Symplectic spreads from twisted fields

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1 Introduction

A *symplectic spread* of $PG(2n+1,q)$ is a spread of the symplectic polar space $W(2n+1,q)$ defined by a nonsingular alternating bilinear form on a $(2n+2)-$ dimensional vector space over $GF(q)$, i.e., a set of $q^{n+1}+1$ pairwise disjoint maximal totally isotropic subspaces. Note that a symplectic spread of $PG(3,q)$ is equivalent, under the Klein correspondence, to an ovoid of the quadric $Q(4,q)$.

For any q and n, a regular spread of $PG(2n+1,q)$ provides an example of a symplectic spread ([4], [15], [16]). Many examples of symplectic spreads of $PG(2n+1,q)$ are known when q and n are even ([6], [7]). There are also examples having $q=2^{2e+1}$ and $n=1$ ([13] Chap. 4), and these lead to further examples having $q=2^{2e+1}$ and n any odd integer by the method indicated at the start of §2. When q is odd, relatively few examples of nonregular spreads of $W(2n+1,q)$ have been published; and each of them arises from a translation plane of dimension 2 over its kernel ([8], [14]; cf. §2 below).

In this note we prove that the spreads defined by some of Albert's twisted fields [1] are symplectic; we also observe that this is true for the spread of the Hering plane of order 27 [5]. Moreover, we will see that the symplectic spreads of $W(5,q)$ associated with commutative twisted fields of dimension 3 over $GF(q)$, q odd, arise from some partial ovoids and partial spreads of $W(7,q)$ related to a description of the generalized hexagon ${}^3D_4(q)$ in $PG(9,q)$ (see [12]).

As we write this paper, it appears that every known finite translation plane arising from a symplectic spread has the property that its dimension over its kernel is 2 or odd.

2 Known examples

Let $F = GF(q)$ and $V = F^{2n+2}$. Let F have degree t over a subfield $K = GF(s)$, and let tr denote the trace map $F \to K$. If $(\ \ ,\ \ )$ is a nonsingular alternating bilinear form on V, then $[x, y] = tr(x, y)$ (with $x, y \in V$) defines a nonsingular alternating bilinear form on the $K-$space V. Any symplectic spread Σ of the polar space $W(2n+1, q)$ associated with the bilinear form $(\ \ ,\ \ )$ can also be regarded as a symplectic spread Σ^* of $W(2t(n+1)-1, s)$ with respect to $[\ \ ,\ \ ]$ (see [6] and [7]). The translation planes associated with Σ and Σ^* are identical. The simplest example of this construction arises when $n = 0$ and Σ is a regular spread.

For q odd, the only published examples of symplectic spreads are those constructed via the above procedure starting from one of the following:

a) the regular spread of $PG(1, q)$;

b) the spread of $PG(3, q)$ associated with the Knuth semifield defined by $(a, b) \circ (c, d) = (ac + db^\sigma m, bc + ad)$, where m is a nonsquare in $F = GF(q)$ and σ is a nontrivial automorphism of F (see [8] §5);

c) the spread of $PG(3, 3^{2e+1})$ represented, on the Klein quadric, by an ovoid of $Q(4, 3^{2e+1})$ constructed in [8] using the Ree group ${}^2G_2(3^{2e+1})$;

d) the spreads of $PG(3, 3^e)$, $e \geq 3$, represented, on the Klein quadric, by the ovoid of $Q(4, 3^e)$ constructed by Payne and Thas in [14].

3 Twisted fields

Take any finite field E of odd order, and a nontrivial automorphism ρ such that $-1 \notin E^{\rho-1}$. Note that a nontrivial automorphism ρ behaves in this manner if and only if E has odd degree over the fixed field E_ρ of ρ.

Let $F = GF(q)$ be a subfield of E_ρ, and set $n = [E : F]$. Then the subspaces

$$\{ (0, y) \mid y \in E \}$$
$$\{ (x, mx^{\rho^{-1}} + m^\rho x^\rho \mid x \in E \} \quad (m \in E)$$

of the $F-$space E^2 form a spread Σ of the corresponding projective space $PG(2n-1, q)$; namely, $mx^{\rho^{-1}} + m^\rho x^\rho = 0$ implies that $mx = 0$ since $-1 \notin E^{\rho-1}$.

Moreover, this spread is symplectic with respect to the following alternating form:

$$((x_1,y_1),(x_2,y_2)) = tr(x_1y_2 - x_2y_1),$$

where tr denotes the trace map $E \to F$. Namely, all of the subspaces under consideration are totally isotropic:

$$tr(x\{my^{\rho^{-1}} + m^\rho y^\rho\}) - tr(y\{mx^{\rho^{-1}} + m^\rho x^\rho\})$$
$$= tr(mxy^{\rho^{-1}}) - tr(m^\rho x^\rho y) + tr(m^\rho xy^\rho) - tr(mx^{\rho^{-1}}y)$$
$$= tr(m^\rho x^\rho y) - tr(m^\rho x^\rho y) + tr(m^\rho xy^\rho) - tr(m^\rho xy^\rho) = 0.$$

This spread arises from a semifield. A presemifield for it is $(E,+,\circ)$, where $m \circ x = mx^{\rho^{-1}} + m^\rho x^\rho$ for $m,x \in E$; and this is isotopic to the presemifield defined by $m * x = mx + m^\rho x^{\rho^2}$, which produces one of Albert's twisted fields [1]. Note that E_ρ is the kernel of the resulting translation plane.

Remark 1. The following collineations of the above twisted field plane are symplectic transformations:

$$(x,y) \mapsto (x, y + u \circ x)$$
$$(x,y) \mapsto (ax, a^{-1}y)$$

for all $u \in E$, $0 \neq a \in E$. Also, for any $\theta \in \mathrm{Aut}E$ the collineation $(x,y) \mapsto (x^\theta, y^\theta)$ preserves the alternating form up to the field automorphism θ, and yields a collineation. The three types of mappings just described generate the full translation complement of the plane [1].

Remark 2. One could try to generalize these examples as follows. Start with two automorphisms $\rho \neq 1$ and σ of E, and nonzero elements $a, \ell \in E$; again assume that $F \subseteq E_\rho$ and $-1 \notin E^{\rho-1}$. Then the $F-$subspaces

$$\{\,(0,y) \mid y \in E\,\}$$
$$\{\,(x, \ell^{\rho-1}a^\rho m^{\sigma\rho}x^\rho + am^\sigma x^{\rho^{-1}} \mid x \in E\,\} \quad (m \in E)$$

of E^2 form a spread that is symplectic with respect to the alternating form $((x_1,y_1),(x_2,y_2)) = tr(\ell(x_1y_2 - x_2y_1))$, where tr is as before. However, this spread is equivalent to the previous one: since

$$\ell^{\rho-1}a^\rho m^{\sigma\rho}x^\rho + am^\sigma x^{\rho^{-1}} = \ell^{-1}\{(\ell am^\sigma)(x^{\rho^{-1}}) + (\ell am^\sigma)^\rho (x^{\rho^{-1}})^{\rho^2}\},$$

we obtain the same twisted field as before.

Remark 3. If ρ has order 3, then the above presemifield is isotopic to the one defined by $m \cdot x = m^\rho x^{\rho^2} + x^\rho m^{\rho^2}$, which produces a commutative twisted field of dimension 3 over its centre (compare §5).

Remark 4. As F is a subfield of the centre E_ρ of the twisted field, Σ is S_∞-regular[1] by [10] Teorema 5.

[1]Let Σ be a spread of any projective space. Let $A, B \in \Sigma$ with $A \neq B$. We recall that

4 The Hering plane of order 27

Let $F = GF(3)$, and define an alternating form on $V = F^6$ by $((x_i),(y_i)) = x_1y_6 - x_6y_1 + x_2y_5 - x_5y_2 + x_3y_4 - x_4y_3$. Hering [5] defined a subgroup $G \cong SL(2,13)$ of $GL(6,3)$, and showed that $\Sigma = \{S_\infty g \mid g \in G\}$ is a spread of $PG(5,3)$, where $S_\infty = F \times F \times F \times 0 \times 0 \times 0$. In [5], G is given as the group generated by three matrices r, h, s, and it is an easy calculation to check that these all preserve the above alternating form. Since S_∞ is totally isotropic with respect to this form, Σ is symplectic. Dempwolff [3] has shown that this spread, as well as the case $n = q = 3$ appearing in §3, are (up to isomorphism) the only nonregular spreads in $W(5,3)$.

5 A remark on the generalized hexagon ${}^3D_4(q)$

Let $q = p^r$ be any odd prime power. Let $F = GF(q)$ and $E = GF(q^3)$, and let tr and N denote the trace and norm maps $E \to F$, so that $tr(a) = a + a^q + a^{q^2}$ and $N(a) = a^{1+q+q^2}$ for $a \in E$. Define a nonsingular alternating bilinear form on the 8-dimensional $F-$space $V = F \times E \times E \times F$ by

$$((\alpha_1, b_1, c_1, \delta_1), (\alpha_2, b_2, c_2, \delta_2)) = \alpha_1\delta_2 - \alpha_2\delta_1 - tr(b_1c_2 - c_1b_2),$$

in order to produce a symplectic geometry $W(7,q)$.

Write $\tilde{E} = E \cup \{\infty\}$. Let $p_\infty = \langle(0,0,0,1)\rangle$ and $p_a = \langle(1, a, a^{q+q^2}, N(a))\rangle$ with $a \in E$. The set $\mathcal{O}_3 = \{\, p_i \mid i \in \tilde{E} \,\}$ is a partial ovoid of $W(7,q)$ [12].

For $a \in E$, let

$$T_\infty = \{\, (0, 0, c, \delta) \mid c \in E, \delta \in F \,\}$$

$$T_a = \{\, (\alpha, b, -\alpha a^{q+q^2} + a^q b^{q^2} + a^{q^2} b^q, \; -2\alpha N(a) + tr(a^{q+q^2} b)) \mid \alpha \in F, b \in E \,\}.$$

Each T_i is a totally isotropic 3-space, and $\Sigma = \{T_i \mid i \in \tilde{E}\}$ is a partial spread of $W(7,q)$ (see [12]). Properties of the partial ovoid $\mathcal{O}_3$ and the partial spread Σ have been discussed in detail in [12].[2] Here we only recall the following:

Σ is called an $(A,B)-$*regular spread* if, for any $C \in \Sigma$, there exists a regulus containing A, B and C which consists of elements of Σ. Moreover, Σ is called $A-$*regular* if Σ is $(A,B)-$regular for all $B \in \Sigma - \{A\}$ (for more details, see [10]).

[2]The definitions of $\mathcal{O}_3$ and Σ that appear in [12] are equivalent to the ones mentioned here, under the following change of coordinates: $(\alpha, b, c, \delta) \mapsto (\delta, b^{q^2}, b^q, c, b, c^q, c^{q^2}, \alpha)$ (compare [6] and [11], p. 130, for the case in which q is even). The present descriptions of $\mathcal{O}_3$ and Σ implicitly appear in the construction of the ${}^3D_4(q)$-hexagon as a coset geometry

(1) p_i is the unique point of $\mathcal{O}_3$ incident with T_i; (2) the stabilizer of $\mathcal{O}_3$ in the group $PSp(8,q)$ acts 2-transitively on both $\mathcal{O}_3$ and Σ; and (3) ([12], Lemma 4.5 (4)) for any $i \in \tilde{E}$, any totally isotropic line of $W(7,q)$ through p_i is incident with at most one of the elements of $\Sigma - \{T_i\}$.

Fix any $i \in \tilde{E}$. If $j \in \tilde{E}$, let $U_j = \langle p_i, T_j \cap {p_i}^{\perp} \rangle$. This is a maximal totally isotropic subspace of $W(7,q)$ such that $p_i \in U_j \subset {p_i}^{\perp}$. Moreover, for any distinct $j, h \in \tilde{E}$ we have $U_j \cap U_h = p_i$.

Fix any $j \in \tilde{E} - \{i\}$. The subspace $T = {p_i}^{\perp} \cap {p_j}^{\perp}$ is a 5-dimensional symplectic geometry $W(5,q)$.

Then $\Sigma_i = \{\, U_h \cap T \mid h \in \tilde{E} \,\}$ is a symplectic spread of $PG(5,q)$.

As the stabilizer of $\mathcal{O}_3$ in $PSp(8,q)$ is 2-transitive on $\mathcal{O}_3$, the spread Σ_i does not depend on the choice of the elements i, j in $\tilde{E}$.

It follows that we lose no generality by letting $p_i = p_\infty$ and $p_j = p_0$. Then $T = \{\, (0,b,c,0) \mid b,c \in E \,\}$ and, for all $a \in E$,

$$U_a = \{\, (0, b, a^q b^{q^2} + a^{q^2} b^q, \beta + tr(a^{q+q^2} b) \mid b \in E, \beta \in F \,\}$$

$$S_a = U_a \cap T = \{\, (0, b, a^q b^{q^2} + a^{q^2} b^q, 0) \mid b \in E \,\}.$$

Moreover, as $U_\infty = T_\infty$, we have

$$S_\infty = U_\infty \cap T = \{\, (0,0,c,0) \mid c \in E \,\}.$$

The symplectic spread is $\Sigma_\infty = \{S_i \mid i \in \tilde{E}\}$. It arises from the presemifield mentioned in Remark 3, and hence from a commutative twisted field having dimension 3 over its centre. It was this example that led us to the additional symplectic spreads in §3.

6 Some perfect 1-codes

Let $\Gamma(q)$ be the graph having the totally isotropic planes of $W(5,q)$ as vertices, two vertices F_1 and F_2 being adjacent if and only if $F_1 \cap F_2$ is a line. Then $\Gamma(q)$ is a metrically regular graph. A *perfect* 1-*code* of $\Gamma(q)$ is a subset C of q^3+1 pairwise disjoint totally isotropic planes,[3] i.e., a symplectic spread (for further information, see [15] §3, or [6] §11). Thus, the symplectic spreads in §§3,4 produce perfect 1-codes of $\Gamma(q)$.

indicated in [9].

[3]Then every other vertex of $\Gamma(q)$ has distance 1 from a unique vertex in C.

References

[1] A.A. Albert, On the collineation groups associated with twisted fields, in *Calcutta Math. Soc. Golden Jubilee Commemoration volume* (1958/59), *part* II, 485-497.

[2] P. Dembowski, *Finite Geometries*, Springer-Verlag (1968).

[3] U. Dempwolff, Translation planes of order 27, Preprint 1992.

[4] R.H. Dye, Partitions and their stabilizers for line complexes and quadrics, *Ann. Mat. (4)* **114** (1977), 175-194.

[5] C. Hering, Eine nicht-desarguessche zweifach transitive affine Ebene der Ordnung 27, *Abh. Math. Sem. Hamb.* **34** (1969), 203-208.

[6] W.M. Kantor, Spreads, translation planes and Kerdock sets. I, *SIAM J. Alg. Disc. Meth.* **3** (1982), 151-165.

[7] W.M. Kantor, Spreads, translation planes and Kerdock sets. II, *SIAM J. Alg. Disc. Meth.* **3** (1982), 308-318.

[8] W.M. Kantor, Ovoids and translation planes, *Can. J. Math.* **34** (1982), 1195-1207.

[9] W.M. Kantor, Generalized polygons, SCABs and GABs. *Buildings and the Geometry of Diagrams*, Springer Lecture Notes **1181** (1984), 79-158.

[10] G. Lunardon, Proposizioni configurazionali in una classe di fibrazioni, *Boll. Un. Mat. Ital.* **13-A** (1976), 404-413.

[11] G. Lunardon, Varietà di Segre e ovoidi dello spazio polare $Q^+(7,q)$, *Geom. Ded.* **20** (1984), 121-131.

[12] G. Lunardon, Partial ovoids and generalized hexagons, *Proceedings of Deinze Conference "Finite Geometries and Combinatorics"* (to appear).

[13] H. Lüneburg *Translation Planes*, Springer-Verlag, Berlin, 1980

[14] S.E. Payne and J.A. Thas, Ovoids in $Q(4,q)$, q a power of 3, Preprint 1992.

[15] J.A. Thas, Two infinite classes of perfect codes in metrically regular graphs, *J. Comb. Theory (B)* **23** (1977), 236-238.

[16] J.A. Thas, Ovoids and spreads of finite classical polar spaces, *Geom. Ded.* **10** (1981), 135-144.

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