2-Transitive and flag-transitive designs

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IN MEMORY OF MARSHALL HALL, JR.

0 Introduction.

Throughout this paper $\mathcal{D}$ always will denote a design with v points, $k > 2$ points per line, and $\lambda = 1$ line through any two different points. Let $G \leq \text{Aut}(\mathcal{D})$. I will primarily be interested in the case in which G either is 2-transitive on the points of $\mathcal{D}$ or is transitive on the flags (incident point-line pairs) of $\mathcal{D}$. Note that 2-transitivity implies flag-transitivity since $\lambda = 1$.

The subject matter has been separated partly along historical lines, but more significantly as regards the use of the classification of finite simple groups. §I involves comparatively little in the way of group-theoretic background (in particular, it concerns results noticeably predating the aforementioned classification). §II describes the main results that use properties of simple groups. Finally, §III reverts to a more combinatorial and very much less group-theoretic problem: the construction of new flag-transitive designs.

No attempt has been made to be encyclopedic. See [9, §§2.3, 4.4], [31] and [3] for other surveys of similar material with somewhat different emphases.

I Pre-classification.

The most beautiful result concerning the type of question being considered here is the Ostrom-Wagner Theorem [49]: *If $\mathcal{D}$ is a finite projective plane*

*Research partially supported by NSF Grant DMS 90-01784 and NSA Grant MDA 904-88-H-2040.

I am grateful for very helpful remarks on preliminary versions of this paper made by R. A. Liebler, T. G. Ostrom, A. Wagner and J. Yaqub.

having a 2-transitive collineation group, then $\mathcal{D} \cong PG(2,q)$.[1] This was preceded by some special cases: Ostrom [47] and Hall and Hughes [28] had handled 2-transitive projective planes of nonsquare orders[2]; while slightly later but independently, Wagner [54] had proved that a finite projective plane is desarguesian if it admits a collineation group transitive on ordered quadrangles of points. The groundbreaking precursors [47, 54] of [49] contained germs of ideas used in that paper – and led to the collaboration by those authors. This was the first time 2-transitivity produced a complete classification of finite geometries. Since then the notion of a geometric classification in terms of a group-theoretic hypothesis has become commonplace. That was not the case 35 years ago, and it is a measure of these papers' influence that this type of hypothesis is now regarded as a natural extension of Klein's Erlangen program.[3]

Another fundamental question studied in [49] was that of an affine plane $\mathcal{D}$ admitting a 2-transitive or line-transitive collineation group G. It was shown that $\mathcal{D}$ must be a translation plane if G is 2-transitive on points (the cases in which k is odd and not a square, or $k = 2^{2e+1}$ for some e, had been dealt with earlier in [47, 48]). Under the seemingly weaker hypothesis of line-transitivity, it was first shown that this implies flag-transitivity[4], and then that $\mathcal{D}$ must be a translation plane provided that k satisifes *any* of the following conditions: k is even, k is odd and not a square, or k is a prime power. Later, Wagner [56] proved that k must be a prime power, thereby completing the general case: *if $\mathcal{D}$ is an affine plane admitting a line-transitive collineation group G, then $\mathcal{D}$ is a translation plane and G contains the translation group.* This result contains the Ostrom-Wagner Theorem[5] as a more or less[6] immediate consequence. The classification and construction of flag-transitive affine planes will be discussed further in §§II,III.

[1]Moreover, they also showed that any 2-transitive collineation group of $PG(2,q)$ contains the "little projective group", i.e., $PSL(3,q)$.

[2]Ostrom only considered planes of odd order. This led Hall and Hughes to study the even order version of his theorem. Both results were included in [15, 20.9.8].

[3]On the other hand, Ostrom informs me that, for him, the idea of characterizing desarguesian planes in terms of groups came from Hall's work on cyclic difference sets [14].

[4]A simple counting argument proves this for affine planes and, more generally, when $k|v$.

[5]See [57] for historical comments on this, including the remark that one should refer to the *projective and affine versions of the Ostrom-Wagner Theorem*. In recent conversations, Wagner made the same remark to me, while Ostrom made related comments.

[6]This is really not at all a straightforward consequence: its proof occupies (at least) pp. 366-382 of [15].

Soon after the appearance of [49] Hall conjectured that any Steiner triple system (i.e., $k = 3$) with a 2-transitive group G must be a projective space over $GF(2)$ or an affine space over $GF(3)$ [16, 17]. His approach to this conjecture was to study the nature of the subspaces[7] generated by triangles by investigating the behavior of involutory automorphisms of Steiner triple systems. He showed that a 2-transitive triple system $\mathcal{D}$ must contain a subspace isomorphic to $PG(2,2)$ or $AG(2,3)$, and then used this to prove that *if G is transitive on the ordered quadruples of "independent" points, then $\mathcal{D} \cong PG(d,2)$ or $AG(d,3)$ for some d.* Later, Bruck observed that his work on commutative Moufang loops could be used to obtain the same conclusion from a weaker hypothesis: transitivity on ordered triangles [17]; [9, pp. 100-101]. In [16] there is also a discussion of further connections between designs and 2-transitive groups.

In 1958 Hall proposed the above conjecture on Steiner triple systems to Wagner. Instead of that question, Wagner [55] was led to a related one: whether all 2-transitive collineation groups of finite projective *spaces* could be determined. After a variety of special cases (e.g., [55, 52, 29, 30]) a complete determination of all such groups was finally obtained independently in [46] and [6]. The latter proof is especially combinatorial in unexpected ways, a crucial ingredient being generalized hexagons. Neither of those proofs used any group theory beyond very elementary facts about Sylow subgroups of permutation groups. On the other hand, as will be observed later, the classification of finite simple groups can be used to obtain a far stronger result.

The general study of flag-transitive designs $\mathcal{D}$ was initiated by Higman and McLaughlin [26]. They showed that *a flag-transitive automorphism group G is necessarily primitive on the points of $\mathcal{D}$*. They posed the problem of classifying the finite flag-transitive projective planes, and (by closely following ideas in [49]) proved that such planes are desarguesian if their orders are suitably restricted. Significant progress in the case of projective planes was made much later in [50, 51], where it was shown that the order is a prime power except, perhaps, if G is a Frobenius group and v is prime.

II Post-classification.

The preceding results were obtained with relatively small amounts of group theory – in many cases, little more than one would find in an undergraduate

[7] A *subspace* of a design $\mathcal{D}$ is a set X of points such that the line through any two points of X is contained in X.

course. The more recent trend has been to use information concerning simple groups: first, group-theoretic results are used in order to determine the group G, and then all designs $\mathcal{D}$ corresponding to G are determined[8]. In a sense this trend began around 1973, before the classification of finite simple groups was expected to be completed in this century, but nevertheless in the same spirit as post-classification work.

In 1973 Hering [21] announced[9] *the determination of all affine planes admitting a flag-transitive group that has a noncyclic composition factor isomorphic to an alternating group or a group of Lie type*; see §IIIA for the list of such planes. This was part of a general program of his: the study of subgroups of $GL(d,q)$ of order divisible by some prime s dividing q^d-1 but not dividing q^i-1 whenever $1 \le i < d$.[10] In the case of an affine plane, d is even, s certainly divides $q^{d/2}+1$, and hence s is a factor of the order of any flag-transitive collineation group.

The results in Hering's general program also apply to an even more important situation: determining all *transitive linear groups* (subgroups of $GL(d,q)$ transitive on the nonzero vectors of the underlying vector space). The proof of his results on such groups has appeared in [22, 23] in the generic case, when some noncyclic composition factor is an alternating group or a group of Lie type; while in [23, p. 164] Hering states that the unpublished manuscript [24] "contains the corresponding investigation for the known sporadic groups". An independent proof of the enumeration of all transitive linear groups is presented in [39, Appendix 1], also using the classification of finite simple groups. Note that such an enumeration is equivalent to the determination of all point-transitive collineation groups of finite projective spaces (and hence, in particular, contains as a very special case the much more elementary result in [46, 6] concerning 2-transitive collineation groups).

Any finite 2-transitive group either has a simple normal subgroup or an elementary abelian regular normal subgroup [5, p. 202]. The classification of finite simple groups produces a list in each of these cases. When there is a simple normal subgroup most of the work towards such a list is contained in [8]; the case in which there is an elementary abelian regular normal subgroup is equivalent to obtaining a list of all transitive linear groups, a question that

[8] For example, three 3^6-point examples in IIIA arise from the same 2-transitive group. In the case of solvable groups, numerous examples can arise from the same group, as is seen in §§IIIB,C.

[9] Apparently only a small portion [20] of the proof of this has appeared in print.

[10] Such a prime divisor of q^d-1 almost always exists [58]. The only exceptions are: $d=2$ when q is a Mersenne prime, and $d=6, q=2$.

was just discussed. Both of these are lists presented, for example, in [33].

Hall returned to the question of 2-transitive Steiner triple systems $\mathcal{D}$ in [18], and proved his earlier conjecture: *$\mathcal{D} \cong PG(d,2)$ or $AG(d,3)$ for some d.* His proof used the classification of 2-transitive groups having a simple normal subgroup, since the case in which there is an elementary abelian regular normal subgroup had already been handled in [13]. The same result concerning 2-transitive Steiner triple systems was obtained independently and at the same time by Key and Shult [38], using a method similar to Hall's: the stabilizer G_{xy} in G of two points x, y must fix a third point z (the third one on the line L through x and y); and 2-transitive groups can then be checked to see which have this property and then tested to see whether $\{x, y, z\}^G$ is the set of lines of a design. At essentially the same time the determination of all 2-transitive designs with $\lambda = 1$ and arbitrary k was given in [33], using the same basic approach but slightly more about the nature of G_{xy} in each 2-transitive group. In that more general setting, $L - \{x, y\}$ must be a union of orbits of G_{xy} and its size $k - 2$ must satisfy the conditions $k - 1 | v - 1$ and $v \geq k^2 - k + 1$; any such union is then readily examined to see if, together with $\{x, y\}$, it produces a design.[11] Also at roughly the same time a special case of the result for general k was proved in [7] for a model-theoretic application – again using the classification of 2-transitive groups.

Flag-transitive projective planes were almost determined in [35]: *such a plane is desarguesian if G is not a Frobenius group of prime degree* (the Frobenius case remains elusive, but presumably occurs only for $PG(2, 2)$ and $PG(2, 8)$). The fact that the order of such a plane can be assumed to be a prime power [50, 51] was not used. Instead, group-theoretic results were used in order to obtain a list of the odd degree primitive permutation representations of all nonsporadic nearly simple[12] groups – a result obtained independently in [40]. This result does not itself use the classification of finite simple groups, but of course for applications the sporadic groups must be handled individually. In the case of a projective plane, v is odd and G is point-primitive, and these were the only assumptions ultimately made: *a projective plane $\mathcal{D}$ admitting a point-primitive collineation group G is desarguesian except, perhaps, if G is either a regular or Frobenius group of prime degree*[13] [35]. The corresponding result concerning affine planes was

[11] Unlike [18, 38], in [33] lists of all 2-transitive groups were used, not just of those having simple normal subgroups.

[12] A group H is *nearly simple* if $S \leq H \leq \mathrm{Aut}\, S$ for a nonabelian simple group S.

[13] The regular and Frobenius cases are the same as that of a (planar) difference set in a

conjectured in [34]: *an affine plane admitting a point-primitive collineation group is a translation plane.* When v is even this is essentially contained in [49], while for odd v it was proved in [27] using the aforementioned result on primitive permutation groups of odd degree.

It is remarkable that an almost complete determination of all flag-transitive designs $\mathcal{D}$ has now been announced [4]: *either $\mathcal{D}$ must be one of the examples listed below in §IIIA, or v is a prime power and G is isomorphic to a subgroup of the 1-dimensional affine group* $A\Gamma L(1, v) = \{z \mapsto az^{\varphi} + b \mid a \neq 0,\, b \in GF(v),\, \varphi \in \mathrm{Aut}\, GF(v)\}$. (Thus, either $\mathcal{D}$ is known or G is dull!) This result, containing the main ones in [18, 38, 33, 35] as very special cases, uses far more knowledge of the structure of simple groups than in those papers. In [35] it was first necessary to reduce to the case of a nearly simple group G, which was made relatively easy due to the behavior of involutions on a projective plane. In the general case the reduction was more involved [3], and led to very different situations in which G *either* has an elementary abelian regular normal subgroup *or* is nearly simple. Both of these cases were very difficult to handle, involving numerous results concerning the maximal subgroups and the representations of the finite simple groups. The geometry of the situation is, of course, not lost; but the crucial ingredient becomes the group-theoretic analysis.

III Post-post-classification: examples.

III.A The classification in [4] consists of the following list:

projective and affine spaces;

the designs with $v = q^3 + 1$ and $k = q + 1$ associated with the natural 2-transitive permutation representations of the unitary groups $PSU(3, q)$ or the Ree groups ${}^2G_2(q)$;

designs associated with the action of $PSL(2, q), q$ even, on the set of $v = \frac{1}{2}q(q-1)$ non-secants of a conic ($k = \frac{1}{2}q$);

affine planes:

the nearfield plane of order $k = 9$ [9, p. 230], [12],

Hering's plane of order $k = 27$ [19],

group of prime order [14].

Lüneburg-Tits planes of order $k = 2^{2(2e+1)}$, one for each $e \geq 1$ [41];

two designs with $v = 729$ and $k = 9$ arising from the occurrence of $SL(2, 13)$ as a transitive linear subgroup of $GL(6, 3)$ [25];

or the flag-transitive group G lies in affine group $A\Gamma L(1, v)$, where v is a power of a prime p.

The last case on this list produces open and important problems. Unlike the situation with most of the examples listed above, in this case the same group can act flag-transitively on many different designs (see below). *Is there a hope for a complete classification of these designs?* I suspect not. My evidence is the number of examples described below, together with their somewhat wild appearance. In any event, this case involves significantly *less* group theory than in either of the preceding two sections! Instead, it depends on properties of finite fields: the set of points can and will be viewed as the set of elements of the field $F = GF(v)$. The lines will be all of the translates of the set of lines through 0. In most cases (but not all, see (1) below), when F is viewed as a vector space over a suitable subfield $K = GF(q)$, each line through 0 will be a K-subspace.

Therefore, for most of the constructions below a set of $\mathcal{S}$ of K-subspaces of F will be given, frequently obtained as distortions of a subfield $L = GF(k) = GF(q^n)$ of F. All of the translates of the members of $\mathcal{S}$ produce an incidence structure $\mathcal{D}(\mathcal{S})$, and this is a design with $\lambda = 1$ iff the following conditions hold: each member of $\mathcal{S}$ has size k, $|\mathcal{S}| = (v-1)/(k-1)$, and distinct members of $\mathcal{S}$ have only 0 in common.

For $\alpha \in F^*$ let $\tilde{\alpha}$ denote the K-linear transformation $z \mapsto \alpha z$ from F to itself.

III.B Affine planes.

Here $F = GF(v) = GF(q^{2n})$; let "bar" denote its involutory automorphism. Let $s \in F^*$ have order $(q^n + 1)(q - 1)$. Constructions (i-iii) below produce nondesarguesian flag-transitive planes of order k for every non-prime prime power k except in the cases $k = 4$ or 8 (when all affine planes are desarguesian) and $k = 2^{2^i}$ for some $i \geq 2$. These and the ones in the preceding subsection are the only known flag-transitive nondesarguesian affine planes. There is no such plane when $k = 16$ [10], but it seems likely that such planes exist when i is sufficiently large.

(i) If $v = q^{2n}$, q is even, $n > 1$ is odd, $r \in GF(q^2) - K$, $T: L \to K$ is the trace map, and $h(x) = T(x) + rx$ for $x \in L$, then

$$\mathcal{S}_r = \{s^i h(L) \mid 0 \le i \le q^n\}$$

produces an affine plane $\mathcal{D}(\mathcal{S}_r)$ *having the flag-transitive group* $\{z \mapsto s'z+w \mid s' \in \langle s \rangle, w \in F\}$ [32]. Each such plane is nondesarguesian if $q^n > 8$. *Elements* $r, r' \in GF(q^2) - K$ *produce isomorphic planes iff* $r'+1 = k(r+1)^\varphi$ *for some* $k \in K^*$, $\varphi \in \operatorname{Aut} GF(q^2)$. (See (2) below for a sketch of proofs of these assertions in a more general setting.)

However, there are no other known examples of nondesarguesian affine planes of even order having solvable flag-transitive groups. Note that each of these planes admits a *sharply* flag-transitive collineation group $\{z \mapsto s'z+w \mid s' \in \langle s^{q-1} \rangle, w \in F\}$ (since $(q^n+1, q-1) = 1$).

The above planes arose in a coding-theoretic context. They have an unusual property: there is a nonsingular alternating form on the K-space F (namely, $(x,y) \mapsto T(x\overline{y} + \overline{x}y)$) that vanishes on each member of $\mathcal{S}_r$. The possible implications of this property for the internal structure of an affine plane have remained an open question for 10 years.

(ii) If q and n are odd, $b \in F$ satisfies $\overline{b} = -b$, $1 \ne \sigma \in \operatorname{Gal}(L/K)$, and $h(x) = x + bx^\sigma$ for $x \in F$, then

$$\mathcal{S}_{b,\sigma} = \{s^i h(L) \mid 0 \le i \le q^n\}$$

produces an affine plane $\mathcal{D}(\mathcal{S}_{b,\sigma})$ *having the flag-transitive group* $\{z \mapsto s'z + w \mid s' \in \langle s \rangle, w \in F\}$ [36]. This description generalizes one due to Suetake [53]. In [36] it is shown that *if* $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{c,\tau})$ *then* $\tau = \sigma^{\pm 1}$; $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{b^{-1},\sigma^{-1}})$; *and* $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{c,\sigma})$ *iff* $c = \alpha^{1-\sigma} b^\varphi$ *for some* $\alpha \in L^*$, $\varphi \in \operatorname{Aut} F$. See (3) below for a sketch of the proofs of similar assertions in a more general setting.

An example of a plane $\mathcal{D}(\mathcal{S}_{b,\sigma})$ with $q^n = 27$ is given in [43]; see [53]. A flag-transitive plane of order $q^n = 125$ is presented in [44] that has an element inducing a transitive cyclic group on the line at infinity; presumably this plane is another instance of a plane $\mathcal{D}(\mathcal{S}_{b,\sigma})$, but this remains to be proved.

(iii) If $q^n \equiv 1 \pmod 4$, if b, L and h are as in (ii), and if $\sigma \in \operatorname{Gal}(F/K)$ is nontrivial on L, then

$$\mathcal{S}'_{b,\sigma} = \{s^{2i} h(L) \mid 0 \le i \le \tfrac{1}{2}(q^n - 1)\} \cup \{s^{2i} h(bL) \mid 0 \le i \le \tfrac{1}{2}(q^n - 1)\}$$

produces an affine plane $\mathcal{D}(\mathcal{S}'_{b,\sigma})$ having a flag-transitive group generated by $\{z \mapsto s'z + w \mid s' \in \langle s^2 \rangle,\, w \in F\}$ and $z \mapsto bz^\sigma$ [36]. This description again generalizes one due to Suetake [53]. In [36] it is shown that *if the fixed field of σ is properly contained in L then there is no cyclic collineation group transitive on the line at infinity* (such a group clearly exists in (ii)). It is also shown that *if $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{c,\tau})$ then $\tau = \sigma^{\pm 1}$*; $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{b^{-1},\sigma^{-1}})$; and $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{c,\sigma})$ iff $c = \alpha^{1-\sigma} b^\varphi$ *for some* $\alpha \in L^* \cup bL^*$, $\varphi \in \mathrm{Aut}\, F$.

When $n = 2$, F can be viewed as a 4-dimensional vector space over K. This case has received the most attention, and was the first to produce examples. In [12] two flag-transitive planes of order 25 were constructed and shown to be the only nondesarguesian flag-transitive planes of that order. The general case when $n = 2$ is found in [42, 1, 2]. *Every nondesarguesian affine plane of order q^2 admitting a flag-transitive group lying in $A\Gamma L(1, q^4)$, and for which $\mathcal{S}$ is a set of 2-dimensional $GF(q)$-spaces, is isomorphic to a plane $\mathcal{D}(\mathcal{S}'_{b,q})$.* Namely, q cannot be even [53]; while for q odd this assertion is essentially contained in [1, 2] with a different description of the sets $\mathcal{S}'_{b,q}$.

(iv) If q^n, b, L, s and h are as in (ii) (so that n is odd), if $\sigma \in \mathrm{Gal}(F/K)$ is nontrivial on L, and if $\mu \in F^*$ is such that $\mu\overline{\mu}$ is fixed by σ but is not a square in L, then

$$\mathcal{S}''_{b,\sigma} = \{s^{2i} h(L) \mid 0 \le i \le \tfrac{1}{2}(q^n - 1)\} \cup \{\mu s^{2i} \overline{h(L)} \mid 0 \le i \le \tfrac{1}{2}(q^n - 1)\}$$

produces an affine plane $\mathcal{D}(\mathcal{S}''_{b,\sigma})$ having a flag-transitive group generated by $\{z \mapsto s'z + w \mid s' \in \langle s^2 \rangle,\, w \in F\}$ and $z \mapsto \mu\overline{z}$ [37] (compare (6) below). This construction does not depend on the choice of μ, and produces the same planes as in (iii) when $q^n \equiv 1 \pmod 4$. If the fixed field of σ is properly contained in L then there is no cyclic collineation group transitive on the line at infinity. There is again an isomorphism criterion similar to those seen earlier.

There is an affine plane of order 27 constructed in [45] having a flag-transitive group lying in $A\Gamma L(1, 27^2)$ and having no element inducing a transitive cyclic group on the line at infinity. I suspect that this plane is one of those constructed in (iv).

III.C Designs that are not planes.

The remainder of this paper is concerned with the construction of designs, other than planes, that admit flag-transitive automorphism groups contained in $A\Gamma L(1, v)$. This question has received surprisingly little attention. The only known construction methods depend heavily on delicate arithmetic questions involving finite fields. The construction below in (1) makes this very plain, while those in (2) and (6) seem to have too many potential variations. Consequently, a complete classification seems doubtful.

Remark on automorphism groups. Assume that $\mathcal{D} = \mathcal{D}(\mathcal{S})$ is as above, and that $\mathcal{D}$ is not an affine space or plane. Since $\mathrm{Aut}\,\mathcal{D}$ is flag-transitive, $\mathrm{Aut}\,\mathcal{D} \leq A\Gamma L(1,v)$ by [4]. This is useful for the discussion of the question of isomorphisms among the designs constructed below. (Note, however, that much less than the full strength of [4] is needed for the determination of $\mathrm{Aut}\,\mathcal{D}$ – and even less is needed for the applications to isomorphism questions.) In the case of affine planes the situation is noticeably simpler: all that will actually be required is that the group of translations $z \mapsto z + w$ is normal in $\mathrm{Aut}\,\mathcal{D}$, and this is standard in the case of an affine plane. It would be interesting to have a simple, purely geometric, canonical description of the group of translations for the designs constructed below. (Here is a simple description that is not quite geometric enough and only works when $q > 2$. For any point x let $G_{(x)}$ denote the group of all automorphisms of $\mathcal{D}(\mathcal{S})$ fixing all lines through x. If $x \neq y$, $1 \neq g \in G_{(x)}$ and $1 \neq h \in G_{(y)}$, then $g^{-1}h^{-1}gh$ is a nontrivial translation; moreover, each nontrivial translation arises in this manner.)

Remark on isomorphism testing. The following is a strategy for determining whether or not two of these designs $\mathcal{D} = \mathcal{D}(\mathcal{S})$ and $\mathcal{D}'$ are isomorphic (compare [36]). WLOG $\mathcal{D}$ and $\mathcal{D}'$ have the same set of points and even the same automorphism group. If $g{:}\mathcal{D} \to \mathcal{D}'$ is an isomorphism then $(\mathrm{Aut}\,\mathcal{D})^g = \mathrm{Aut}\,\mathcal{D}' = \mathrm{Aut}\,\mathcal{D}$. WLOG g fixes 0. Assume that $\mathcal{D}$ is not an affine space or plane. By the preceding paragraph $\mathrm{Aut}\,\mathcal{D} \leq A\Gamma L(1,v)$, and hence g conjugates the unique minimal normal subgroup $\{z \mapsto z + w \mid w \in F\}$ of $\mathrm{Aut}\,\mathcal{D}$ to itself.

There is a subgroup $\langle s_0 \rangle$ of F^* of prime order that lies in no proper subfield of F (by [58]; cf. [10]). Then $|s_0|$ divides $(v-1)/(k-1)$ and hence also $|\mathrm{Aut}\,\mathcal{D}|$, so that $s_0 \in \mathrm{Aut}\,\mathcal{D}$ in each case below. Since $GL(F)$ has cyclic Sylow $|s_0|$-subgroups, g can be adjusted in order to have it normalize $\langle s_0 \rangle$.[14] Then g has the form $z \mapsto \alpha z^{\varphi}$ with $\alpha \in F^*$, $\varphi \in \mathrm{Aut}\,F$. After a further adjustment of g, WLOG $\varphi = 1$. Now it is just a question of calculating with the specific definitions of $\mathcal{D}$ and $\mathcal{D}'$ in order to see whether some such α can exist. An example of such a calculation is given in (3).

(1) Assume that $k \geq 3$ and $k(k-1) | v - 1$. Let B be the subgroup of

[14] In fact, this is automatic since it is already known that $\mathrm{Aut}\,\mathcal{D} \leq A\Gamma L(1,v)$.

F^* of order k. For some choices of F and B it is known that the set of all images of B under the transformations $z \mapsto a^{k-1}z + w$, $a \in F^*$, $w \in F$, form a design with $\lambda = 1$. (Examples where this phenomenon occurs are "Netto triple systems" having $k = 3$ and $v \equiv 7 \pmod{12}$. For other examples, see [4].)

The conditions required for this construction to produce a design are extremely delicate. For example, when $v - 1 = k(k-1)$ the construction would produce a flag-transitive projective plane (and does so when $v = 7$ or 73) – and every *sharply* flag-transitive projective plane arises in this manner.

(2) It is occasionally possible to impose the structure of a flag-transitive design on each of the lines of a flag-transitive design – at least when the latter design is an affine space.[15]

(2a) Let $F = GF(v) = GF(q^{2nd}) \supset E = GF(q^{2n})$, and let

$$G = \{z \mapsto tz + w \mid w \in F,\ t \in F^* \text{ and } |t| \text{ divides } (q^{2nd} - 1)(q-1)/(q^n - 1)\}.$$

Then $|G_E| = q^{2n}(q^n + 1)(q - 1)$ iff $((q^{2nd} - 1)(q-1)/(q^n - 1), q^{2n} - 1) = (q^n + 1)(q - 1)$; and this occurs iff $(d, (q^n - 1)/(q - 1)) = 1$. In this situation, E^G is the set of lines of $AG(d, q^{2n})$. Note that G_E acts faithfully on E.

Now equip E with the structure of a nondesarguesian affine plane admitting G_E as a flag-transitive group (examples have been given above in (i) and (ii)). Temporarily call the lines of the affine plane E "sublines". Clearly, each subline is contained in just one line, namely, E. Consequently, the images of the sublines under G produce a design $\mathcal{D}$ admitting G as a flag-transitive group.

Once again there are arithmetic conditions involved in this construction, but there is an additional variable: the specific affine plane inserted into E. *Claim: The subspace E of $\mathcal{D}$ can be recovered from $\mathcal{D}$.* For, $\mathrm{Aut}\,\mathcal{D} \le A\Gamma L(1, q^{2nd})$, so that there is only one subgroup of $\mathrm{Aut}\,\mathcal{D}$ isomorphic to G; and all subgroups isomorphic to G_E are conjugate. But E is a subspace of $\mathcal{D}$ and is invariant under the group G_E, while G_E has exactly one point-orbit of size $|E|$. This proves the Claim. It follows that *nonisomorphic planes produce nonisomorphic designs.*

This simple approach also can be applied to examples in (1), but it merely gives another design arising in (1).

(2b) The above construction can be varied somewhat, using a slightly different group G so as to make G_E coincide with the flag-transitive group

[15] In [4] this is called my "inflation trick". However, "free lunch trick" would be a more appropriate description.

arising in (iii). For example, let $F = GF(v) = GF(q^{2nd}) \supset E = GF(q^{2n}) \supset L = GF(q^n)$, where $q^n \equiv 1 \pmod 4$; let $\sigma \in \mathrm{Gal}(F/K)$ be nontrivial on L; let $b \in E$ with $b^{q^n} = -b$; and let G be generated by $\{z \mapsto tz + w \mid w \in F, t \in F^*$ and $|t|$ divides $(q^{2nd} - 1)(q - 1)/2(q^n - 1)\}$ and $z \mapsto bz^\sigma$. Then G_E is isomorphic to the group in (iii) iff $((q^{2nd} - 1)(q - 1)/2(q^n - 1), q^{2n} - 1) = (q^n + 1)(q - 1)/2$; and this occurs if, for example, n is odd and $(d, (q^n - 1)/(q - 1)) = 1$. In this situation, E^G is the set of lines of $AG(d, q^{2n})$, and any plane from (iii) can be inserted into E exactly as in (2a). Once again, nonisomorphic planes from (iii) produce nonisomorphic designs, each of which is not isomorphic to any constructed in (2a) if the fixed field of σ is properly contained in $GF(q^n)$.

(3) The construction of planes in (i) can be generalized as follows.

Consider a prime p, powers $q > 1$ and $f > 1$ of p, and an integer $n > 1$ such that $p \nmid n$. Let

$$F = GF(v) = GF(q^{fn}),\ L = GF(q^n),\ F_0 = GF(q^f) \text{ and } K = GF(q).$$

Assume that $((q^n - 1)/(q - 1), f - 1) = 1$ (if $f \le q$ this states that $(n, f - 1) = 1$). There are norm maps $N: F \to L$ and $N_0: F_0 \to K$ and trace maps $T: L \to K$ and $T_0: F_0 \to K$. Let $s \in F$ have order $(q^{fn} - 1)(q - 1)/(q^n - 1)$, so $N(s) \in K$.

Let $r \in F_0 - K$ be such that the polynomial identity $N_0(x + r) = x^f + T_0(N_0(r)/r)x + N_0(r)$ holds in $F[x]$ (e.g., if $f = 3$ then this condition is simply that $T_0(r) = 0$). If $u \in L$ then $N(u + r) = \Pi_{i=0}^{f-1}(u + r^{q^{ni}}) = u^f + T_0(N_0(r)/r)u^{f-1} + N_0(r)$. (For, since $(f, n) = 1$, $\{q^{ni} \mid 0 \le i < f\}$ and $\{q^i \mid 0 \le i < f\}$ both induce all the elements of $\mathrm{Gal}(F_0/K)$.)

Claim: $\mathcal{S}_r = \{s^i(\mathrm{Ker}\, T + rK) \mid 0 \le i < (q^{fn} - 1)/(q^n - 1)\}$ *produces a design* $\mathcal{D}(\mathcal{S}_r)$ *with* $\lambda = 1$ *and flag-transitive group* $\{z \mapsto s'z + w \mid s' \in \langle s \rangle, w \in F\}$. (When $f = 2$ these designs are just the planes in (i) with $r + 1$ in place of r: in that case $T(x) + rx = (T(x) + x) + (r + 1)x$ with $T(x) + x \in \mathrm{Ker}\, T$.)

To see this, first note that $\dim_K(\mathrm{Ker}\, T + rK) = n$ (since $\mathrm{Ker}\, T \subset L$, $r \in F_0 - K$ and $F_0 \cap L = K$). Consequently, it suffices to show that the conditions $0 \ne u + rk = s^i(u' + rk')$, where $u, u' \in \mathrm{Ker}\, T$, $k, k' \in K$, and $0 < i < (q^{fn} - 1)/(q^n - 1)$, lead to a contradiction. Since $u, k \in L$, applying N yields

$$u^f + T_0(N_0(r)/r)uk^{f-1} + N_0(r)k^f = N(s^i)\{u'^f + T_0(N_0(r)/r)u'k'^{f-1} + N_0(r)k'^f\}$$

$$(u^f - N(s^i)u'^f) + T_0(N_0(r)/r)(uk^{f-1} - N(s^i)u'k'^{f-1}) + N_0(r)(k^f - N(s^i)k'^f) = 0$$

where $N(s^i) \in K$. Now apply T: $n(N_0(r)(k^f - N(s^i)k'^f)) = 0$ (since $(\operatorname{Ker} T)^f = \operatorname{Ker} T$ and $T(x) = nx$ for $x \in K$). However, $p \not| n$. Thus, $k = yk'$ where $y^f = N(s^i)$ and $y \in K$. It follows that

$$
\begin{aligned}
&(u - yu')^f + T_0(N_0(r)/r(y^{f-1}k'^{f-1}(u - yu') \\
&\quad = (u^f - y^f u'^f) + T_0(N_0(r)/r)(uy^{f-1}k'^{f-1} - y^f u'k'^{f-1}) = 0.
\end{aligned}
$$

Now $(u - yu')^{f-1}$ is 0 or $-T_0(N_0(r)/r)y^{f-1}k'^{f-1}$ and so lies in K. But $((q^n - 1)/(q - 1), f - 1) = 1$ by hypothesis, so that $u - yu' \in K \cap \operatorname{Ker} T = 0$. Now our original equation $u + rk = s^i(u' + rk')$ asserts that $y = s^i$, so that $s^i \in K$. Then $i(q-1) \equiv 0 \pmod{(q^{fn} - 1)(q - 1)/(q^n - 1)}$, which contradicts the fact that $0 < i < (q^{fn} - 1)/(q^n - 1)$. This proves the Claim.

Claim: When $n > 2$, $\mathcal{D}(\mathcal{S}_r) \cong \mathcal{D}(\mathcal{Y}_{r'})$ iff $r' = \alpha r^{\varphi}$ for some $\alpha \in K^*$, $\varphi \in$ $\operatorname{Aut} GF(q^f)$. For, one direction is easy. Suppose that $\mathcal{D}(\mathcal{S}_r) \cong \mathcal{D}(\mathcal{S}_{r'})$. By the remark on isomorphism testing, WLOG an isomorphism g has the form $z \mapsto \alpha z$ for some $\alpha \in F^*$. It suffices to show that $rK = r'K$, so assume that $rK \neq r'K$. By replacing g by $g\tilde{s}^i$ for some i, WLOG $\alpha(\operatorname{Ker} T + rK) = \operatorname{Ker} T + r'K$. Then $\dim_K(\alpha \operatorname{Ker} T \cap \operatorname{Ker} T) = (n - 1) - 1 > 0$, so that $\alpha \in L$. Also, $r' = \alpha(u + rk)$ for some $u \in \operatorname{Ker} T$, $k \in K$. Then $r' = (\alpha u)^{q^{fi}} + r(\alpha k)^{q^{fi}}$ for $0 \le i < n$, where $\alpha u, \alpha k \in L$. Adding these equations yields $nr' = k_1 + rk_2$, where $k_1, k_2 \in K$ and $nr' \neq 0$. Then $0 = (k_1 - n\alpha u) + r(k_2 - n\alpha k)$, while $r \notin L$, so that $n\alpha k = k_2 \in K$. Then $\alpha \in K$, $\operatorname{Ker} T + rK = \operatorname{Ker} T + r'K$, and hence (by intersecting with F_0) $rK = r'K$, which is not the case. This proves the Claim. It is not difficult to check that the Claim also holds when $n = 2$.

Claim: $\mathcal{D}(\mathcal{S}_r)$ is not an affine space $AG(f, q^n)$. For, suppose it is. Note that the subspaces s^iL, $0 \le i < (q^{fn} - 1)/(q^n - 1)$, are the lines through 0 of an $AG(f, q^n)$. By the remark on isomorphism testing, an isomorphism g may be assumed to have the form $z \mapsto \alpha z$ for some $\alpha \in F^*$. Multiply g by a power of $\tilde{s}$ in order to have $(\operatorname{Ker} T + rK)^g = L$. Then $\alpha(\operatorname{Ker} T + rk) = L$, so that $\alpha \in L$ and $\alpha r \in L$, where $r \notin L$. This proves the Claim.

As in (i), each of these designs admits a *sharply* flag-transitive automorphism group $\{z \mapsto s'z + w \mid s' \in \langle s^{q-1} \rangle, w \in F\}$ since $((q^{fn} - 1)/(q^n - 1), q - 1) = (f, q - 1) = 1)$.

Are many designs obtained in this manner? Do many elements $r \in GF(q^f)$ satisfy the identity $N_0(x + r) = x^f + T_0(N_0(r)/r)x + N_0(r)$? I do not know any answer. Here is one easy example: if $f = p$ then $x^p - x - c$, $c \in GF(p)^*$, is irreducible over $GF(p^e) = GF(q)$ whenever $p \not| e$ [11, p. 29], and hence has a root $r \in GF(q^p)$. Then r behaves as desired – but different choices of c produce elements r in the same coset of $GF(p)^*$ in $GF(q^p)^*$, so up to isomorphism only one design is obtained in this manner for each choice of p and e.

(4) The construction in (ii) also can be generalized as follows.

Let $m, n > 1$, $(m, n) = 1$, $m|q-1$, and $F = GF(q^{mn}) \supset L = GF(q^n) \supset K = GF(q)$; let $\omega \in K^*$ have order m and $s \in F^*$ have order $(q^{mn}-1)(q-1)/(q^n-1)$; let $1 \neq \sigma \in \mathrm{Gal}(L/K)$ and let $\theta \in \mathrm{Gal}(F/K)$ be the q^nth power map; and let $b \in F$ be such that $b^\theta = \omega b$. Write $h(x) = x - bx^\sigma$ for $x \in F$.

Claim: $\mathcal{S}_{b,\sigma} = \{s^i h(L) \mid 0 \le i < (q^{mn}-1)/(q^n-1)\}$ *produces a design* $\mathcal{D}(\mathcal{S}_{b,\sigma})$ *with* $\lambda = 1$ *and flag-transitive group* $\{z \mapsto s'z + w \mid s' \in \langle s \rangle, w \in F\}$. (When $m = 2$ this is the same construction as in (ii).)

Let T and N be the trace and norm maps $F \to L$. The identity $\Pi_{i=0}^{m-1}(x - b^{\theta^i}) = x^m - b^m$ holds in $F[x]$ since the roots on both sides are $\omega^i b$, $0 \le i < m$. In particular, $T(b) = 0$ and $N(b) = -(-b)^m$. Also, for any $u, u' \in L$, $N(u - bu') = \Pi_{i=0}^{m-1}(u - b^{\theta^i}u') = u^m - b^m u'^m$.

Now we return to the Claim. First of all, h is injective: if $x - bx^\sigma = 0$ with $x \in L$ then $mx = T(x) = T(bx^\sigma) = T(b)x^\sigma = 0$, so that $x = 0$ since $m|q-1$. Next, $\langle \tilde{s} \rangle$ is transitive on $\mathcal{S}_{b,\sigma}$, so it suffices to assume that $0 \neq h(x) = s^i h(y) \in h(L) \cap s^i h(L)$, where $x, y \in L$ and $0 < i < (q^{nm}-1)/(q^n-1)$, and derive a contradiction. Apply N: $x^m - b^m x^{\sigma m} = N(s^i)(y^m - b^m y^{\sigma m})$, so that $x^m - N(s^i)y^m = b^m(x^m - N(s^i)y^m)^\sigma$ since $N(s) \in K$ is fixed by σ. If $x^m - N(s^i)y^m \neq 0$ then $b^m(x^m - N(s^i)y^m)^{\sigma-1} = 1$, where $m|\sigma - 1$ by hypothesis, so that $b(x^m - N(s^i)y^m)^{(\sigma-1)/m} \in \langle \omega \rangle \subset K$, whereas $b \notin L$. Thus $x^m - N(s^i)y = 0$. Now $(x/y)^m = N(s^i) \in K$, but $(m, (q^n-1)/(q-1))|(q-1, m, n) = 1$, so $x/y = k \in K$. It follows that $ky - b(ky)^\sigma = s^i(y - by^\sigma)$. Then $k = s^i$, so that $s^{i(q-1)} = 1$, which is impossible since $|s| = (q^{mn}-1)(q-1)/(q^n-1)$ and $0 < i < (q^{mn}-1)/(q^n-1)$. This proves the Claim.

Moreover, if $m > 2$ then the following are not difficult to prove (as in [36]): $\mathcal{D}(\mathcal{S}_{b,\sigma})$ *is not an affine space*, $\mathcal{D}(\mathcal{S}_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}_{b,\tau})$ *iff* $\sigma = \tau$ *and* $c = \alpha^{1-\sigma} b^\varphi$ *for some* $\alpha \in L^*, \varphi \in \mathrm{Aut}\, F$.

(5) The construction in (iii) also can be generalized as follows.

Again let $F = GF(q^{mn}) \supset L = GF(q^n) \supset K = GF(q)$, where $m|q-1$; and assume that $q^n \equiv 1 \pmod{2m}$ if m is even. Let $\sigma \in \mathrm{Gal}(F/K)$, and assume that σ acts nontrivially on L. Let θ, ω, s, b, h and N be as in (4). Then

$$\mathcal{S}'_{b,\sigma} = \{s^{mi} h(b^j L) \mid 0 \le j < m,\ 0 \le i < (q^{mn}-1)/m(q^n-1)\}$$

produces a design $\mathcal{D}(\mathcal{S}_{b,\sigma})$ *with* $\lambda = 1$ *and flag-transitive* group generated by $\{z \mapsto s'z + w \mid s' \in \langle s^m \rangle, w \in F\}$ and $z \mapsto bz^\sigma$. (When $m = 2$ this is the same as (iii).)

For, once again $\mathcal{S}'_{b,\sigma}$ consists of n-dimensional K-spaces. Clearly $\langle \bar{s}^m \rangle$ has just m orbits on $\mathcal{S}'_{b,\sigma}$. The transformation $z \mapsto bz^\sigma$ sends $s^{mi}([b^j x] - b[b^{j\sigma} x^\sigma])$ to $s^{mi\sigma}([bb^{j\sigma} x^\sigma] - b[bb^{j\sigma} x^\sigma]^\sigma)$, where $bb^{j\sigma} \in b^{j+1}L$ since $(b^\sigma)^\theta = (\omega b)^\sigma = \omega b^\sigma$; and $h(b^m L) = h(L)$ since $b^m \in L$. This proves flag-transitivity.

Consequently, it suffices to consider the possibility that $h(x) = s^{mi} h(b^j y)$ for some $x, y \in L^*$ and some i, j such that $0 \le i < (q^{nm} - 1)/m(q^n - 1)$ and $0 \le j < m$. First note that

$$\begin{aligned} N(h(b^j y)) &= N(b^j y - bb^{j\sigma} y^\sigma) = N(b^j) N(y - [bb^{j\sigma}/b^j] y^\sigma) \\ &= (-1(-b)^m)^j (y^m - [bb^{j\sigma}/b^j]^m y^{\sigma m}) \end{aligned}$$

since $[bb^{j\sigma}/b^j]^\theta = \omega[bb^{j\sigma}/b^j]$.

Now $x^m - b^m x^{\sigma m} = N(s^{mi})(-1)^{j(m-1)}(b^{jm} y^m - b^m b^{jm\sigma} y^{\sigma m})$, so that

$$x^m - (-1)^{j(m-1)} N(s^{mi})(b^j y)^m = b^m \{x^m - (-1)^{j(m-1)} N(s^{mi})(b^j y)^m\}^\sigma.$$

If $x^m - (-1)^{j(m-1)} N(s^{mi})(b^j y)^m \ne 0$ then $b^m(\{x^m - (-1)^{j(m-1)} N(s^{mi})(b^j y)^m\}^{(\sigma-1)/m})^m = 1$, and hence $b\{x^m - (-1)^{j(m-1)} N(s^{mi})(b^j y)^m\}^{(\sigma-1)/m} \in K$ (since K contains all mth roots of 1), whereas $x, y, b^m \in L$ and $b \notin L$.

Thus, $x^m = (-1)^{j(m-1)} N(s^{mi})(b^j y)^m$. But $(-1)^{j(m-1)} = \ell^m$ for some $\ell \in L$ (as $q^n \equiv 1 \pmod{2m}$ if $m-1$ is odd; use $\ell = 1$ if $j(m-1)$ is even), so that $(x/\ell N(s^i) b^j y)^m = 1$ and hence $x/\ell N(s^i) b^j y \in K$. Then $b^j \in L$ while $(b^j)^\theta = b^j \omega^j$, which successively imply: $j = 0$, $\ell = 1$, $x = ky$ with $k \in K$, $ky - bky^\sigma = s^{mi}(b^j y - b(b^j y)^\sigma) = s^{mi}(y - by^\sigma) \ne 0$, $k = s^{mi}$, and $mi(q-1) \equiv 0 \pmod{(q^{mn}-1)(q-1)/(q^n-1)}$. Since $0 \le i < (q^{mn} - 1)/m(q^n - 1)$, it follows that $i = j = 0$ and hence $h(L) = s^{mi} h(b^j L)$.

Consequently, $\mathcal{D}(\mathcal{S}'_{b,\sigma})$ is a design with $\lambda = 1$. Clearly, the conditions on the parameters q, n, m are slightly different from those in (4) when m is even, so some examples here do not appear in (4). However, there is a more significant difference between (4) and (5): *if the fixed field of* σ *is properly contained in* L, *then* $\mathcal{D}(\mathcal{S}'_{b,\sigma})$ *has no automorphism fixing* 0 *and cyclically permuting* $\mathcal{S}'_{b,\sigma}$. Also, *for* $m > 2$, if $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{c,\tau})$ *then* $\sigma = \tau$ *on* L *and* $c = \alpha^{1-\sigma} b^\varphi$ *for some* $\alpha \in \langle b \rangle L^*$, $\varphi \in \mathrm{Aut}\, F$; *while if* b *and* c *are related in this manner then* $\mathcal{D}(\mathcal{S}'_{b,\sigma}) \cong \mathcal{D}(\mathcal{S}'_{c,\sigma})$. These are not hard to prove; see [36], where questions of this sort are examined in overly minute detail for the planes (iii).

(6) In order to generalize the construction in (iv), again let $F = GF(q^{mn}) \supset L = GF(q^n) \supset K = GF(q)$ with $m | q-1$, and let θ, ω, s, b, h and N be as in (4). Lastly, let $\mu \in F^*$ be such that $N(\mu)$ is fixed by σ but $N(\mu)^j$ is not an mth power in L whenever $1 \le j < m$. Then

$$\mathcal{S}''_{b,\sigma} = \{s^{mi}\mu^{1+\theta+\cdots+\theta^{j-1}}h(L)^{\theta^j} \mid 1 \le j \le m, 0 \le i < (q^{mn}-1)/m(q^n-1)\}$$

produces a design $\mathcal{D}(\mathcal{S}''_{b,\sigma})$ *with* $\lambda = 1$ *and flag-transitive group* generated by $\{z \mapsto s'z + w \mid s' \in \langle s^m \rangle, w \in F\}$ and $z \mapsto \mu z^\theta$. For, once again we find that $\mathcal{S}''_{b,\sigma}$ consists of n-dimensional subspaces, $\mu^{1+\theta+\cdots+\theta^{m-1}}h(L)^{\theta^m} = h(L)$ since $\mu^{1+\theta+\cdots+\theta^{m-1}}h(L)^{\theta^m} \in L$, and $\mu(\mu^{1+\theta+\cdots+\theta^{j-1}}h(L)^{\theta^j})^\theta = \mu^{1+\theta+\cdots+\theta^j}h(L)^{\theta^{j+1}}$. This implies flag-transitivity, and leaves us to consider the possibility that $h(x) = s^{mi}\mu' h(y)^{\theta^j}$ for some i, j, where $\mu' = \mu^{1+\theta+\cdots+\theta^{j-1}}$. This time we find that

$$x^m - b^m x^{\sigma m} = N(s^{mi})N(\mu')(y^m - b^m y^{\sigma m})^{\theta^j} = N(s^{mi})N(\mu')(y^m - b^m y^{\sigma m})$$

since $(b^m)^{\theta^j} = b^m$. Then $x^m - N(s^{mi})N(\mu')y^m = b^m(x^m - N(s^{mi})N(\mu')y^m)^\sigma$ since $N(\mu') = N(\mu)^j$ is assumed to be fixed by σ. As usual, it follows that $x^m = N(s^{mi})N(\mu')y^m$, so that $N(\mu') = N(\mu)^j$ is an mth power in L. By hypothesis, $j = m$. Now $(x/N(s^i)N(\mu)y)^m = 1$, and as in (5) we deduce successively that $x = kN(s^i)N(\mu)y$ with $k \in K$, $kN(s^i)N(\mu)y - bkN(s^i)N(\mu)y^\sigma = s^{mi}N(\mu)(y - by^\sigma)$ since $\mu' = N(\mu)$ is fixed by σ while $N(s) \in K$, $s^{mi} = kN(s^i) \in K$, $i = 0$, and hence $s^{mi}\mu' h(L)^{\theta^m} = h(L)$. Consequently, $\mathcal{D}(\mathcal{S}''_{b,\sigma})$ is a design with $\lambda = 1$, as required. However, the isomorphisms among these designs, and the extent to which they do not already arise from previous constructions, remain to be studied.

(7) Now the examples in (2) can be generalized further using (3-6). The analogue of (2a) is as follows. Let $m > 2$, $n > 1$, $F = GF(q^{mnd}) \supset E = GF(q^{mn})$, and let $G = \{z \mapsto tz + w \mid w \in F, t \in F^* \text{ and } |t| \text{ divides } (q^{mnd}-1)(q-1)/(q^n-1)\}$. Then $|G_E| = q^{mn}(q^{mn}-1)(q-1)/(q^n-1)$ iff $((q^{mnd}-1)(q-1)/(q^n-1), q^{mn}-1) = (q^{mn}-1)(q-1)/(q^n-1)$; and once again this occurs if $(d, (q^n-1)/(q-1)) = 1$. In this situation, E^G is the set of lines of $AG(d, q^{mn})$.

Now equip E with the structure of a design with $\lambda = 1$ admitting G_E as a flag-transitive group (e.g., one of those given above); temporarily call the lines of E "sublines". The images of the sublines under G again produce a design $\mathcal{D}$ admitting G as a flag-transitive group. As before, if nonisomorphic flag-transitive designs are inserted into E then nonisomorphic flag-transitive designs $\mathcal{D}$ are obtained.

There is a similar version of (2b).

It will come as no surprise that all of designs obtained in this manner from those constructed earlier can easily be described directly.

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