $\mathcal{T}_n$ as a superalgebra and consider its supermodules instead of modules. For this reason in the remainder of the article all modules will in fact be supermodules without further comment. We refer the reader to [BK$_2$], [K, Part II] for explanation of these basic ideas.

A partition $\lambda \vdash n$ is called *p-strict* if all its repeated parts are divisible by p. For each p-strict partition $\lambda \vdash n$, we introduce in section 5 a notion of Specht 'module' $S(\lambda)$ for $\mathcal{T}_n$; it is actually a virtual module. On the other hand, the irreducible modules $D(\mu)$ for $\mathcal{T}_n$ are labelled by restricted p-strict partitions $\mu \vdash n$, see [BK$_2$]. In section 2 we define analogues of the ladders, regularization λ^R, shadow $\mathrm{sh}(\mu)$, and shadow node. Our main result is then as follows.

Theorem 1.2.

(i) *For any p-strict $\lambda \vdash n$, $D(\lambda^R)$ appears as a composition factor of $S(\lambda)$ with multiplicity 1, and for any other composition factor $D(\mu)$ of $S(\lambda)$ we have that $\mu < \lambda^R$.*

(ii) *For a restricted p-strict $\mu \vdash n$, let the outer ladder of μ be of size m and residue i. Then $D(\mathrm{sh}(\mu))$ appears as a composition factor of the restriction $D(\mu)\downarrow_{\mathcal{T}_{n-1}}$ with multiplicity $2m$ if $i \neq 0$ and $(n - h_{p'}(\lambda))$ is odd, and with multiplicity m otherwise. Moreover, for any other composition factor $D(\nu)$ of $D(\mu)\downarrow_{\mathcal{T}_{n-1}}$ we have that $\nu < \mathrm{sh}(\mu)$.*

Note that all $>$'s and $<$'s are interchanged in the two theorem above. This has to do with the labelling we are using for irreducible $\mathcal{T}_n$-modules; the analogously labelled irreducible S_n-modules would be $D_\lambda := D^{\lambda^t} \otimes \mathrm{sgn}$.

Finally, we note that the identification of the two labellings of the irreducible $\mathcal{T}_n$-modules by the set $\mathscr{RP}_p(n)$ from [BK$_2$] and [BK$_1$] is still lacking. Here we work with the labelling of [BK$_2$]. Theorem 1.2 is a step towards the desired identification.

2. Ladders, Shadows and Regularization

We will use the same notation as in [BK$_1$, BK$_2$, K]. In particular we have:
$p = 2\ell + 1$;
$\mathscr{P}_p(n)$ is the set of p-strict partitions of n;
$\mathscr{RP}_p(n)$ is the set of restricted p-strict partitions of n;
$h(\lambda)$ is the number of non-zero parts of a partition λ;
$h_{p'}(\lambda)$ is the number of parts of a partition λ which are not divisible by p.

As usual, the Young diagram of a partition λ (identified with λ itself) can be considered as a subset of the set $Q := \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ of nodes. If $A = (i, j)$ is a node, we say that A is in the row i and column j. If $B = (i', j')$ is another node, we say that A is to the left of B (or B is to the right of A) if $j < j'$.

We denote the n-tuple $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{Z}^n$, with 1 in the ith position, by ε_i. We write α_i for $\varepsilon_i - \varepsilon_{i+1}$, $1 \leq i < n$. For $\lambda, \mu \vdash n$, we have that $\lambda \geq \mu$ in the dominance ordering if and only if $\lambda - \mu = \sum_{i=1}^{n-1} m_i \alpha_i$ for some non-negative integers $m_1, \dots, m_{n-1}$.

Let $j \in \mathbb{Z}_{>0}$ be a column number. It can be written uniquely in the form

$$j = mp + \ell + 1 \pm k, \quad m, k \in \mathbb{Z},\ 0 \leq k \leq \ell.$$

The *residue* of j is then defined to be $\ell - k$, written $\mathrm{res}\, j = \ell - k$. The *residue* of $A = (i, j)$, written $\mathrm{res}\, A$, is defined to be $\mathrm{res}\, j$. So $\mathrm{res}\, A \in \{0, 1, \dots, \ell\}$ for any A, and the residue of a node depends only on its column. The *residue content* $\mathrm{cont}(\lambda)$ of a p-strict partition λ means the tuple $(c_0, c_1, \dots, c_\ell)$ where c_i is the number of nodes of residue i in λ.

Following [Ja₁, LT], we define certain sets of nodes called ladders. Fix any $j \geq 1$. The *jth ladder* L_j is defined as follows. If $\operatorname{res} j \neq 0$ then

$$L_j = \{(i, j-(i-1)p) \mid 1 \leq i \leq \lceil j/p \rceil\}.$$

If $\operatorname{res} j = 0$ then $j = mp$ or $mp+1$ for some $m \in \mathbb{Z}$, and in this case we set

$$L_j = \{(i, mp-(i-1)p) \mid 1 \leq i \leq m\} \cup \{(i, mp+1-(i-1)p) \mid 1 \leq i \leq m+1\}.$$

If $L = L_j$, we write simply $\operatorname{res} L$ for $\operatorname{res} j$. Note that all nodes of L are of residue $\operatorname{res} L$, and that ladders of residue 0 are twice as wide as the others. A k-element subset of a ladder is called *complete* if it consists of the k leftmost nodes of the ladder. We say that the ladder $L = L_j$ is the *outer* ladder for λ if $\lambda \cap L \neq \varnothing$ but $\lambda \cap L_{j+1} = \varnothing$. In this case, the rightmost node on $\lambda \cap L$ will be called the *shadow node* of λ. If λ is restricted (i.e. $\lambda \in \mathscr{RP}_p(n)$) and A is the shadow node of λ then $\lambda - A$ is also restricted. We will sometimes write $\operatorname{sh}(\lambda)$ for $\lambda - A$.

Lemma 2.1. *Let $\lambda \in \mathscr{P}_p(n)$. Then λ is restricted if and only if for every ladder L, the intersection $L \cap \lambda$ is a complete subset of L.*

Proof. If $L \cap \lambda$ is not complete, then there are nodes $A = (i, j)$, and $B = (i-1, j+x)$ on the ladder L such that $A \notin \lambda$ and $B \in \lambda$. If $x \geq p$ this implies that $\lambda_{i-1} - \lambda_i > p$, i.e. λ is not restricted. Otherwise $\operatorname{res} L = 0$, $A = (i, mp+1)$, and $B = (i-1, (m+1)p)$. In this case $\lambda_{i-1} - \lambda_i \geq p$, and the equality holds only if p divides λ_{i-1}. So again λ is not restricted. The argument can be easily reversed. □

For every p-strict partition λ we define its *regularization* λ^R to be the set of nodes such that for every ladder L, $\lambda^R \cap L$ consists of the leftmost $|\lambda \cap L|$ nodes on L. In other words, λ^R is obtained by shifting the nodes of λ along the ladders to the left as far as they can go.

Example 2.2. If $p = 5$ and $\lambda = (12, 5)$ then $\lambda^R = (9, 6, 2)$:

$\lambda =$

0	1	2	1	0	0	1	2	1	0	0	1
0	1	2	1	0							

, $\lambda^R =$

0	1	2	1	0	0	1	2	1
0	1	2	1	0	0			
0	1							

.

Proposition 2.3. *Let $\lambda \in \mathscr{P}_p(n)$. Then $\lambda^R \in \mathscr{RP}_p(n)$. Moreover $\lambda = \lambda^R$ if and only if λ is restricted.*

Proof. In view of Lemma 2.1, all we need to check is that $\lambda^R \in \mathscr{P}_p(n)$, which is left as an exercise. □

3. P-functions

Let G be the algebraic supergroup $Q(n)$, see [BK₃]. If M is a (finite dimensional) G-module, there is a notion of its formal character, see [B, §3]:

$$\operatorname{ch} M = \sum_{\lambda \in X(T)} (\dim M_\lambda) x^\lambda \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

We are only going to consider polynomial representations of G of degree n in the sense of [BK$_3$, §10]. For such modules, the formal character $\mathrm{ch}\, M$ is a homogeneous symmetric polynomial of degree n in $x_1, \dots, x_n$.

Let $\mathrm{Ch}(n) \subset \mathbb{Z}[x_1, \dots, x_n]$ be the $\mathbb{Z}$-submodule spanned by all formal characters $\mathrm{ch}\, M$, where M is an arbitrary polynomial representation of G of degree n. Then $\mathrm{Ch}(n)$ has a $\mathbb{Z}$-basis given by the characters

$$L_\lambda := \mathrm{ch}\, L(\lambda) \tag{3.1}$$

of irreducible modules as λ runs through $\mathscr{P}_p(n)$, see [BK$_3$, 10.4].

As in [B, §4], for any $\lambda \in \mathscr{P}_p(n)$ define a virtual module

$$E(\lambda) = \sum_{i \geq 0} (-1)^i H^i(\lambda), \tag{3.2}$$

and its formal character

$$E_\lambda = \sum_{i \geq 0} (-1)^i \mathrm{ch}\, H^i(\lambda). \tag{3.3}$$

Let $\lambda \in \mathscr{P}_p(n)$ and $P_\lambda = P_\lambda(x_1, \dots, x_n)$ be the Schur's P-function, obtained by taking $t = -1$ in [M, III(2.2)]. Set

$$e(\lambda) := \left\lfloor \frac{h_{p'}(\lambda) + 1}{2} \right\rfloor, \tag{3.4}$$

cf. [BK$_3$, 6.4]. We use the following result proved in [B, 4.3, 6.3].

Theorem 3.1. *Let $\lambda \in \mathscr{P}_p(n)$. Then $E_\lambda = 2^{e(\lambda)} P_\lambda$. Moreover,*

$$E_\lambda = L_\lambda + \sum_{\mu \in \mathscr{P}_p(n)} c_{\lambda\mu} L_\mu,$$

with $c_{\lambda\mu} \in \mathbb{Z}$ such that $c_{\lambda\mu} = 0$ unless $\mu < \lambda$ and $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$.

Corollary 3.2. *$\{E_\lambda \mid \lambda \in \mathscr{P}_p(n)\}$ is a $\mathbb{Z}$-basis of $\mathrm{Ch}(n)$.*

Let $f = f(x_1, \dots, x_n) \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Define

$$f{\downarrow} = (f{\downarrow})(x_1, \dots, x_{n-1})$$

to be the x_n-coefficient of f. It follows from (3.6) below that the restriction of $\downarrow$ defines a map from $\mathrm{Ch}(n)$ to $\mathrm{Ch}(n-1)$.

Definition 3.3. Let $\lambda \in \mathscr{P}_p(n)$. A node $B = (i, j) \in \lambda$ is called *branching* if $\lambda - B$ is a partition (of $n - 1$), and either $j = 1$ or the part $j - 1$ appears even number of times in λ.

It is easy to see that if $\lambda \in \mathscr{P}_p(n)$ and $B \in \lambda$ is branching then $\lambda - B \in \mathscr{P}_p(n-1)$. The following result is a special case of [M, III(5.5′),(5.14′)].

Proposition 3.4. *Let $\lambda \in \mathscr{P}_p(n)$, and let $B_1, B_2, \dots, B_k$ be the branching nodes of λ. Then*

$$P_\lambda{\downarrow} = \sum_{i=1}^{k} a_i P_{\lambda - B_i},$$

where $a_i = 1$ *if* $h(\lambda - B_i) < h(\lambda)$, *and* $a_i = 2$ *otherwise.*

Corollary 3.5. *Let* $\lambda \in \mathscr{P}_p(n)$, *and let* $B_1, B_2, \dots, B_k$ *be the branching nodes of* λ. *Then*

$$E_\lambda{\downarrow} = \sum_{i=1}^{k} a_i 2^{e(\lambda)-e(\lambda-B_i)} E_{\lambda-B_i},$$

where $a_i = 1$ *if* $h(\lambda - B_i) < h(\lambda)$, *and* $a_i = 2$ *otherwise.*

We now explain what the operation $\downarrow$ corresponds to at the level of irreducible modules. Set

$$L(\lambda)_j := \bigoplus \{L(\lambda)_\mu \mid \mu \text{ is a weight with } \mu_n = j\}.$$

Then we have a decomposition

$$\operatorname{res}^{Q(n)}_{Q(n-1)} L(\lambda) = \bigoplus_{j \geq 0} L(\lambda)_j, \tag{3.5}$$

Now,

$$L_\lambda{\downarrow} = \operatorname{ch} L(\lambda)_1. \tag{3.6}$$

Let $\operatorname{Ch}(n)'$ be the $\mathbb{Z}$-submodule of $\operatorname{Ch}(n)$ spanned by $\{L_\lambda \mid \lambda \in \mathscr{RP}_p(n)\}$, and let

$$\operatorname{Ch}(n) \to \operatorname{Ch}(n)', \ f \mapsto f'$$

be the natural projection, i.e. the linear map such that for $\lambda \in \mathscr{P}_p(n)$ we have $L'_\lambda = L_\lambda$ if λ is restricted, $L'_\lambda = 0$ otherwise.

Lemma 3.6. *Let* $f \in \operatorname{Ch}(n)$. *Then* $(f'{\downarrow})' = (f{\downarrow})'$.

Proof. As all the maps are linear, it suffices to check the lemma for $f = L_\lambda$, $\lambda \in \mathscr{P}_p(n)$. If λ is restricted, then $L_\lambda = L'_\lambda$, and the result is clear. Assume then that λ is not restricted. In this case $L'_\lambda = 0$, so we have to check that $(L_\lambda{\downarrow})' = 0$ or that $(\operatorname{ch} L(\lambda)_1)' = 0$, see (3.6). As λ is not restricted, Steinberg's tensor product theorem [BK$_3$, 9.9] implies that

$$L(\lambda) = L(\lambda(0)) \otimes L(\lambda(1))^{[1]},$$

where $\lambda = \lambda(0) + p\lambda(1)$, $\lambda(0)$ is restricted, and $\mu = (\mu_1 \geq \cdots \geq \mu_n \geq 0)$ is a *non-zero* dominant weight for $GL(n)$. As $L(\lambda(1))^{[1]}_\mu = 0$ for any μ with $\mu_n = 1$, we have

$$L(\lambda)_1 = L(\lambda(0))_1 \otimes L(\lambda(1))^{[1]}_0.$$

Therefore, no composition factor of the $Q(n-1)$-module $L(\lambda)_1$ is restricted, which proves that $(\operatorname{ch} L(\lambda)_1)' = 0$. □

4. Leading terms

We need three technical combinatorial lemmas to prove our main result.

Lemma 4.1. *Let $\lambda \in \mathscr{P}_p(n)$ and A, B be nodes of λ such that $\lambda - A, \lambda - B \in \mathscr{P}_p(n-1)$. Assume that $A \in L_i$, $B \in L_j$, and the ladder L_i is strictly to the right of the ladder L_j. Then $(\lambda - A)^R > (\lambda - B)^R$.*

Proof. We have $(\lambda - A)^R = \lambda^R - A'$ and $(\lambda - B)^R = \lambda^R - B'$, where A' and B' are the rightmost nodes of $\lambda^R \cap L_i$ and $\lambda^R \cap L_j$, respectively. Let A' be in the row k and B' be in the row l. We need to prove that $k > l$. Assume this is not the case. Then, by Lemma 2.1, the lth row of λ^R contains a node from the ladder L_i. This node lies to the right of B', which contradicts the fact that $\lambda^R - B'$ is a partition. $\square$

Lemma 4.2. *Let $\lambda, \mu \in \mathscr{RP}_p(n)$. If $\mu \leq \lambda$ then* $\mathrm{sh}(\mu) \leq \mathrm{sh}(\lambda)$. *Moreover if* $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$ *then $\mu \neq \lambda$ implies* $\mathrm{sh}(\mu) \neq \mathrm{sh}(\lambda)$.

Proof. Let the shadow node A of λ be in row i and the shadow node B of μ be in row j. We have to prove that $\mu - \varepsilon_j \leq \lambda - \varepsilon_i$. If $i \geq j$, then $\mu \leq \lambda \leq \lambda + \varepsilon_j - \varepsilon_i$, whence $\mu - \varepsilon_j \leq \lambda - \varepsilon_i$, as required. Now let $i < j$. Write $\mu = \lambda - \sum_{k=1}^{n-1} m_k \alpha_k$. Then $\mu - \varepsilon_j = \lambda - \varepsilon_i + (\varepsilon_i - \varepsilon_j) - \sum_{k=1}^{n-1} m_k \alpha_k$. So, we see that $\mu - \varepsilon_j \leq \lambda - \varepsilon_i$, unless $m_r = 0$ for some $i \leq r < j$. It follows that $\mu_r - \mu_{r+1} \geq \lambda_r - \lambda_{r+1} = p$. As μ is restricted, we now deduce that $m_k = 0$ for all $k > r$. Similarly, $m_k = 0$ for all $i \leq k < r$. This shows that $\mu_k = \lambda_k$ for all $k > i$, which contradicts the assumption that $j > i$.

Finally, if $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$, suppose that $\lambda \neq \mu$. We may assume without loss of generality that $i < j$. Then, because λ is restricted and A is its shadow node, $\lambda_j + 1 = \mu_j$ implies that $\mathrm{res}\, B \neq \mathrm{res}\, A$. So $\mathrm{cont}(\lambda - A) \neq \mathrm{cont}(\mu - B)$, hence $\lambda - A \neq \mu - B$. $\square$

Lemma 4.3. *Let $\lambda, \mu \in \mathscr{RP}_p(n)$,* $\mathrm{cont}(\lambda) = \mathrm{cont}(\mu)$*, and B be the shadow node of μ.*

(i) *If A is the shadow node of λ and* $\mathrm{res}\, A = \mathrm{res}\, B$*, then* $\mathrm{sh}(\mu) < \mathrm{sh}(\lambda)$ *implies $\mu < \lambda$.*

(ii) *Let L be the outer ladder for λ, L' be a ladder strictly to the left of L, and A be the rightmost node in $\lambda \cap L'$. Assume that $\lambda - A \in \mathscr{RP}_p(n-1)$. Then* $\mathrm{sh}(\mu) \leq \lambda - A$ *implies $\mu < \lambda$.*

Proof. First of all note that $\lambda \neq \mu$. Indeed, in (i), $\lambda = \mu$ would imply $\lambda - A = \mu - B$, giving a contradiction. In (ii), in view of Lemma 4.1, $\lambda = \mu$ implies $\mu - B > \lambda - A$, giving a contradiction again.

Let A be in row i and B be in row j. We can write $\mu - \varepsilon_j = \lambda - \varepsilon_i - \sum_{k=1}^{n-1} m_k \alpha_k$ for non-negative coefficients $m_k \in \mathbb{Z}$. If $j \geq i$, this implies that $\mu \leq \lambda$, whence $\mu < \lambda$ as we have already noticed that $\lambda \neq \mu$. So suppose that $i > j$. Again, it suffices to show that $\mu \leq \lambda$. This is certainly the case unless $m_r = 0$ for some $j \leq r < i$. As B is the shadow node of μ and

$\lambda - \varepsilon_i$ is restricted, it now follows that $m_k = 0$ for all $k \geq r$. In particular, $\lambda_i = \mu_i + 1$ and $\lambda_k = \mu_k$ for $k > i$. To complete the proof of (i), we now get that $\operatorname{res} A \neq \operatorname{res} B$, a contradiction. To complete the proof of (ii), the fact that λ is restricted implies that $\lambda_{i+1} = 0$. So A must be on the outer ladder of λ (using restrictedness of λ again), which is a contradiction. □

Theorem 4.4. *Let $\lambda \in \mathscr{P}_p(n)$, L be the outer ladder of λ, and $m = |\lambda \cap L|$.*

(i) *If λ is restricted and A is the shadow node of λ, then*

$$(L_\lambda\downarrow)' = d(\lambda) L_{\lambda - A} + \sum_{\substack{\mu \in \mathscr{RP}_p(n-1) \\ \mu < \lambda - A}} b_{\lambda,\mu} L_\mu$$

where $d(\lambda) = m$ if $\operatorname{res} L = 0$ and $h_{p'(\lambda)}$ is even, and $d(\lambda) = 2m$ otherwise.

(ii) *We have*

$$E'_\lambda = L_{\lambda^R} + \sum_{\substack{\mu \in \mathscr{RP}_p(n) \\ \mu < \lambda^R}} c_{\lambda\mu} L_\mu.$$

Proof. Let $B_1, \dots, B_k$ be the branching nodes of λ, and assume that $B_i \in L$ if and only if $1 \leq i \leq t$ (note t does not have to equal m if $\operatorname{res} L = 0$). We prove the theorem by induction on n, the induction base being clear. Assume the theorem is correct for $n - 1$.

We prove (i) for n. Note that $(\lambda - B_i)^R = \lambda - A$ for $1 \leq i \leq t$, and $(\lambda - B_i)^R < (\lambda - A)$ for $i > t$, in view of Lemma 4.1. Now, using Corollary 3.5 and inductive hypothesis (part (ii)), one can deduce that $L_{\lambda - A}$ appears in $(E_\lambda\downarrow)'$ exactly $d(\lambda)$ times, and we have $\nu < \lambda - A$ for every other L_ν appearing in $(E_\lambda\downarrow)'$.

On the other hand, by Theorem 3.1, we have that

$$E_\lambda = L_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu} L_\mu,$$

where $c_{\lambda\mu} = 0$, unless $\operatorname{cont}(\mu) = \operatorname{cont}(\lambda)$. So, by Lemma 3.6,

$$(E_\lambda\downarrow)' = (E'_\lambda\downarrow)' = (L_\lambda\downarrow)' + \sum_{\substack{\mu \in \mathscr{RP}_p(n) \\ \mu < \lambda}} c_{\lambda\mu} (L_\mu\downarrow)'. \tag{4.1}$$

Now, we apply induction on the dominance order on $\mathscr{RP}_p(n)$. By inductive hypothesis, we may assume that for any $\mu \in \mathscr{RP}_p(n)$ with $\mu < \lambda$, one has $(L_\mu\downarrow)' = d(\mu) L_{\operatorname{sh}(\mu)} + (*)$, where $(*)$ stands for a linear combination of L_ν with $\nu < \operatorname{sh}(\mu)$. Now it suffices to apply Lemma 4.2.

We prove (ii) for n. Let A be the shadow node for λ^R. Note that $(\lambda - B_i)^R = \lambda^R - A$ for $1 \leq i \leq t$, and $(\lambda - B_i)^R < \lambda^R - A$ in view of Lemma 4.1.

So, by Lemma 3.6, Corollary 3.5 and inductive hypothesis, we have

$$\begin{aligned}(E'_\lambda\!\downarrow)' = (E_\lambda\!\downarrow)' &= \sum_{i=1}^{k} a_i 2^{e(\lambda)-e(\lambda-B_i)} E'_{\lambda-B_i} \\ &= \sum_{i=1}^{t} a_i 2^{e(\lambda)-e(\lambda-B_i)} L_{\lambda^R-A} + (*),\end{aligned} \tag{4.2}$$

where $(*)$ stands for the terms L_ν with $\nu < \lambda - A$.

We now prove that

$$\sum_{i=1}^{t} a_i 2^{e(\lambda)-e(\lambda-B_i)} = d(\lambda^R). \tag{4.3}$$

If $\operatorname{res} L \neq 0$, then (4.3) is clear. Now let $\operatorname{res} L = 0$. In this case the row containing B_i $(1 \le i \le t)$ might have one or two nodes from $\lambda \cap L$. If it has one, we say that B_i is of the first type, and if it has two, we say that B_i is of the second type. Let $C_1, \dots, C_x$ and $D_1, \dots, D_y$ be the type one and type two nodes among $B_1, \dots, B_t$, respectively. We have $x + y = t$ and $x + 2y = m$. Note that $h_{p'}(\lambda - D_j) = h_{p'}(\lambda) - 1$, and

$$h_{p'}(\lambda - C_i) = \begin{cases} h_{p'}(\lambda) + 1 & \text{if } C_i \text{ is not in the first column} \\ h_{p'}(\lambda) - 1 & \text{otherwise.} \end{cases}$$

This implies (4.3) if we use $h_{p'}(\lambda) \equiv h_{p'}(\lambda^R) \pmod 2$ [K, (22.9), (22.10)].

Now, (4.2), (4.3), part (i) (already proved for n), and Lemma 4.2 imply that L_{λ^R} appears in E'_λ exactly once. It remains to prove that $\mu < \lambda^R$ for any other L_μ appearing in E'_λ. Take any such L_μ. Applying (4.2), part (i) and Lemma 4.2 again, we deduce that $\operatorname{sh}(\mu) < \lambda^R - A$. If $\operatorname{cont}(\operatorname{sh}(\mu)) = \operatorname{cont}(\lambda^R - A)$ then $\mu < \lambda^R$ by Lemma 4.3(i). Otherwise, note by (4.2), that $L_{\operatorname{sh}(\mu)}$ must appear in some $E_{\lambda-B_i}$ for $\operatorname{res} B_i \neq \operatorname{res} A$. By inductive hypothesis, $\operatorname{sh}(\mu) \le \lambda^R - C$, where C is the rightmost node of λ^R on the ladder containing B_i, i.e. $(\lambda - B_i)^R = \lambda^R - C$. As $\operatorname{res} B_i \neq \operatorname{res} A$, this ladder is to the left of the outer ladder L. So $\mu < \lambda^R$ by Lemma 4.3(ii). $\square$

5. Translation to $\mathcal{T}_n$-modules

Recall from [BK$_2$, §10] (cf. also [BK$_3$, §10]) the Schur functor $M \mapsto \xi_\omega M$ going from polynomial $Q(n)$-modules of degree n to modules over the Sergeev algebra $\mathcal{Y}_n$. By [BK$_3$, 10.2], we have $\xi_\omega L(\lambda) = 0$ if λ is not restricted, and if λ is restricted then $\xi_\omega L(\lambda) = M(\lambda)$, an irreducible $\mathcal{Y}_n$-module of the same type as $L(\lambda)$. Moreover,

$$\{M(\lambda) \mid \lambda \in \mathscr{RP}_p(n)\}$$

is a complete set of irreducible $\mathcal{Y}_n$-modules up to isomorphism.

Finally, there is an exact functor $\mathfrak{G}_n$ from $\mathcal{Y}_n$-modules to $\mathcal{T}_n$-modules, see [K, 13.2] or [BK$_2$, §3]. If n is even then $\mathfrak{G}_n$ is an equivalence categories, in particular $\mathfrak{G}_n M(\lambda) = D(\lambda)$, an irreducible $\mathcal{T}_n$-module of the same type as

$M(\lambda)$. If n is odd and $h_{p'}(\lambda)$ is even, then $\mathfrak{G}_n M(\lambda) = D(\lambda)$, an irreducible $\mathcal{T}_n$-module of type $\mathtt{Q}$. If n is odd and $h_{p'}(\lambda)$ is odd, then $\mathfrak{G}_n M(\lambda) = D(\lambda) \oplus D(\lambda)$, where $D(\lambda)$ is an irreducible $\mathcal{T}_n$-module of type $\mathtt{M}$. Moreover,

$$\{D(\lambda) \mid \lambda \in \mathscr{RP}_p(n)\}$$

is a complete set of irreducible $\mathcal{T}_n$-modules up to isomorphism.

For any $\lambda \in \mathscr{P}_p(n)$, we define the *Specht 'module'* to be the virtual module

$$S(\lambda) := \mathfrak{G}_n(\xi_\omega E(\lambda)),$$

obtained by the application of the Schur functor $M \mapsto \xi_\omega M$ followed by the functor $\mathfrak{G}_n$ to the virtual module $E(\lambda)$ defined in (3.2). (One can also interpret the Specht 'module' as a complex whose Euler characteristic is $S(\lambda)$ but we will not pursue this here.)

The discussion above, [K, (13.11),(13.15)] and Theorem 4.4 imply Theorem 1.2.

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