CORRIGENDA TO 'LINEAR AND PROJECTIVE REPRESENTATIONS OF SYMMETRIC GROUPS'

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We are grateful to Steffen Koenig and Steffen Oppermann for pointing out that there is a gap in the proof of Lemma 6.3.2 of [1]. We do not know at the moment whether Lemma 6.3.2 is correct or not. However, we claim that it is not needed anywhere in the book if the following changes are made.

1) Drop Lemma 6.3.2.

2) Amend Lemmas 6.3.3 and 8.4.3 as follows.

Lemma 6.3.3. *Take $a, b \in F$ with $a \neq b$ and set $k = k_{ab}$. Let $M \in \mathcal{H}_n$-mod be irreducible, and $\varepsilon = \varepsilon_a(M)$.*

(i) *There exists a unique integer r with $0 \leq r \leq k$ such that for every $m \geq 0$ we have*
$$\varepsilon_a(\tilde{f}_a^m \tilde{f}_b M) = m + \varepsilon - r.$$

(ii) *Assume $m \geq k$. Then a copy of $\tilde{f}_a^m \tilde{f}_b M$ appears in the head of*
$$\mathrm{ind}\, \tilde{f}_a^{m-k} M \boxtimes L(a^r b a^{k-r}),$$
where r is an in (i).

(iii) *Assume $0 \leq m < k \leq m + \varepsilon$. Then a copy of $\tilde{f}_a^m \tilde{f}_b M$ appears in the head of*
$$\mathrm{ind}\, \tilde{e}_a^{k-m} M \boxtimes L(a^r b a^{k-r}),$$
where r is an in (i).

Proof. Write $M = \tilde{f}_a^\varepsilon N$ for an irreducible $\mathcal{H}_{n-\varepsilon}$-module N with $\varepsilon_a(N) = 0$. It suffices to prove (i) for any fixed choice of m, the conclusion for all other $m \geq 0$ then follows immediately by (5.12). So take $m \geq 0$ with $k \leq m + \varepsilon$. Note that $\tilde{f}_a^m \tilde{f}_b M = \tilde{f}_a^m \tilde{f}_b \tilde{f}_a^\varepsilon N$ is a quotient of
$$\begin{cases} \mathrm{ind}\, N \boxtimes L(a^\varepsilon) \boxtimes L(b) \boxtimes L(a)^{\boxtimes k} \boxtimes L(a^{m-k}) & \text{if } m \geq k, \\ \mathrm{ind}\, N \boxtimes L(a^{m+\varepsilon-k}) \boxtimes L(a)^{\boxtimes (k-m)} \boxtimes L(b) \boxtimes L(a^m) & \text{if } m < k, \end{cases}$$
which by Lemma 6.3.1 has a filtration with factors isomorphic to
$$\begin{cases} F_r := \mathrm{ind}\, N \boxtimes L(a^\varepsilon) \boxtimes L(a^r b a^{k-r}) \boxtimes L(a^{m-k}) & \text{if } m \geq k, \\ F_r := \mathrm{ind}\, N \boxtimes L(a^{m+\varepsilon-k}) \boxtimes L(a^r b a^{k-r}) & \text{if } m < k, \end{cases}$$
for $0 \leq r \leq k$, each appearing with some multiplicity. So $\tilde{f}_a^m \tilde{f}_b M$ is a quotient of some such factor, and to prove (i) it remains to show that $\varepsilon_a(L) = \varepsilon + m - r$ for any irreducible quotient L of F_r. The inequality $\varepsilon_a(L) \leq \varepsilon + m - r$ is clear from the Shuffle Lemma. On the other hand, by transitivity of induction and Lemma 6.3.1, $F_r \cong \mathrm{ind}\, N \boxtimes (\mathrm{ind}\, L(a^r b a^{k-r}) \boxtimes L(a^{\varepsilon+m-k}))$.

So by Frobenius reciprocity, the irreducible module $N \boxtimes (\mathrm{ind}\, L(a^r b a^{k-r}) \boxtimes L(a^{\varepsilon+m-k}))$ is contained in $\mathrm{res}_{n-\varepsilon,m+1+\varepsilon} L$. Hence $\varepsilon_a(L) \geq \varepsilon + m - r$.

To complete the proof of (ii) and (iii), by Lemma 5.21, we also have

$$F_r \cong \mathrm{ind}\, N \boxtimes L(a^{\varepsilon+m-k}) \boxtimes L(a^r b a^{k-r}),$$

and by the Shuffle Lemma, the only composition factors K of F_r with $\varepsilon_a(K) = \varepsilon + m - r$ come from its quotient

$$\mathrm{ind}\, \tilde{f}_a^{m-k+\varepsilon} N \boxtimes L(a^r b a^{k-r}).$$

The latter is $\mathrm{ind}\, \tilde{f}_a^{m-k} M \boxtimes L(a^r b a^{k-r})$ if $m \geq k$ and $\mathrm{ind}\, \tilde{e}_a^{k-m} M \boxtimes L(a^r b a^{k-r})$ otherwise. $\square$

Lemma 8.4.3 *Let $i, j \in I$ with $i \neq j$. Suppose that M is an irreducible $\mathcal{H}_n^\lambda$-module such that $\varphi_j^\lambda(M) > 0$. Then*

$$\varphi_i^\lambda(\tilde{f}_j M) - \varepsilon_i^\lambda(\tilde{f}_j M) \leq \varphi_i^\lambda(M) - \varepsilon_i^\lambda(M) - \langle h_i, \alpha_j \rangle.$$

Proof. Set

$$\varepsilon := \varepsilon_i^\lambda(M), \;\; \varphi := \varphi_i^\lambda(M), \;\; k := -\langle h_i, \alpha_j \rangle.$$

By Lemma 6.3.3, there exist unique $r, s \geq 0$ with $r + s = k$ such that $\varepsilon_i(\tilde{f}_j M) = \varepsilon - r$. We need to show that $\varphi_i^\lambda(\tilde{f}_j M) \leq \varphi + s$, which follows if we can show that $\mathrm{pr}^\lambda \tilde{f}_i^m \tilde{f}_j M = 0$ for all $m > \varphi + s$. It suffices to prove that

$$\varepsilon_i^*(\tilde{f}_i^m \tilde{f}_j M) \geq \varepsilon_i^*(\tilde{f}_i^{m-s} M) \tag{8.18}$$

for all $m > \varphi + s$. Indeed, by the definition of φ, we have $\mathrm{pr}^\lambda \tilde{f}_i^{m-s} M = 0$ for any $m > \varphi + s$. In view of Corollary 7.4.1, this means that $\varepsilon_j^*(\tilde{f}_i^{m-s} M) > \langle h_j, \lambda \rangle$ for some $j \in I$. But by Lemma 8.4.2, such j can only equal i. Thus $\varepsilon_i^*(\tilde{f}_i^{m-s} M) > \langle h_i, \lambda \rangle$ for all $m > \varphi + s$. So (8.18) implies that $\varepsilon_i^*(\tilde{f}_i^m \tilde{f}_j M) > \langle h_i, \lambda \rangle$ for all $m > \varphi + s$, hence by Corollary 7.4.1 once more, $\mathrm{pr}^\lambda \tilde{f}_i^m \tilde{f}_j M = 0$ as required.

To prove (8.18), note that $k \leq m + \varepsilon$, so by Lemma 6.3.3(ii),(iii) there is a surjection

$$\mathrm{ind}_{n-\varepsilon,\varepsilon+m-k,k+1}^{n+m+1} N \boxtimes L(i^{\varepsilon+m-k}) \boxtimes L(i^r j i^s) \twoheadrightarrow \tilde{f}_i^m \tilde{f}_j M,$$

where $N = \tilde{e}_i^\varepsilon M$. By Lemma 6.2.1, $\mathrm{res}_{r,s+1}^{r+s+1} L(i^r j i^s) \cong L(i^r) \boxtimes L(j i^s)$. Hence there is a surjection $\mathrm{ind}_{r,s+1}^{r+s+1} L(i^r) \boxtimes L(j i^s) \twoheadrightarrow L(i^r j i^s)$. Combining, we have proved existence of a surjection

$$\mathrm{ind}_{n-\varepsilon,\varepsilon+m-s,s+1}^{n+m+1} N \boxtimes L(i^{\varepsilon+m-s}) \boxtimes L(j i^s) \twoheadrightarrow \tilde{f}_i^m \tilde{f}_j M.$$

Hence by Frobenius reciprocity there is a non-zero map

$$(\mathrm{ind}_{n-\varepsilon,\varepsilon+m-s}^{n+m-s} N \boxtimes L(i^{\varepsilon+m-s})) \boxtimes L(j i^s) \to \mathrm{res}_{n+m-s,s+1}^{n+m+1} \tilde{f}_i^m \tilde{f}_j M.$$

Since the left-hand module has irreducible cosocle $\tilde{f}_i^{m-s} M \boxtimes L(ji^s)$, we deduce that $\tilde{f}_i^m \tilde{f}_j M$ has a constituent isomorphic to $\tilde{f}_i^{m-s} M$ on restriction to the subalgebra $\mathcal{H}_{n+m-s} \subseteq \mathcal{H}_{n+m+1}$. This implies the claim. □

Similar changes need to be made to Lemmas 18.3.2, 18.3.3 and 19.6.3 in Part II.

References

[1] A. Kleshchev, *Linear and Projective Representations of Symmetric Groups*, CUP, 2005.

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