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Plane Geometries Associated with Certain 2-Transitive Groups*

WILLIAM M. KANTOR

Department of Mathematics, University of Oregon, Eugene, Oregon 97403

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1. *Introduction*

Let G be a permutation group 2-transitive on a finite set. Assume that the stabilizer of two points fixes exactly $k \geqslant 3$ points, but not all points. Let L denote this set of k points, and call the distinct sets L^g, $g \in G$, *lines*. Then all lines have k points, and any two different points are in a unique line. This means that we have a design $\mathscr{D}$ (with $\lambda = 1$). The only known possibilities for $\mathscr{D}$ are affine spaces, the projective spaces $PG(d, 2)$, and one example with 28 points when $k = 4$.

A *subspace* of $\mathscr{D}$ is a set Δ of points and at least two lines such that any two different points of Δ are on a line of Δ, and all points on a line of Δ are in Δ. In [11], Hall considered the case $k = 3$. His main result was that $\mathscr{D}$ contains as a subspace a seven point projective plane or a nine point affine plane; this means, intuitively, that these classical planes are the building blocks of $\mathscr{D}$. Our purpose is to generalize this result.

THEOREM 1. *There is a subspace Δ and a subgroup H of G fixing Δ, such that one of the following holds for Δ and the group $\bar{H}$ induced by H on Δ.*

(i) *$|\Delta| = k^i$ for some $i \geqslant 2$; $\bar{H}$ is 2-transitive on Δ, has a regular normal subgroup, and has no involution fixing more than one point. (In this case, Δ can be taken to be the set of fixed points of a Sylow 2-subgroup of the stabilizer in G of two points.)*

(ii) *Δ is an affine translation plane of order k, $\bar{H}$ contains the translation group, and each line in Δ is fixed pointwise by an involution in $\bar{H}$.*

(iii) *Δ is projective plane of order 2, $k = 3$, and $\bar{H}$ is $PSL(3, 2)$ or the stabilizer of a line in $PSL(3, 2)$.*

(iv) *Δ is isomorphic to a certain design $\mathscr{D}(k)$ with k a power of 2, $|\Delta| = k(2k - 1)$, and $\bar{H} \approx PSL(2, 2k)$.*

For each $k = 2^e$, there exists a unique design $\mathscr{D}(k)$ mentioned in (iv).

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In fact, $\mathscr{D}(k)$ is the design determined by the dual of the complement of a completed conic in $PG(2, 2k)$ (see (6.3) for a group-theoretic description). The design $\mathscr{D}(4)$ is the one with 28 points mentioned earlier. Any affine plane of odd order coordinatized by a semifield behaves as in (ii).

Note that, when k is prime, each translation plane of order k is desarguesian; here, in situations (i) and (ii), $AG(2, k)$ is a subplane of Δ. Also, the possible groups $\bar{H}$ in (i) have been explicitly determined by Hering [13] and Huppert [15].

It appears to be very difficult to determine $\mathscr{D}$ starting from the subspaces Δ guaranteed by Theorem 1. In view of the 2-transitivity of G, there will be many images of such a Δ under G. What is now needed is a method of tying all these subspaces together. Moreover, even when $\mathscr{D}$ is known to be $AG(d, k)$ or $PG(d, 2)$, only very meager results are known concerning the possible groups (see, e.g., [17, Section 4]).

From our proof it is not difficult to deduce the following consequence of Theorem 1.

COROLLARY. *Let G be a group 3-transitive on a finite set, in which the stabilizer of three points fixes exactly $k + 1 \geqslant 4$ points, but not all points. Then there is a set Δ of points such that the stabilizer H of Δ induces a group $\bar{H}$ on Δ having the following properties.*

(i) *If three points are in Δ, so are all $k + 1$ points fixed by the stabilizer in G of the three points.*

(ii) *Either* (a) $k = 4$, $|\Delta| = 8$, *and $\bar{H}$ is a subgroup of the holomorph of $E = Z_2 \times Z_2 \times Z_2$ containing the normalizer of a Klein group in E; or* (b) $|\Delta| = k^i + 1$, *and $\bar{H} \geqslant PSL(2, k^i)$ for some $i \geqslant 2$.*

(iii) *If a Sylow 2-subgroup of the stabilizer in G of three points fixes exactly $k + 1$ points, then either* (iia) *holds, or* (iib) *holds with $|\Delta| = k^2 + 1$ and $\bar{H} \trianglerighteq PGL(2, k^2)\langle a\rangle$, where a is an involution induced by the involutory field automorphism of $GF(k^2)$.*

In view of a beautiful result of Nagao [18], there is no point to examining the 4-transitive generalization of this situation. We remark that Nagao's argument is very elementary, except when the stabilizer of four points fixes exactly five points. Here, he uses the Feit–Thompson theorem. However, we note that this case can be easily handled by means of the reduction argument used in Hall [11], Theorem 2.2, or (3.1) together with the case $k = 3$ of Theorem 2.

The proof of Theorem 1 involves a blend of geometric, combinatorial, and group-theoretic methods. Most of the time, only the Feit–Thompson theorem is needed, not any more recent deep group-theoretic classification theorems. The glaring exception here is at the end of Section 5, where [1] seems to be needed. Nevertheless, we have tried to avoid recent classification theorems when possible.

In the proof, we first reduce to a suitable subspace of $\mathscr{D}$ and section of G. The cases k odd and k even must then be dealt with separately. When k is odd, Theorem 1 is a consequence of the next result.

THEOREM 2. *Let G be an automorphism group of a design $\mathscr{D}$ with $\lambda = 1$ and k odd, satisfying the following conditions: for any two different points x and y, the stabilizer of x and y fixes the line L through them pointwise and has even order; the group induced on L by the set stabilizer of L has at most one involution fixing x; and no nontrivial element fixes three noncollinear points. Then $\mathscr{D}$ is $PG(2, 2)$ and G contains the stabilizer of a line, $\mathscr{D}$ is $PG(3, 2)$ and $G \approx A_7$, or $\mathscr{D}$ is an affine translation plane and G contains the translation group.*

At a very early stage of the proof of Theorem 2, it is known that G has no elementary abelian subgroup of order 8. However, the geometry is necessary (and sufficient) in order to handle the case where $Z^*(G) \neq 1$. On the other hand, when $Z^*(G) = 1$ it seems necessary to apply a classification theorem [1]. Throughout the main part of the proof, essential use is made of a result of Harada [12] (see (2.1)).

At several parts of the proof of Theorem 2, it is shown that $k = 3$. It also turned out that the case $k = 3$ could be handled by our methods using relatively elementary group theory. We have therefore given a new proof of Hall's result (see Section 4); however, while this proof is fairly elementary, it certainly requires much more background than Hall's original proof.

The following result implies Theorem 1 when k is even.

THEOREM 3. *Let G be an automorphism group of a design $\mathscr{D}$ with $\lambda = 1$ and $k = 2^e$, satisfying the following conditions: for any two different point x and y, the stabilizer of x and y fixes the line L through x and y pointwise, has even order, and has no involution fixing a point off of L; and the group induced on L by the set stabilizer of L has no element of order 4. Then either $\mathscr{D}$ is an affine translation plane and G contains the translation group, or $\mathscr{D}$ is $\mathscr{D}(k)$ and $G \trianglerighteq PSL(2, 2k)$.*

The proof of Theorem 3 is perhaps more elementary than that of Theorem 2. In the case of Theorem 3, a great-deal of attention is given to the structure of the stabilizer of a point. A crucial idea in the proof is borrowed from Bender [3, Lemma 4.2].

Theorem 3 provides a proof of Theorem 4 of Harada [12] quite different from his proof. On the other hand, if, in Theorem 2, G is 2-transitive on points, that theorem is weaker than [12], Theorem 1. We note that Theorems 1, 2, and 3 contain all the results of [16]. However, the proofs use ideas from [16].

The remainder of the paper is divided into two chapters, dealing with the cases k odd and even, respectively. Each chapter begins with background material. Then general reductions and combinatorial arguments are presented. Finally, Theorems 2 and 3 are proved.

We have also included an appendix concerning Harada's theorem [12] on 2-transitive groups.

I. k ODD

2. *Background*

We will use the geometric terminology of Dembowski [7] and the group-theoretic terminology of Gorenstein [10].

As usual, $Z^*(G)$ is defined by: $Z^*(G) \geqslant O(G)$ and $Z^*(G)/O(G) = Z(G/O(G))$.

If G is a permutation group on a set Ω, and $\Delta \subseteq \Omega$, the groups $G(\Delta)$, G_Δ, and $G_\Delta{}^\Delta$ are defined as follows:

$G(\Delta)$ is the pointwise stabilizer of Δ;

G_Δ is the set stabilizer of Δ; and

$G_\Delta{}^\Delta = G_\Delta/G(\Delta)$ is the permutation group induced by G_Δ on Δ.

In addition, if Δ' is another set (or a point), $G_{\Delta\Delta'} = G_\Delta \cap G_{\Delta'}$.

THEOREM 2.1 (Harada [12]). *Let G be a group and H a proper subgroup. Assume*: $|H \cap H^g|$ *is odd whenever* $g \in G - N_G(H)$; $N_G(H)/H$ *has cyclic or generalized quaternion Sylow 2-subgroups, and* $N_G(H) - H$ *contains an involution conjugate in G to an involution of H. If* $Z^*(G) = O(G)$, *then either* (i) *a Sylow 2-subgroup of G must be dihedral, quasidihedral*, $Z_{2^m} \times Z_{2^m}$, *or* $Z_{2^m} \wr Z_2$ *for some m; or* (ii) *G has a normal subgroup N of index 2 such that* $|H \cap N|$ *is odd.*

LEMMA 2.2 (Wielandt [21], p. 27). *Let G be a transitive permutation group having a regular normal subgroup N. If X is any subset of G fixing at least one point, then $C_G(X)$ is transitive on the fixed points of X.*

LEMMA 2.3 (Brauer–Wielandt; see Wielandt [20]). *Let* $\langle t, u\rangle$ *be a Klein* *group acting on a group H of odd order. If* $C_H(\langle t, u\rangle) = 1$, *then* $|H| = |C_H(t)|\,|C_H(u)|\,|C_H(tu)|$.

LEMMA 2.4 (Thompson's transfer lemma; see Gorenstein [10], p. 265, Ex. 3(i)). *Let S be a Sylow 2-subgroup of G,* $S = \langle t\rangle S_0$ *with t an involution and* $t \notin S_0 \lhd S$, *and suppose that no element of* S_0 *is conjugate in G to t. Then* $G = \langle t\rangle H$ *with* $t \notin H$.

LEMMA 2.5. *Let S be a set, and* $n \geqslant 3$ *an integer. Let* $\theta: S \to 2^S$ *be a mapping such that*

(i) $|x^\theta| = n + 1$ *for all x,*

(ii) $y \in x^\theta$ *implies* $x \in y^\theta$ *and* $x, y \in z^\theta$ *for some z,*

(iii) $x \notin x^\theta$ *for all x, and*

(iv) $|x^\theta \cap y^\theta| \leqslant 1$ *whenever* $x \neq y$.

Then $|S| > n^2 + n + 1$.

Proof. Assume $|S| \leqslant n^2 + n + 1$. Call the subsets x^θ, $x \in S$, lines. As on pp. 138–139 of [7], this produces a projective plane of order n. Then θ is a polarity, and (iii) contradicts a theorem of Baer [2].

LEMMA 2.6. *Let* $\mathscr{D}$ *be a design with* $k = 3$ *and* $\lambda = 1$. *Let K be a Klein group of automorphisms fixing some line L pointwise and semiregular off of L. Let* $x \notin L$ *and semiregular off of L. Then* $L \cup x^K$ *is a subspace PG(2, 2) of* $\mathscr{D}$.

Proof. Let $L = \{a, b, c\}$ and $K = \{1, t, u, tu\}$. Clearly, K is regular on x^K. Then u must move xx^t, as otherwise tu would fix x. Also, t fixes xx^t, so we may assume $\{a, x, x^t\}$ is a line. Thus, $\{a, x^u, x^{tu}\}$ is also a line. Since u fixes xx^u and $a \notin xx^u$, we may assume $\{b, x, x^u\}$ is a line. Apply t, and find that $\{b, x^t, x^{tu}\}$ is a line. Finally, tu fixes xx^{tu}, and $a, b \notin xx^{tu}$. Hence, $c \in xx^{tu}$, so $c \in x^u x^t$ also. This proves the lemma.

The preceding proof is a simple special case of the arguments used by Hall in [11].

3. *Reduction and Preliminary Results*

We will use the following notation throughout this paper. $\mathscr{D}$ is a design with $\lambda = 1$. $\mathscr{D}$ has v points, and $r = (v - 1)/(k - 1)$ lines per point. G is an automorphism group of $\mathscr{D}$ such that the stabilizer G_{xy} of two points x, y fixes the line through them pointwise.

Points will be denoted x, y; lines L, M; and involutions s, t, u.

$G(L)$, G_L, and $G_L{}^L$ were defined in Section 2.

By abuse of language, each subspace of $\mathscr{D}$ will be identified with its set of points.

PROPOSITION 3.1. *Theorem* 1 *follows from Theorems* 2 *and* 3.

Proof. Let S be a 2-group maximal with respect to fixing three noncollinear points; possibly $S = 1$. The set Δ of fixed points of S is a subspace of the design $\mathscr{D}$ defined at the start of Section 1.

Suppose first that S is Sylow in $G(L)$ for one—and hence all—$L \subset \Delta$. Then $N_G(S)^\Delta$ is 2-transitive and has no involution fixing more than one point. By results of Bender [3, 4], it follows that Theorem 1(i) holds or $N_G(S)^\Delta \trianglerighteq PSL(2, q)$ with $q > 3$, $Sz(q)$, or $PSU(3, q)$ with $q > 2$. But in each of the latter cases, the stabilizer of two points fixes only two points.

We may thus suppose that S is not Sylow in $G(L)$ for any $L \subset \Delta$. Then $S \lhd S_1 \leqslant G(L)$ with $| S_1 : S | = 2$. By the maximality of S, $S_1{}^\Delta$ is an involution whose set of fixed points is L. For the same reason, $N_G(S)^\Delta$ has no involution fixing more than k points.

It is clear that $G_L{}^L$ is a sharply 2-transitive group. Consequently, if k is even the hypotheses of Theorem 3 hold. Thus, Theorem 3 implies Theorem 1 in this case.

Suppose k is odd. Then the hypotheses of Theorem 2 hold, except for the statement concerning subspaces. Let Δ' be a minimal subspace of Δ, and set $H = G_{\Delta'}$. We will show that Theorem 2 applies to Δ' and $H^{\Delta'}$.

Let $L \subset \Delta'$, and let $t \in N_G(S)^\Delta(L)$ be an involution. Take any point $y \in \Delta' - L$, and let L_1 be the line of Δ through y and y^t. Then t interchanges y and y^t, so it fixes L_1. Since k is odd, t fixes some point x of L_1. Then $x \in L$. Thus, $y^t \in \Delta'$ since $x, y \in \Delta'$. This means that $\Delta'^t = \Delta'$.

Consequently, Theorem 2 applies, so Theorem 1 holds.

We now begin the proof of Theorem 2.

Throughout the proofs of Theorems 2 and 3, L will denote any line and x any point of L.

An involution t fixing exactly k points will be called *axial.* Its line of fixed points is called its *axis*, and is denoted A_t.

Throughout the remainder of Chapter II, k will be odd. For the sake of convenience, we will assume that $\mathscr{D}$ and G provide a counterexample to Theorem 2. In particular, $v > 7$.

We may assume that G is generated by its axial involutions.

The properties of G in the next lemma will be used frequently and usually without reference.

LEMMA 3.2.

(i) *$G(L)$ is semiregular off of L.*

(ii) *If $G_L{}^L$ moves every point, then $G_L{}^L$ has one Frobenius orbit, and is regular on each remaining orbit.*

(iii) *If t is an involution in $G_{xL} - G(L)$, then $t^L \in Z(G_{xL}^L)$.*

(iv) *If two different involutions fix the same two lines, one of the involutions must act trivially on one of the lines.*

Proof.

(i) and (iii) are hypotheses of Theorem 2.

(ii) If $g \in G_L$ fixes two points of L, then $g \in G(L)$.

(iv) Let t and u be the involutions, and L and M the lines. Then $L \cap M$ is a point x since k is odd. By (i), we cannot have $t^L = u^L$ and $t^M = u^M$, so the assertion follows from (iii).

LEMMA 3.3. *Let $t \in G(L)$ be an involution.*

(i) *t fixes exactly $r - 1$ lines $\neq L$, each of which meets L.*

(ii) *t centralizes exactly $r - 1$ axial involutions not in $G(L)$.*

(iii) *$C_G(t)$ is transitive on $M(t) = \{x \in L \mid x^s = x$ for an involution $s \in C_G(t)$, $s \notin G(L)\}$, and induces a semiregular group on $L - M(t)$.*

(iv) *$|M(t)| = m(t)$ divides k.*

(v) *If $x \in M(t)$, t fixes exactly $(r - 1)/m(t)$ lines $\neq L$ on x.*

(vi) *$|C_G(t) \cap G(L)| \geqslant (r - 1)/m(t) \geqslant (r - 1)/k$.*

Proof.

(i) If $y^t \neq y$, then t fixes the line through y and y^t.

(ii) By (i), there are at least $r - 1$ such involutions. If t centralizes $u, u' \in G(L')$, then u and u' agree on L, so $uu' \in G(L) \cap G(L') = 1$.

(iii), (iv) These follow from (3.2(ii),(iii)).

(v) This follows from the transitivity of $C_G(t)$ on $M(t)$.

(vi) Let $x \in M(t)$. There are $(r - 1)/m(t)$ involutions in $C_G(t)_x$ not in $G(L)$. Fix one of these involutions s, and consider the $(r - 1)/m(t)$ elements ss', where s' runs through all these involutions. Clearly, $ss' \in C_G(t) \cap G(L)$.

LEMMA 3.4. $r - 1 \neq k$.

Proof. Suppose $r - 1 = k$. Then our design $\mathscr{D}$ is an affine plane, and each axial involution is a homology. By (3.3), we can find a Klein group $\langle t, u\rangle$ with t and u axial and $A_t \neq A_u$. It is then easy to check that tu is a dilatation of the plane (compare Dembowski [7, p. 120]). Also, since t is a homology, it fixes exactly one line $\neq A_t$ per point of A_t. By (3.3(v)), $m(t) = k$. Since t can be any axial involution in G, it follows that G is transitive. Thus, each point is the center of an involutory dilatation. The pointwise stabilizer H of the line at infinity now acts on the affine points as a Frobenius group, and the Frobenius kernel of H consists of translations. Hence, $\mathscr{D}$ is a translation plane.

LEMMA 3.5. *Each involution in G fixes exactly k points.*

Proof. Let s be an involution fixing fewer than k points. Then s fixes just one point x. If $y \neq x$, then s fixes the line through y and y^s, and hence fixes one of its k points. Consequently, s fixes each line L through x. Since G_{xL}^L has a unique involution for each such L, we must have $s \in Z(G_x)$.

Now let $t \in G_x$ be an involution fixing k points. Since $s \in C_G(t)$, t fixes $(r-1)/m(t)$ lines $\neq A_t$ on x (by (3.3)). However, s and t fix at most 2 common lines (by (3.2iv)). Thus, $(r-1)/m(t) + 1 \leqslant 2$, so $r - 1 \leqslant m(t) \leqslant k$ where $k \leqslant r$. In fact, if $m(t) < k$ then $m(t) \leqslant k/3$, so $r - 1 \leqslant k/3 \leqslant r/3$. Thus, $m(t) = k$ divides $r - 1$, so $r - 1 = k$. This contradicts (3.4).

LEMMA 3.6. *Suppose there is a point x such that G_x fixes some line L on x. Then $r - 1 = 2k$ and $C_G(t)$ is transitive on A_t for each involution $t \in G_x - G(L)$.*

Proof. Let $t \in G_x$ be an involution with $A_t \neq L$. By (3.3) and (3.5), there are exactly $(r-1)/m(t)$ involutions in $C_G(t)_x$ not in $G(A_t)$. If u and u' are two of these not in $G(L)$, then $uu' \in G(A_t)$ and $uu' \in G(L)$. Thus, $u = u'$, so $(r-1)/m(t) \leqslant 2$.

Now $m(t) = k$ (as otherwise $k - 1 \leqslant r - 1 \leqslant 2m(t) \leqslant 2(k/3)$ since $m(t) \mid k$, so $k = 3$, $r \leqslant 3$, and $\mathscr{D}$ is $PG(2, 2)$, which we are assuming is not the case). That is, $C_G(t)$ is transitive on A_t for each involution $t \in G_x - G(L)$. Moreover, $(r-1)/k$ is an integer, so $r - 1 = 2k$ by (3.4).

LEMMA 3.7. *If G_x moves each line on x for each x, then $m(t) > 1$ for each involution t.*

Proof. If $m(t) = 1$ for some involution t, then t must fix each line on some $x \in A_t$. Let $g \in G_x$ move A_t. Then $t^g \neq t$ fixes all lines on x. Consequently, there is at most one line $\neq A_t$, $(A_t)^g$ on x. Then $k \leqslant r \leqslant 3$, and $\mathscr{D}$ is the seven point projective plane.

4. *Hall's Theorem*

We now digress from the main part of the proof of Theorem 2 to give a proof of Hall's special case of it: $k = 3$.

By (3.5), all involutions are axial.

LEMMA 4.1. *Assume that, for some involution t, $m(t) = 1$. Let x be the point of $M = A_t$ on each fixed line of t. Then the following hold.*

(i) $v = 15$.

(ii) *If $x \in L \neq M$, and $u \in G(L)_M$ is an involution, then u fixes exactly 3 lines on x.*

(iii) *G_x fixes M.*

(iv) *G is transitive on points.*

(v) *For each line L on x, $G_L{}^L$ is transitive.*

Proof. By (3.3), t fixes L, and u fixes $(v - 3)/6$ or $(v - 3)/2$ lines $\neq L$ on x. By (3.2iv), t and u fix at most one common line other than L and M. Since $(v - 3)/2 \leqslant 2$ is excluded, $(v - 3)/6 \leqslant 2$. This proves (i) and (ii). Moreover, by (ii), M is the only line on x fixed pointwise by a conjugate of t, so (iii) holds. By (3.6), x^G contains all points outside of M. Let $x' \in M - \{x\}$, and $x' \in L' \neq M$. Then an involution in $G(L')$ moves the third point of M outside of M or to x. This proves (iv).

Finally, if $L \neq M$ then (v) holds by (3.3). Consider $G_M{}^M$. Let $x' \in M - \{x\}$, and let $x' = x^g$ with $g \in G$. Then t^g fixes all lines on x'. If $(t^g)^M \neq 1$, then $G_M{}^M$ is transitive by (3.3). If $t^g \in G(M)$ then $G^M_{x'M} \neq$ by (3.3), and again $G_M{}^M$ is transitive.

LEMMA 4.2. $v = 15$.

Proof. By (3.3) and (4.1), we may assume that $m(t) = 3$ for every involution t. Fix x, and count the triples (L, M, t) such $L \cap M = \{x\}$ and t is an involution fixing L and M, but fixing neither pointwise. By (3.2iv), there are at most $r(r - 1) \cdot 1$ triples. Also, each $t \in G_x$ fixes $(r - 1)/m(t)$ lines $L \neq A_t$, so there are at least $r \cdot \frac{1}{3}(r - 1)(\frac{1}{3}(r - 1) - 1)$ triples. Thus, $r \leqslant 13$.

By (3.3), each t centralizes exactly $(r - 1)/3$ axial involutions $s \notin G(L)$. But st is also axial. Hence, $(r - 1)/3$ is even. It follows that $r = 7$ or 13.

The possibility $r = 13$ is eliminated as follows. In this case, the two counts of the triples (L, M, t) yield the same result. Hence, each line is the axis of a unique involution, and any two lines on x are fixed by an involution having neither as axis. For each L on x, let L^θ denote the set of fixed lines $\neq L$ on x of the involution in $G(L)$. Then $| L^\theta | = (r - 1)/3 = 4$ and

$L \notin L^\theta$. If $M \in L^\theta$ then the unique involution t in $G(L)$ centralizes the unique involution u in $G(M)$ and $L, M \in A^\theta_{tu}$. Finally, if $L \neq M$ then $|L^\theta \cap M^\theta| \leqslant 1$ by (3.2iv). By (2.5), necessarily $r > 3^2 + 3 + 1$. This contradiction proves (4.2).

The remainder of the case $k = 3$ thus involves a detailed examination of the possibility $v = 15$.

LEMMA 4.3. *G_x moves all lines on x.*

Proof. Let G_x fix a line M on x, and set $\mathscr{M} = M^G$.

By (3.3) and (4.1iv,v), G is transitive on points and $G_M{}^M$ is transitive. Thus, $\mathscr{M}$ is a complete system of imprimitivity for G. In particular, $|\mathscr{M}| = v/3 = 5$. Now $G^{\mathscr{M}}$ is a transitive subgroup of S_5. If t is an involution with $A_t \notin \mathscr{M}$, then (by (3.3)) t fixes $(r-1)/k = 3$ lines in $\mathscr{M}$. Thus, $G^{\mathscr{M}} \approx S_5$.

Let K be the kernel of the action of G on $\mathscr{M}$. Then K is an elementary abelian 3-group of order $\leqslant 9$ (as a $3'$-element of K must fix at least 5 points, and hence is trivial). Since A_5 acts on K, G has a normal subgroup H of index 2 with $K \leqslant Z(H)$ and $H/K \approx A_5$. Thus, $H = A \times K$ with $A \approx A_5$.

Clearly, A has 3 orbits of length 5 permuted transitively by G. Let x and y be distinct points in one of these orbits. Then A has 3 involutions interchanging x and y. Thus, each of these involutions must fix the third point z on the line L through x and y. But $L \notin \mathscr{M}$, so each of these involutions fixes a different line in $\mathscr{M}$. This is a contradiction.

COROLLARY 4.4. *For each involution t, $C_G(t)$ is transitive on A_t and t fixes exactly 3 lines through each point of A_t.*

Proof. (4.3) and (4.1(iii)) imply the first assertion. The second follows from (3.3) and (4.2).

LEMMA 4.5. *For some line L, $G(L)$ contains a normal Klein group.*

Proof. Suppose first that, for each line L on x, $G(L)$ contains a unique involution t. Let t^θ consist of the 3 lines on x fixed by t (see (4.4)). If $s \in C_G(t) - G(A_t)$ then $t^\theta = \{A_t, A_s, A_{st}\} = s^\theta$. It follows that the sets t^θ partition the 7 lines on x into sets of 3, which is absurd.

Thus, some $G(L)$ has at least 2 involutions. Suppose $G(L)$ has no normal Klein group. Note that $|G(L)|$ divides $v - 3 = 12$. Thus, $G(L)$ must have a normal Sylow 3-subgroup P of order 3. Then $P \lhd G_L$. Let $t \in G(L)$ and $u \in C_G(t) - G(L)$ be involutions. Then $\langle t, u\rangle$ normalizes P. Here, u cannot centralize P (as otherwise P would fix $L \cup A_u$ pointwise). Similarly, tu cannot centralize P. Thus, t centralizes P. However, this is true for each involution $t \in G(L)$, and hence $G(L)$ has a normal Klein group.

Completion of the Proof When $k = 3$.

By (2.6), it suffices to show that every line behaves as in (4.5).

Let L be as in (4.5). Then $|G(L)|$ divides $v - 3 = 12$. The proof of (3.3vi) shows that $|G(L)| \geqslant r - 1 = 6$. Thus, $|G(L)| = 12$ and $G(L)$ is regular on the points not in L. In particular, if $x \in L$ then $G(L)$ is transitive on the lines $\neq L$ through x. Since G_x moves L, it follows that G_x is transitive on the lines through x. This completes the proof.

5. *The General Case of Theorem* 2

LEMMA 5.1. *If $Z^*(G) = O(G)$, then one of the following holds.*

(i) *A Sylow 2-subgroup of G is dihedral, quasidihedral, wreathed $Z_{2^m} \wr Z_2$, or $Z_{2^m} \times Z_{2^m}$ for some m.*

(ii) *G has a proper normal subgroup K having a strongly embedded subgroup K_L for some line L.*

Proof. Let K be the subgroup $O^{2'}(G)$ generated by all elements of odd order. Let $z \in K$ be an involution, and set $H = G(A_z)$.

Suppose z has a K-conjugate z' such that $zz' = z'z \neq 1$ and $A_z \neq A_{z'}$. Apply (2.1) to G and H (see (3.2)). If (2.1(i)) holds, we are finished. If (2.1(ii)) holds, then G has a normal subgroup N of index 2 with $z \notin N$; however, by definition $K \leqslant N$, so this cannot occur.

Thus, we may assume that, for each involution $z \in K$ and each $z' \in (z^K - \{z\}) \cap C_K(z)$, we have $A_z = A_{z'}$. Let S be a Sylow 2-subgroup of K, choose $z \in Z(S)$, and set $L = A_z$. Then z fixes only $L \in L^K$; for if z fixed $L' \neq L$ in L^K, then z would centralize an element of $z^K \cap K(L')$, contrary to our assumption. If all involutions in K are conjugate, this completes the proof of (ii).

Suppose there is a second class t^K of involutions in K. Assume that t commutes with some $t' \in t^K - \{t\}$. We may assume $\langle t, t' \rangle \leqslant S$. Since $A_t = A_{t'}$ by the second paragraph of this proof, necessarily $A_t = A_{tt'}$, and since $\langle t, t' \rangle$ acts on L, necessarily $\langle t, t' \rangle \leqslant K(L)$. Now let $z' \in z^K \cap K(L')$ for $L' \in L^K - \{L\}$. Then z' and t are not conjugate in $\langle z', t \rangle$, so $\langle z', t \rangle$ has central involution u. Clearly u fixes $A_t = L$. Then $\langle t, u \rangle$ centralizes some $z'' \in z^K \cap K(L)$, and $\langle t, z'' \rangle$ acts on A_u, so $A_u = L$. But $z' \in z^K$ now fixes $L = A_z$, which is not possible.

Consequently, t commutes with none of its K-conjugates. Then $t \in Z^*(K) \cap S \leqslant Z(S)$ (Glauberman [9]). Interchanging t and z, we see that all involutions in K are in $Z^*(K)$. However, $G = KS_1$ with $S_1 > S$ Sylow in G. Then S_1 centralizes some involution $u \in Z(S)$, and hence $u \in Z^*(G)$ by [9].

LEMMA 5.2. *For each line L, G_L is transitive on L.*

Proof. Deny! Call a line L "good" if G_L is transitive on L, and "bad" otherwise. Call a point x "bad" if, for some line L on x, $G_{xL} - G(L)$ has no involution. Let $\mathscr{B}$ denote the set of bad points; bad points will be indicated by the letter b, and bad lines by B.

Note that $\mathscr{B} \neq \varnothing$. For, let B be any bad line. Then by (3.2), $G_B{}^B$ has a semiregular orbit on B, and each point b of this orbit is bad. Moreover, $G_{bB} = G(B)$.

G_b is transitive on the lines on b. For, there is a bad line B on b such that $G_{bB} = G(B)$. Let $s \in G(B)$ be an involution. Here, s fixes just the one line B on b. Then G_B contains a Sylow 2-subgroup of G_x, which may be assumed to have s in its center. Take any involution $t \in G_x$. By using a suitable conjugate of t, we may assume that t centralizes s. The choice of B now implies that $A_t = B$. This means that each line on x can be moved to B using an element of G_x.

It follows that G is transitive on the set $\mathscr{L}$ of bad lines. Thus, $| B \cap \mathscr{B} |$ is independent of the bad line B, so the bad points and lines form a design with $v^* = | \mathscr{B} |$, $k^* = | B \cap \mathscr{B} |$, $r^* = r$, and $\lambda^* = 1$. Then transitivity on bad lines implies transitivity on $\mathscr{B}$ [7, p. 78].

G_b acts faithfully as a Frobenius group on the r lines B on b. For, a nontrivial element of $G_{bB} = G(B)$ cannot fix a second line on b. It follows that $G_{bB} = G(B)$ has a unique involution.

Since G_b is transitive on the lines on b, G is transitive on the pairs (b, B) with $b \in B \cap \mathscr{B}$, so G_B is transitive on $B \cap \mathscr{B}$. Thus, $G^{\mathscr{B}}$ is primitive [14; 7, p. 79]. Then $O(G^{\mathscr{B}}) = 1$ (as otherwise, $O(G^{\mathscr{B}})$ would be transitive on the $v^* = 1 + r(k^* - 1) \equiv 1 + 1 \pmod 2$ bad points). Since the pointwise stabilizer of $\mathscr{B}$ fixes each line meeting $\mathscr{B}$, it is trivial. Hence, $O(G) = 1$, so also $Z^*(G) = 1$.

The number of bad lines is $v^*r/k^* = r\{1 + r(k^* - 1)\}/k^*$, so good lines exist. Hence, G has at least two classes of involutions. Since $Z^*(G) = 1$, (5.1) applies. If (5.1(i)) holds, then by (2.4) G must have dihedral Sylow 2-subgroups, so there is an involution u such that $\langle u \rangle$ is Sylow in $G(A_u)$. If (5.1(ii)) holds, then there is again such a u (cf. (2.1)).

Let u be as above. By (3.2ii) and (3.3(ii)), a Sylow 2-subgroup of $C_G(u)$ has just 3 involutions. Hence, $C_G(u)$ has a single class of Klein groups, and thus just two classes of involutions $\neq u$, both of the same size. By (3.3), u fixes $(v^* - k^*)/k^*$ or v^*/k^* bad lines $\neq A_u$, and hence exactly that number of involutions having bad axes. Thus, $C_G(u) - \{u\}$ contains exactly $(v^* - k^*)/k^*$ or v^*/k^* involutions having good axes. By (3.3), $r - 1 = 2v^*/k^*$ or $2(v^* - k^*)/k^*$. An easy calculation shows that neither equation can hold.

LEMMA 5.3. *For each point x, G_x moves all lines on x.*

Proof. Deny! By (3.6), $r - 1 = 2k$. Let G_x fix M, and set $\mathcal{M} = M^G$.

G is transitive on points. For, x^G contains the complement of M by (3.6). If L_1 is any line meeting $M - \{x\}$, then either $G(L_1)$ moves M—and the transitivity of G is clear—or $G(L_1)$ always fixes M, in which case $G_M{}^M$ is transitive.

$\mathcal{M}$ is a complete system of imprimitivity for G. To see this, it suffices to prove that $G_M{}^M$ is transitive. Let $x' \in M - \{x\}$. Then $x' = x^g$ for some $g \in G$. If g can always be chosen in G_M, our assertion follows. So suppose $M^g \neq M$. Then $G_{x'}$ fixes M^g. In particular, $G(M)$ fixes M^g. Then an involution in $G(M)$ centralizes an involution in $G(M^g)$, so that $| G^M_{x'M} |$ is even. By (3.2(ii)), x' and x are in the same G_M-orbit, so g can indeed be chosen inside G_M, as desired.

In particular, $| \mathcal{M} | = v/k = 2k - 1$.

If $x \in L \neq M$, $G(L)$ acts on M, so $G(L)$ has a unique involution t. Clearly, G_M contains a Sylow 2-subgroup S of G, so $G(M)$ contains an involution $s \in Z(S)$. We may assume $t \in S$.

Suppose first that $Z^*(G) = O(G)$, so that (5.1) applies. We know G has at least two classes of involutions. Hence, if (5.1(i)) holds then G has a normal subgroup not containing t. Since t was arbitrary, (5.1(ii)) holds here. Thus, in any case, we can apply Bender's theorem [4] to the group K of (5.1(ii)). Since $Z^*(G) = O(G)$, it follows that $Z(S \cap K)$ is elementary abelian of order $\geqslant 4$. Moreover, it is easy to see that, if $t \in S$, then $| C_{Z(S\cap K)}(t)|^2 \geqslant | Z(S \cap K)|$. The only possibility is $K^{\mathcal{M}} \approx A_5$, so $2k - 1 = 5$. Now (4.3) yields a contradiction.

Thus, $Z^*(G) > O(G)$. If $t \in Z^*(G)$, then S has no elementary abelian subgroup of order 8 (since t is the unique involution in $G(A_t)$). Thus, $\langle s, t\rangle = \Omega_1(S)$, and hence $s \in Z^*(G)$.

We may now assume $s \in Z^*(G)$. Then $O(G)$ is transitive on $\mathcal{M}$. Let $K \leqslant O(G)$ be the kernel of the action of G on $\mathcal{M}$. Since $O(G)$ is solvable (by [8]), G has a normal subgroup $H > K$ with H/K a q-group for some $q \mid | \mathcal{M} | = 2k - 1$. Since $| K |$ clearly divides k^{2k-1}, $G = KN_G(Q)$ for a Sylow q-subgroup Q of H. We may assume $S \leqslant N_G(Q)$.

Let $u \in S$ be an involution. Then $C_Q(u)$ acts on A_u. Since $(| C_Q(u)|, k(k-1)(v-k)) = 1$, necessarily $C_Q(u) = 1$. Thus, $\langle s, t\rangle$ is fixed-point-free on Q, which is absurd.

Let π be the set of prime divisors of k.

LEMMA 5.4. *Suppose G has a nontrivial normal π-subgroup N. Then*

(i) *N is semiregular*;

(ii) *N is intransitive*; *and*

(iii) *For each line L, $G(L)$ has a unique involution.*

Proof. By the Feit–Thompson theorem [8], N is solvable. We first consider the case where N is semiregular. Let L be any line, and $x \in L$. By (5.3), G_x must move L, and hence so must N. By (2.2), $N_L = C_N(G(L.))$ Thus, $G(L)$ acts on the nontrivial π-group $N/C_N(G(L))$. A nontrivial π'-element of $G(L)$ cannot centralize any element of $N - C_N(G(L))$ since $G(L) \cap G(L)^g = 1$ for $g \notin G_L$. In particular, if $t \in G(L)$ is an involution then t inverts $N/C_N(G(L))$. Since the kernel of the action of $G(L)$ on $N/C_N(G(L))$ is a π-group, $G(L) = C_{G(L)}(t)\, O_\pi(G(L))$.

Take any involution $u \in G_L - G(L)$ centralizing t. Then $\langle t, u\rangle$ acts on $X = O_\pi(G(L))$. Also, $A_t \cap A_u$ is a point x. Since $C_X(u)$ acts on $A_u - \{x\}$ and $(|X|, k-1) = 1$, we must have $C_X(u) = 1$. Similarly, $C_X(tu) = 1$. Thus, t centralizes X. Since $G(L)/X$ has a unique involution, so does $G(L)$.

Thus, if N is semiregular, then (iii) holds. To see that (ii) also holds in the case, assume N is transitive. Let $\langle t, u\rangle \leqslant G_x$ be a Klein group. By (2.2), $|C_N(t)| = |C_N(u)| = |C_N(tu)| = k$. Hence, by (2.3), $|N| = k^3$. Now $v = k^3$ and $r = (v-1)/(k-1) = k^2 + k + 1$. Let S be the set of lines on x, and define θ: $S \to 2^S$ by: A_t^θ is the set of fixed lines $\neq A_t$ of t on x. Since (iii) is known to hold, θ is well-defined. By (3.3i), $|A_t^\theta| = (r-1)/k = k+1$. Also, for any distinct involutions $s, t \in G_x$, $|A_s^\theta \cap A_t^\theta| \leqslant 1$ (cf. (3.2iv)); and if s fixes A^t then $st = ts$, so t fixes A_s and $A_s, A_t \in A_{st}^\theta$. Thus, (2.5) applies, and yields a contradiction.

Consequently, if (i) holds, then so do (ii) and (iii). Suppose now that N is not semiregular, and choose such an N with $|N|$ minimal. Let $M \lhd N$ be a normal subgroup of G maximal with respect to being semiregular. Then N/M is a p-group for some $p \in \pi$.

We are assuming that $N_x \neq 1$ for some point x. Here, N_x is a p-group. If N_x fixes a point $\neq x$, its set of fixed points is a line L on x, and then G_x fixes L. Consequently, by (5.3), N_x fixes only x, so $p \mid v - 1$. On the other hand, M is semiregular, so $|M| \mid v$. Consequently, N_x is Sylow in N, so $G = N_G(N_x)N = G_x M$. By (5.2), M is transitive. Since M is a semiregular normal π-subgroup of G, this contradicts the first part of the proof of (5.4). Hence, (i) must hold.

LEMMA 5.5. $Z^*(G) = O(G)$.

Proof. Suppose $Z^*(G) > O(G)$. By the Feit–Thompson theorem [8], $Z^*(G)$ is solvable. By (5.4), $O_\pi(G)$ is semiregular. Let H be a normal subgroup of G such that $H > O_\pi(G)$ and $H/O_\pi(G)$ is an elementary abelian q-group for some prime $q \notin \pi$. If Q is a Sylow q-subgroup of H, then $G = HN_G(Q) = O_\pi(G)\, N_G(Q)$.

Q is not semiregular. For, there is a Klein group $\langle t, u\rangle$ acting on Q, so we may assume $C_Q(t) \neq 1$. Suppose Q is semiregular. Then $C_Q(t)$ is semiregular on A_t, and this contradicts $q \notin \pi$.

Q must fix a point. For, if not, certainly $q \mid v$. There is a point x such that $Q_x \neq 1$. Then $q \nmid v - 1$ implies that Q_x fixes exactly k points, and hence is semiregular on $v - k$ points. Consequently, $q \mid v - k$, and hence $q \mid k$, whereas $q \notin \pi$.

Suppose Q fixes just one point x. Then $N_G(Q) \leqslant G_x$, so $G = O_\pi(G)\, N_G(Q) = O_\pi(G) G_x$. Thus, $O_\pi(G)$ is transitive, and this contradicts (5.4).

We may thus assume Q fixes a line L pointwise, so $G = O_\pi(G) G_L$. By (5.4iii), $G = O_\pi(G)\, C_G(s)$ for some involution $s \in Z^*(G)$.

Since G is transitive, the number conjugates of s fixing a given point x is $k \mid G\colon C_G(s)|/v = k \mid O_\pi(G)\colon C_{O_\pi(G)}(s)|/v$. By (5.4), $O_\pi(G)$ is intransitive. It follows that s has fewer than k conjugates fixing x.

Take any involution t not conjugate to s. Then, for each $x \in A_t$, G_x has a Klein subgroup $\langle s', t\rangle$ with $s' \in s^G$. By (5.4)(iii), $A_{s'} \neq A_t$. Now (5.2) implies that $\langle s^G \cap C_G(t)\rangle \leqslant \langle s'\rangle\, O_\pi(G)$ is transitive on A_t.

Take any point $y \in A_t$, and any point $x \neq y$. We will show that $O_\pi(G)$ has an element moving x to y. Since there are fewer than k conjugates of s fixing x, some involution t' not conjugate to s fixes x and a point of A_t. Now $O_\pi(G) \cap C_G(t')$ and $O_\pi(G) \cap C_G(t)$ allow us to move x to y.

Consequently, $O_\pi(G)$ is transitive. This contradicts (5.4).

LEMMA 5.6. $Z^*(G) = 1$.

Proof. Suppose $O(G) \neq 1$, and let H be a nontrivial normal subgroup of G of prime power order. (H exists by the Feit–Thompson theorem [8].) Since G is transitive, $|H|$ divides v. Then H_x fixes a line pointwise, so $H_x = 1$ (as otherwise G_x fixes a line, contradicting (5.3)). G contains a Klein group, so $C_H(t) \neq 1$ for some involution t. Then $C_H(t)$ is semiregular on A_t, and hence H is a π-group.

By (5.4(iii)), each $G(L)$ has a unique involution. Hence, G has 2-rank 2 by (3.2(ii)). By (5.1), (2.4), and [9], G has a Klein group $\langle t, u\rangle$ with t, u, and tu conjugate, such that $C_G(t)$ contains a Sylow 2-group of G. Write $|C_H(t)| = |C_H(u)| = |C_H(tu)| = \mu$, so $\mu \neq 1$.

If s is any involution, then s commutes with some conjugate s' of itself. We may assume that $\langle s, s'\rangle \leqslant C_G(t)$, and then that $ss' = t$. Also, $C_H(\langle t, u\rangle) = 1 = C_H(\langle s, t\rangle)$ by (5.4(iii)). Thus, by (2.3), $|H| = \mu^3 = \mu\, |C_H(s)|^2$.

It follows that each line meets each orbit of H in 0 or μ points. We can now use a counting argument of Higman and McLaughlin [14, Theorem 7] (see also [7, p. 79]), to show that H is transitive. This contradicts (5.4).

Conclusion of the Proof of Theorem 2.

By (5.6), we may apply (5.1). Suppose (5.1(ii)) holds, and that $t \notin K$. Then $| K(A_t)|$ is odd (cf. (2.1)). Now t centralizes at most two elements in the center of a Sylow 2-subgroup of K. By Bender [4], $K \approx A_5$, so $G \approx S_5$. This is easy to eliminate.

Thus, a Sylow 2-subgroup S of G is as in (5.1(i)). Also, G has no normal Klein group since G is transitive. Hence, by Brauer [5] and Alperin–Brauer–Gorenstein [1], since G is generated by its involutions, $G \approx PSL(2, q)$, $PGL(2, q)$, $PSL(3, q)$, $PSU(3, q)$, A_7, or M_{11} (for some odd prime power q).

Let s be an involution in $Z(S)$, and $L = A_s$. Then $C_G(s) \leqslant G_L$. In almost every case, $C_G(s) \leqslant H < G$ implies that $s \in Z^*(H)$. The only exceptions are: A_7, $PSL(2, 5)$, $PSL(2, 9)$. Each of these is easily eliminated. (Recall that $\mathscr{D}$ is assumed not to be $PG(2, 2)$ or $PG(3, 2)$.)

Thus, $s \in Z^*(G_L)$, so $G_L = G(L)\, C_G(s)$ by the Frattini argument. Then $C_G(s)$ has $G_L{}^L$ as a homomorphic image, where (by (5.2)) $G_L{}^L$ is a Frobenius group of even order whose kernel has order k. Hence, if $G \not\approx PSL(2, q)$, $PGL(2, q)$, we must have $k = 3$, so Section 4 applies.

We may thus assume that $G \approx PSL(2, q)$ or $PGL(2, q)$, so $C_G(s) = G_L$. Let $q \equiv \epsilon \pmod 4$, where $\epsilon = \pm 1$. Then $k \mid q - \epsilon$, so $q > 9$. Now $G(A_s) \lhd C_G(s)$ implies that s is the only involution in $G(A_s)$. If t is an involution not conjugate to s, $C_G(t)$ is again a maximal subgroup, and as above $k \mid q + \epsilon$, which is absurd. Thus, $G \approx PSL(2, q)$.

Moreover, there are exactly $vr/k = \frac{1}{2}q(q + \epsilon)$ lines. By (3.3(ii)), $r - 1 = \frac{1}{2}(q - \epsilon)$. Since $v = 1 + r(k - 1)$ and $k \mid r - 1$ (by (3.3v) and (5.2)), we obtain $\frac{1}{2}q(q + \epsilon) = vr/k = (1 + \frac{1}{2}(q - \epsilon + 2)(k - 1))\, \frac{1}{2}(q - \epsilon + 2)/k$, so $q - \epsilon + 2 \mid \frac{1}{2}q(q + \epsilon)$. Then $\epsilon = 1$, so $qk = 1 + \frac{1}{2}(q + 1)(k - 1)$. Hence, $(q - 1)(k + 1) = 0$, which is ridiculous.

II. k Even

6. *Background*

The following key lemma is a slight extension of Lemma 2.7 of Bender [4].

LEMMA 6.1. *Let Y be a group having an elementary abelian Sylow 2-subgroup S, and write $X = O(Y)$. Let p be an odd prime, and $A \leqslant Y$ an abelian p-group normalized by X but not contained in X. Then $S \leqslant C_Y(X)$ if either*

(i) *$N_Y(S)$ is irreducible on S and $S \leqslant \langle A^Y \rangle$, or*

(ii) *$SX \lhd Y$ and $C_{SX/X}(A) = 1$.*

Proof. Let Y be a minimal counterexample. We first show that (i) holds if (ii) does. Thus, let $1 < S_1 < S$ be such that A normalizes S_1X. Write $Y_1 = AS_1X$. Then $O(Y_1) \geqslant X$, but $A \nleqslant O(Y_1)$ (as otherwise $[S_1, A] \leqslant S_1X \cap O(Y_1) = X$). Also, $O(Y_1) = (A \cap O(Y_1))X$ normalizes A. If SX/X is not A-irreducible, it is completely reducible under A, while the minimality of Y yields $S_1 \leqslant C_G(X)$; thus, $S \leqslant C_G(X)$ in this case. Consequently, SX/X is A-irreducible. Since $Y = N_Y(X)$, this proves that (i) holds.

Now assume (i). We must show that $C_S(X) \neq 1$.

By the Feit–Thompson theorem [8], X is solvable. Suppose $M = O_{p'}(X)$ is nontrivial. Then $[A, M] \leqslant A \cap M = 1$. Also, Y/M satisfies the conditions of the lemma, so S centralizes X/M by the minimality of Y. By hypothesis, $C_Y(M) \geqslant \langle A^Y \rangle \geqslant S$. Thus, S centralizes X/M, M, and hence also X.

Consequently, $O_{p'}(X) = 1$. Then $O_{p'}(Y) \nleqslant X$ would imply that $O_{p'}(Y) = O_2(Y)$ centralizes X. Thus, $O_{p'}(Y) = 1$.

Let $F = O_p(X)$. Then $\bar{Y} = Y/F$ and $\bar{A} = AF/F$ act on $\bar{F} = F/\Phi(F)$. Here $O_p(\bar{Y}) = 1$ (as $O_p(Y) \leqslant X$). By hypothesis, $[\bar{F}, \bar{A}, \bar{A}] = 1$.

However, $\bar{Y}$ is p-stable [10, p. 234], so $\bar{A} = 1$. Then $A \leqslant F \leqslant X$, which is not the case.

THEOREM 6.2 (Buekenhout [6]). *Let $\mathscr{D}$ be a design with $\lambda = 1$ and $k \geqslant 4$. Assume that each triangle is contained in a subspace which is an affine plane of order k. Then $\mathscr{D}$ is an affine space.*

LEMMA 6.3. *For each $e \geqslant 2$, there is a unique design $\mathscr{D}(2^e)$ satisfying the following conditions.*

(i) $\lambda = 1$, $k = 2^e$, $r = 2^{e-1} + 1$.

(ii) *There is an automorphism group G of $\mathscr{D}(2^e)$ isomorphic to $PSL(2, 2^{e+1})$.*

(iii) *G is transitive on incident point-line pairs.*

(iv) *G_L is a Sylow 2-subgroup of G.*

(v) $|G(L)| = 2$.

Proof. Write $q = 2^{e+1}$. Assume $\mathscr{D}(2^e)$ exists. Lines correspond to involutions, so there are $(q^2 - 1)k/r = \frac{1}{2}q(q - 1)$ points. Thus, G_x is dihedral of order $2(q + 1)$. Clearly, the line fixed pointwise by an involution t is on x if and only if $t \in G_x$.

It remains to prove existence. Start with $G = PSL(2, q)$. Let points be subgroups of order $2(q+1)$, let lines be involutions, and let incidence correspond to containment. There are then $q^2 - 1$ lines, $v = \frac{1}{2}q(q-1)$ points, $q + 1$ lines per point, and hence $\frac{1}{2}q$ points per line. G_x is transitive on the lines on the point x.

Suppose x and y are distinct points. Since $O(G_x)$ is a T.I. group, $| G_x \cap G_y | \leqslant 2$. That is, x and y are on at most one line.

Fix a point x. There are $q + 1$ lines on x, each having $\frac{1}{2}q - 1$ points $\neq x$. No two of these lines meet except at x. Thus, these lines cover $1 + (q+1)(\frac{1}{2}q - 1) = v$ points.

This proves (i)–(iii). Since $G_L \leqslant C_G(t)$ if the line L corresponds to t, (iv) holds, and then (v) is clear.

LEMMA 6.4.

(i) *$\mathscr{D}(2^e)$ has no proper subspace.*

(ii) *$H = P\Gamma L(2, 2^{e+1})$ is an automorphism group of $\mathscr{D}(2^e)$.*

(iii) *For each point x, H_x is not 2-transitive on the lines through x.*

(iv) *$| H(L)| = 2$ for each line L.*

Proof.

(i) Let Δ be such a subspace. Let $y \notin \Delta$. Each of the lines xy, $x \in \Delta$, meets Δ just once. Thus, $2k + 1 \geqslant | \Delta |$. On the other hand, since Δ has at least k lines per point, $| \Delta | \geqslant 1 + k(k-1)$. This is impossible.

(ii) This follows immediately from the proof of existence in (6.3).

(iii) Suppose H_x is 2-transitive. Then $2^{e+1} = r - 1 \mid | H_x |$. Also, $v = k(2k-1)$, so $2^e \mid | H: H_x |$. Hence, $2^{2e+1} \mid | H |$. Since $2^{2e+1} \nmid 2^{e+1}(e+1)$, this is impossible.

(iv) This also follows from the construction.

LEMMA 6.5. *Let $\mathscr{D}$ and G be as in Theorem 3, where $\mathscr{D}$ is a translation plane, G contains the translation group, and $G(L)$ has no Klein group. Then*

(i) *G has a normal subgroup H of index 2 such that $| H(L)|$ is odd; and*

(ii) *$O(G(L))$ fixes a line $L' \neq L$ and acts faithfully on L', so $| O(G(L))|$ divides $k - 1$.*

Proof.

(i) By Dembowski [7, p. 188], a Sylow 2-subgroup Q of $G(L)$ has order 2 and is normal in $G(L)$. Now (i) follows from (2.4).

(ii) By Dembowski [7, p. 172], $O(G(L))$ consists of homologies, so (ii) holds.

7. *Preliminary Results*

By (3.1), when k is even Theorem 1 follows from Theorem 3.

Let $\mathscr{D}$ and G provide a counterexample to Theorem 3 with minimal $|G|$. Then G is generated by its axial involutions. Recall that $k = 2^e$.

As in Chapter I, L is any line and $x \in L$. We will use the following additional notation:

t is an involution in $G(L)$,

S is a Sylow 2-subgroup of $C_G(t)$,

$Q = S \cap G(L)$, and

F is the Fitting subgroup of G_x.

We note that one of the hypotheses of Theorem 3 implies that if u is an involution in $C_G(t)$ with axis $L' \neq L$, then t is not a square in $C_G(u)$.

LEMMA 7.1. *Each involution $t \in G(L)$ fixes exactly $(v - k)/k$ lines $\neq L$.*

Proof. If $z \notin L$, then t fixes the line through z and z^t. Also, no two fixed lines of t can meet.

LEMMA 7.2.

(i) *G_x is transitive on the lines on x.*

(ii) *If G_x contains a Klein group, then $\langle t^{G_x} \rangle$ acts on the lines on x as $PSL(2, 2^f)$, $Sz(2^f)$, or $PSU(3, 2^f)$, in its usual 2-transitive representation.*

(iii) *G is transitive on the ordered pairs (x, L) with $x \in L$.*

(iv) *All involutions fixing k points are conjugate.*

Proof. For each line L on x and involution $t \in G(L)$, t cannot fix another line on x. Sylow's theorem implies (i), and Bender's theorem [4] implies (ii). Moreover, G is transitive on lines, and hence on points [7, p. 78]. Thus, (iii) holds, and hence so does (iv).

LEMMA 7.3. *$k \mid r - 1$ and $k < r - 1$.*

Proof. Since $v = 1 + r(k - 1)$, (7.1) implies that $k \mid r - 1$. Suppose

$r - 1 = k$. By (7.2(iii)) and Ostrom and Wagner [19], Theorem 7, or Dembowski [7, pp. 214–215], $\mathscr{D}$ is an affine translation plane and G contains the translation group. This contradicts the assumption made at the start of this section.

LEMMA 7.4. *There exist two intersecting lines generating a subspace which is not an affine plane.*

Proof. Deny! By (6.2), $\mathscr{D}$ is an affine space over $GF(k)$. t acts on the translation group V, where V is an elementary abelian 2-group. By (2.2), $k \geqslant (|V|)^{1/2}$, so $k^2 \geqslant v = 1 + r(k-1)$. This contradicts (7.3).

LEMMA 7.5. *$G(L)$ has no Klein group.*

Proof. Deny! Then G_x has a normal subgroup $H = \langle t^{G_x} \rangle$ acting on the lines on x as stated in (7.2ii). In particular, $r - 1 = 2^f$, 2^{2f}, or 2^{3f}, where $f > 1$. Let K be the kernel of the action of G_x on the lines on x. Q is a Sylow 2-subgroup of $H(L)$.

We claim that $X = O(G(L))$ is trivial. For, $XK/K \leqslant O(G_{xL}/K) = 1$, so $X \leqslant K$. It follows that X fixes each line meeting L. Thus, $X = 1$.

In particular, $K \approx K^L$ is a Frobenius complement of odd order. Since $\bar{H} = H/K \cap H$ is simple, it follows that $K \cap H \leqslant Z(H)$. Thus, $Q \lhd G_{xL}$.

We claim that $G(L)$ is semiregular off of L. For, suppose that $1 \neq g \in G(L)$ and that the set Δ of fixed points of g is a subspace properly containing L. By (2.2), $C_Q(g)$ is transitive on the lines $\neq L$ of Δ through x. In particular, it follows that $C_G(g)^\Delta$ and Δ satisfy the hypotheses of Theorem 3. If $|\Delta| = k^2$, the 2-transitivity of G_x implies that any two intersecting lines of $\mathscr{D}$ are contained in an affine plane, and this contradicts (7.4). Thus, Δ is $\mathscr{D}(k)$, and this contradicts (6.4iii).

Let $h \in H_L$ be such that its image $\bar{h} \in \bar{H}$ has order $2^f - 1$. Then h^i fixes exactly one other line $L' \neq L$ on x, whenever $h^i \notin K$. We know that h^i cannot fix pointwise a subspace of more than k points. Choose h^i with $|\bar{h}^i|$ a prime power, and find that $\langle h^i \rangle$ is semiregular on $L - \{x\}$ or $L' - \{x\}$. Thus, $2^f - 1 \mid k - 1 = 2^e - 1$, so $f \mid e$.

By (7.3), $2^e \mid r - 1$ and $2^e < r - 1$. Thus, $\bar{H} \approx Sz(2^f)$ and $e = f$, or $\bar{H} \approx PSU(3, 2^f)$ and $e = f$ or $e = 2f$. Moreover, if $e = f$ then $|G_{xL}^L|$ is divisible by each prime power divisor of $k - 1$, so that $G_L{}^L$ is 2-transitive. By (7.4) and [16], Theorem 3.4, this is impossible.

Thus, $\bar{H} \approx PSU(3, 2^f)$ and $e = 2f$. Replace h above by an element of order dividing $(2^f + 1)/(2^f + 1, 3)$. Then h^i fixes exactly $2^f + 1$ lines on x whenever $h^i \notin K$. We know that h^i fixes at most k points. Hence, as above, we find that $(2^f + 1)/(2^f + 1, 3)$ divides $|G_{LL'}^L|$.

It follows that $(2^{2f} - 1)/3 = (k - 1)/3$ divides the length of each orbit of $G_{LL'}$ on $L' - \{x\}$. The 2-transitivity of G_x implies that each orbit of G_{xL} of points off L has length divisible by $(r-1)(k-1)/3 = (v-k)/3$. For any point $z \notin L$,

$$| G_{zL}\colon G_{xzL} | = | G_L\colon G_{xL} | \, | G_{xL}\colon G_{xzL} | / | G_L\colon G_{zL} |$$
$$= k \cdot (v-k)\alpha/(v-k)\beta = k\alpha/\beta,$$

where $\alpha \leqslant \beta$ and $\alpha, \beta \in \{1/3, 2/3, 1\}$. Hence, each orbit of G_{zL}^L has length divisible by $k/2$.

Let R be a Sylow 2-subgroup of G_{zL}. Then R induces an elementary abelian 2-subgroup of $G_L{}^L$, and hence is elementary abelian. Thus, $| R | \geqslant k/2 = 2^{2f-1}$ where $f > 1$. On the other hand, R is isomorphic to a subgroup of $P\Gamma U(3, 2^f)$. Since an involution in $P\Gamma U(3, 2^f) - PSU(3, 2^f)$ fixes more than one line in the usual permutation representation, $| R | \leqslant 2^f$. This contradiction proves the lemma.

LEMMA 7.6. *If S is elementary abelian, then $N_G(S)$ is transitive on $S \cap t^G$.*

Proof. If $t^g \in S$, then $S^{g^{-1}}$ is Sylow in $C_G(t)$, so $S^{g^{-1}} = S^c$ with $c \in C_G(t)$. Thus, $cg \in N_G(S)$ and $t^g = t^{cg}$.

LEMMA 7.7. *Assume that $t \in Z(G_L)$, and $C_{G(L)}(u) = \langle t \rangle$ whenever $t \neq u \in t^G \cap C_G(t)$. Assume further that $| G(L)|$ divides $r-1$. Label the involutions in $G_L{}^L$ in any way, and let α_i denote the number of elements of $t^G \cap C_G(t)$ inducing the i-th involution. Let δ be the number of nonzero α_i's. Then the following hold.*

(i) $\alpha_i = \beta_i \, |G(L)|/2$ *with* $\beta_i \in \{0, 1, 2\}$.

(ii) $2(k-1) \geqslant 2\delta \geqslant \sum \beta_i = (k-1) \cdot 2(r-1)/k \, |G(L)|$, *where* $2(r-1)/k \, |G(L)|$ *is an integer.*

(iii) *If some $\beta_i = 2$, then $G(L)$ is abelian. Moreover, if $u_i \in t^G \cap C_G(t)$ induces the i-th involution, then u inverts $G(L)$.*

Proof.

(i) Clearly, u has $\frac{1}{2} \, |G(L)|$ conjugates under $\langle u \rangle \, G(L)$. This proves that $uG(L)$ contains $\frac{1}{2} \, |G(L)|$ or $|G(L)|$ conjugates of t.

(ii) By (7.1) and (7.5), $\sum \alpha_i = (v-k)/k$. Hence,

$$\sum \beta_i = (k-1) \cdot 2(r-1)/k \, |G(L)|.$$

Since $k - 1$ is odd and $|G(L)| \mid r - 1$, $2(r-1)/k \mid G(L)|$ is an integer. This proves (ii).

(iii) Here, $u_i G(L)$ consists entirely of involutions.

LEMMA 7.8. *Suppose* $F_L = 1$. *Then*

(i) *t inverts F*;

(ii) $G_x = FC_G(t)_x$;

(iii) *F is abelian, and is regular on the lines on x; and*

(iv) *t is the unique involution in $G(L)$.*

Proof. By (7.2i) and (7.5), $G_x \trianglerighteq \langle t \rangle\, O(G_x) = X$. Here, X is solvable (by [8]), and F is its Fitting subgroup. $C_F(t) \leqslant F \cap G_L = 1$, so t inverts F. If $t' \in X$ is another involution, then $tt' \in C_X(F)$, where $C_X(F) = F$ (by [10], p. 218). Thus, $\langle t^X \rangle = \langle t^F \rangle = \langle t \rangle F$ is transitive and $G_x = FC_G(t)_x$. Also, if $t' \in G(L)$ then $tt' \in F \cap G(L) = 1$.

8. *Elementary Abelian S*

In this section, we will prove:

PROPOSITION 8.0. *S is not elementary abelian.*

Thus, assume that S is elementary abelian. Clearly, $S \cap G(L) = \langle t \rangle$. By the Feit–Thompson theorem [8], G_x is solvable.

LEMMA 8.1. *G_L has a normal subgroup of index 2 not containing t.*

Proof. $G_L/O(G(L)) \trianglerighteq S \cdot O(G(L))/O(G(L))$, and $tO(G(L)) \in Z(G_L/O(G(L)))$. Now apply Maschke's theorem (or transfer).

LEMMA 8.2. *If $S \trianglelefteq G_L$, then not all involutions in S are conjugate.*

Proof. Deny! Then S is Sylow in G, and distinct conjugates of S intersect trivially. Thus, since G is generated by its involutions, $G \approx PSL(2, 2k)$.

Hence, $\mathscr{D}$ is $\mathscr{D}(k)$ by (6.3). However, we have assumed this is not the case.

LEMMA 8.3. *If $N \lhd G_x$ and $N \cap G(L) = 1$ then $N \cap G_L = 1$.*

Proof. Assume that $N \cap G_L \neq 1$. Since $[N, G(L)] = 1$, $C_G(G(L))$ is transitive on L. Also, $S \cap G(L) = \langle t \rangle$. Consequently, there is a normal subgroup $R \times G(L)$ of G_L, with R elementary abelian of order k.

Let ϵ be the number of involutions in $G(L)$. Each coset $rG(L) \neq G(L)$, $r \in R$, contains exactly $\epsilon + 1$ involutions. Count in two ways the ordered triples (u, x, y) with $u \in t^G$ and $x^u = y \neq x$. Since there are vr/k lines, there are $(vr/k)\epsilon \cdot (v - k)$ triples. On the other hand, if $x, y \in L$ are given, then u must be in $RG(L)$, so $u = rg$ with $1 \neq r \in R$ and $g \in G(L)$ an involution or 1; thus, there are at most $v(v - 1)(\epsilon + 1)$ triples.

Hence, $(vr/k)\,\epsilon(v - k) \leqslant v(v - 1)(\epsilon + 1)$. Here, $v - k = (r - 1)(k - 1)$, so $r - 1 \leqslant (1 + 1/\epsilon)k \leqslant 2k$. By (7.3), $k \mid r - 1$ and $k < r - 1$. Thus, $r - 1 = 2k$, $\epsilon = 1$, and all involutions in $RG(L)$ are conjugate to t. Since $\epsilon = 1$, $G_L \trianglerighteq S$. This contradicts (8.2).

LEMMA 8.4. *Let p be a prime and A a nontrivial normal abelian p-subgroup of G_x. Assume that a Sylow p-subgroup P of $G(L)$ is nontrivial, and let Δ be its set of fixed points. Then the following hold.*

(i) $|\Delta| = k$, k^2, *or* $k(2k - 1)$.

(ii) *If* $|\Delta| = k(2k - 1)$, *then* $G_\Delta{}^\Delta \trianglerighteq PSL(2, 2k)$.

(iii) *If* $|\Delta| = k^2$, *then Δ is a translation plane.*

(iv) $C_G(P)_L$ *is transitive on L.*

Proof. If $P \leqslant G(L')$, then $G(L')$ has an involution normalizing P since $4 \nmid |G(L)|$. Thus, if $|\Delta| > k$ then $N_G(P)^\Delta$ satisfies the conditions of Theorem 3. This proves (i)–(iii).

Since $A \nleqslant P$ and P is Sylow in $G(L)$, necessarily $N_A(P) \nleqslant G(\Delta)$ and $N_A(P)^\Delta$ is semiregular on $\Delta - \{x\}$. Clearly, $N_G(P) \cap G(\Delta)$ normalizes $N_A(P)$, and $N_A(P)$ is abelian.

Suppose (ii) holds, and let S be a Sylow 2-subgroup of $N_G(P)$. Then $S \cap G(\Delta) = 1$, and $(N_G(P) \cap N_G(S))^\Delta$ is irreducible on S^Δ. Thus, $N_G(P) \cap N_G(S)$ is irreducible on S. By (6.1i) with $Y = N_G(P)$ and $X = N_G(P) \cap G(\Delta)$, $C_G(P) \geqslant S$. Since $C_G(P)^\Delta \trianglelefteq N_G(P)^\Delta$, it follows that $C_G(P)^\Delta \geqslant PSL(2, 2k)$. In particular, (iv) holds in this case.

If $|\Delta| = k^2$, then (by (6.5)) $N_G(P)$ has a subgroup Y of index 2 such that $Y \cap G(L)$ has odd order. Similarly, if $|\Delta| = k$ then (by (8.1)) $N_G(P)$ also has such a subgroup Y.

Set $X = Y \cap G(\Delta)$. Then X has odd order. Clearly, $N_A(P)$ is an abelian p-group contained in Y, not contained in X, and normalized by X.

Let R be a Sylow 2-subgroup of Y. Then R^Δ is a regular normal subgroup of $N_G(P)^\Delta$. Since $N_A(P)^\Delta$ is semiregular on $\Delta - \{x\}$, it is fixed-point-free on R^Δ. $O(N_G(P)^\Delta) = 1$, so $X = O(Y)$.

Hence, (6.1(ii)) applies, so $R \leqslant C_Y(X) \leqslant C_G(P)$. Since $(R^\Delta)_L$ is transitive on L, this proves (iv).

LEMMA 8.5. *If A is a normal abelian p-subgroup of G_x, then $A \cap G(L) = 1$.*

Proof. Suppose $B = A(L) \neq 1$. If $B \leqslant G(L')$ with $x \in L'$, then $L' = L^g$ with $g \in G_x$ (see (7.2(iii))). Thus, $B \leqslant G(L)^g \cap A^g = B^g$. Consequently, $N_G(B)_x$ is transitive on the lines on x fixed pointwise by B.

By (8.4iv), $C_G(B)_L$ is transitive on L. Consequently, $N_G(B)$ is transitive on the set Γ of fixed points of B. Moreover, B is weakly closed in $G(L)$ with respect to G.

Any two points x_1, y_1 of $\mathscr{D}$ are in exactly one set Γ^g, $g \in G$. (For, by (7.2(iii)), we may assume that $x_1 = x$ and $y_1 \in L$.) Thus, these sets form a new design with $\lambda = 1$. Since A is abelian, it fixes Γ and hence each Γ-line on x; hence so does B. But $B = (A^h)_x$, where $x \neq x^h \in L$ and $h \in G$, so B also fixes each Γ-line on x^h. Since each point not in Γ is the intersection of Γ-lines on x and x^h, it follows that $B = 1$.

LEMMA 8.6. $F_L = 1$.

Proof. Deny! By (8.3), $F(L) \neq 1$. Let p be a prime such that $U_y \neq 1$, where $U = O_p(F)$. Let $A = Z(U)$. By (8.3) and (8.5), A moves L. Thus, if Γ is the set of fixed points of U_y, then $|\Gamma| > k$.

By (8.4(iv)), $C_G(U_y)_L$ is transitive on L. If $x \in L' \subseteq \Gamma$, then $U_y = U(L')$ is normalized by each involution in $G(L')$. The minimality of G thus implies that $|\Gamma| = k^2$ or $k(2k-1)$; moreover, $N_G(U_y)$ has an elementary abelian subgroup R such that R^Γ is regular if $|\Gamma| = k^2$, while $|R| = 2k$ if $|\Gamma| = k(2k-1)$.

Let L' be one of the $(v/k) - (|\Gamma|/k)$ lines fixed by t not in Γ, and let $t' \in G(L')$ centralize t. Since $|\Gamma|$ is even, $\Gamma \cap L' = \varnothing$, so $|\Gamma| = k^2$. We may assume that t' normalizes R, and $t'^\Gamma \in R^\Gamma$. Then $t' \in R$, so $R \leqslant C_G(t')$ and $|R| = k^2$, whereas a Sylow 2-subgroup of $C_G(t')$ is assumed in this section to have order $2k$.

LEMMA 8.7.

(i) *F is regular on the lines on x.*

(ii) *t is the unique involution in $G(L)$.*

(iii) $G_{xL}^L = 1$.

Proof. (8.6) and (7.8) imply (i) and (ii).

Suppose $G_{xL}^L \neq 1$. Write $X = (G_{xL})' \, G(L)F$, so $X \lhd G_x$.

Let $w \in G_{xL}$. We claim that $w^g \equiv w \pmod X$ whenever $w^g \in G_x$ and $g \in G$. This is clear if $g \in G_x$ (since $(G_x)' \leqslant X$) so we may assume that $g \notin G_x$. Then w fixes $x^{g^{-1}} \neq x$. Let L' be the line through x and $x^{g^{-1}}$. Then $w \in G(L')$. Since G_x is transitive on the lines through x, it follows that $w \in X$. Similarly, $w^g \in G(L'^g)$ and $x = (x^{g^{-1}})^g \in L'^g$, so $w^g \in X$. This proves our claim.

Since $|G_x/X| \mid k-1$ and $k-1 \mid v-1$, $(|G: G_x|, |G_x/X|) = 1$. Transferring into G_x/X, we find that G has a proper normal subgroup H of odd index such that $H \cap G_x = X$. Since G is generated by its involutions, this is impossible.

LEMMA 8.8. *$G(L)$ is semiregular off of L.*

Proof. Deny! Let $W \leqslant G(L)$ be a nontrivial q-group (for some prime q) fixing a point off L. Let Δ denote the subspace of fixed points of W.

By (8.7(iii)), each fixed line L' of W is pointwise fixed. By (8.7(ii)), the involution in $G(L')$ centralizes W. Hence, $C_G(W)^\Delta$ satisfies the hypotheses of Theorem 3, so either $|\Delta| = k^2$ or $|\Delta| = k(2k-1)$. If $|\Delta| = k^2$, then (6.5) and (8.7iii) show that $|N_G(W)^\Delta(L)| = 2$. By (6.4iv) and (8.7iii), the same equality holds when $|\Delta| = k(2k-1)$. Consequently, a Sylow q-subgroup of $G(L)$ fixes Δ pointwise, so we may assume that W is Sylow in $G(L)$.

Let $t_1 \in (C_G(t) - \{t\}) \cap t^G$. Since $G(L) = \langle t \rangle \times O(G(L))$, t_1 normalizes some Sylow q-subgroup W_1 of $C_G(t) = G_L$. Consider first the case $W_1 = W$. Here, t_1 acts on Δ. If $|\Delta| = k^2$ and $t_1{}^\Delta$ is in the translation group of $N(W)^\Delta$, then t_1 centralizes an elementary abelian 2-group of order k^2, whereas a Sylow 2-subgroup of $C_G(t_1)$ has order $2k$; thus, $A_{t_1} \subset \Delta$. If $|\Delta| = k(2k-1)$, t_1 must fix points of Δ, and again $A_{t_1} \subset \Delta$. Thus, $W \leqslant G(A_{t_1})$ in either case.

In the case of general W_1, Sylow's theorem shows that $W_1 \leqslant G(A_{t_1})$. Since each point $z \notin L$ is fixed by some conjugate of t (cf. (7.1(i))), z and L lie in a subspace having k^2 or $k(2k-1)$ points. The intersection of subspaces of $\mathscr{D}$ is a subspace, so (by (6.4i)) z and L are in a unique subspace having k^2 or $k(2k-1)$ points.

Moreover, Sylow's theorem implies that all these subspaces have k^2 points—which is impossible by (7.4)—or $k(2k-1)$ points. Hence, all involutions in S are conjugate.

Let U be any Sylow subgroup of $O(G(L))$. We may assume S normalizes U, and then $|S| \geqslant 8$ implies that some involution t' in $S - \langle t \rangle$ centralizes a nontrivial subgroup U_0 of U. Then U_0 fixes a point of the axis of t'. Hence, the first part of this proof shows that U fixes exactly $k(2k-1)$ points. By (6.4i), Δ^g is the set of fixed points of U, for some $g \in G_L$.

We now see that $G(\Delta)$ contains a Sylow q-subgroup for each prime $q \mid |O(G(L))|$. Consequently, $O(G(L))^\Delta = 1$, so $\Delta^g = \Delta$ for all $g \in G(L)$. Since we already know that each point of $\mathscr{D}$ is in Δ^g for some $g \in G(L)$, this is absurd.

LEMMA 8.9.

(i) *$G(L)$ is fixed-point-free on F, so $|G(L)|$ divides $|F| - 1 = v - 1$.*

(ii) *Each element of* $t^G \cap (C_G(t) - \{t\})$ *inverts* $G(L)$.

(iii) $G(L)$ *is cyclic.*

Proof.

(i) By (8.7i) and (8.8), if $1 \neq g \in G(L)$ then $C_F(g) \leqslant F_L = 1$.

(ii) Let $u \in G(L')$ be such an element. Then $C_{G(L)}(u)$ acts on L'. By (8.7(iii)), $C_{O(G(L))}(u) \leqslant G(L')$, where $G(L) \cap G(L') = 1$ by (8.8).

(iii) By (ii), $G(L)$ is abelian, and hence is cyclic by (i).

LEMMA 8.10. $C_G(t)$ *contains an involution not conjugate to* t.

Proof. Otherwise, S is an elementary abelian group of order $2k \geqslant 8$, all of whose involutions are conjugate to t. By (8.9(ii)), $O(G(L)) = 1$. This contradicts (8.2).

Conclusion of the Proof of Proposition 8.0.

We will use the notation of (7.7). By (8.10), $2(k-1) > \sum \beta_i = (k-1) \cdot 2(r-1)/k \mid G(L)|$. Hence, $k - 1 = \sum \beta_i \leqslant 2\delta$, so $\delta \geqslant k/2$. If $u_1, u_2 \in (C_G(t) - \{t\}) \cap t^G$, then $u_1 u_2$ centralizes $G(L)$ by (8.9ii). Hence, $| C_G(G(L))^L | \geqslant k/2$. By (8.7iii) and (8.9(iii)), $C_G(G(L)) = R \times G(L)$ with $| R | \geqslant k/2$ and $| G_L : RG(L)| \leqslant 2$. Since $k < r - 1 = k \mid O(G(L))|$ by (7.3), while u_1 inverts $O(G(L))$, we must have $| R | = k/2$ and $(R\langle t\rangle) \cap t^G = \{t\}$. In particular, $\delta = k/2$.

Now $\sum \beta_i = k - 1 = 2\delta - 1$, so $\beta_i = 0$ or 2 with the exception of a single i for which $\beta_i = 1$. Certainly $| S: R\langle t\rangle| = 2$, and hence $S - R\langle t\rangle$ consists of conjugates of t, except for just one element. By (7.6), $N_G(S)$ has an element g such that $(R\langle t\rangle)^g \neq R\langle t\rangle$. Then $(R\langle t\rangle)^g$ has $k/2$ elements not in $R\langle t\rangle$. If $k/2 > 2$, then $(R\langle t\rangle)^g$ contains more than one conjugate of t, whereas $R\langle t\rangle$ does not.

Hence, $k = 4$ and $| S | = 8$. By (7.6) and (8.7(iii)), $N_G(S)/C_G(S)$ is a 2-group regular on $S \cap t^G$. There are three involutions $r_1, r_2, r_3 \in S$ not conjugate to t, two in $R\langle t\rangle$ and one in $S - R\langle t\rangle$. Then $r_1 r_2 \neq r_3$. $N_G(S)$ fixes some pair $\{r_\alpha, r_\beta\}$, $\alpha \neq \beta$, and hence centralizes $r_\alpha r_\beta \in t^G$, which is ridiculous.

9. *The Case* v/k *Odd*

Proposition 8.0 has the following easy corollaries.

LEMMA 9.1.

(i) v/k *is odd.*

(ii) S *is a Sylow* 2-*subgroup of* G.

Proof.

(i) Suppose v/k is even. S acts on the $(v/k) - 1$ lines $\neq L$ fixed by t (see (7.1)), and hence fixes one of them, say L'. Then S centralizes an involution $u \in G(L')$, so $u \in Z(S)$. Since t is not a square in $C_G(u)$, $|Q| = 2$. Also, S/Q is elementary abelian. If S is abelian, it follows that it is elementary abelian, and then (8.0) applies. If S is nonabelian, then by [10], p. 196, it has an extraspecial subgroup having center $Q = \langle t \rangle$. Then t is a square in $S \leqslant C_G(u)$, which is not the case.

(ii) There are vr/k lines.

LEMMA 9.2. $G_{xL}^L = 1$.

Proof. Deny! By (7.5), $Q = S \cap G(L)$ is cyclic or quaternion. Also, $G_L = G(L)\, N_G(Q)$, so $N_G(Q)^L$ is a Frobenius group whose kernel is an elementary abelian 2-group. In particular, no nontrivial homomorphic image of $N_G(Q)^L$ can be S_3 or a 2-group.

On the other hand, $N_G(Q)/QC_G(Q)$ is isomorphic to a group of outer automorphisms of Q, and hence is A_3, S_3, or a 2-group. Consequently, $QC_G(Q)$ contains a Sylow 2-subgroup of $N_G(Q)$.

By (7.1), we can find $u \in t^G \cap (C_G(t) - \{t\})$, and we may assume that $u \in QC_S(Q)$. Then $u = qc$ with $q \in Q$ and $c \in C_S(Q)$. Clearly, u centralizes q and c. But t is not a square in $C_G(u)$, so $q = 1$ or t. Then $u \in C_G(Q)$, so necessarily $|Q| = 2$.

Similarly, if $u \in Z(S)$, then t cannot be a square in S. Then S is elementary abelian, and this contradicts (8.0).

Thus, S is nonabelian, so $S' = \langle t \rangle$. Then $S = EZ(S)$ with E an extraspecial group ([10], p. 196). If $|E| > 8$ then $C_E(u)$ has an element of order 4, which is again a contradiction. Thus, $|E| = 8$.

Since $G_L \trianglerighteq SG(L)$, $G_L{}^L = N_G(S)^L$ by the Frattini argument. Thus, $N_G(S)_x$ has a subgroup H of odd order inducing a nontrivial fixed-point-free group on $S/\langle t \rangle$. By Maschke's theorem, $S/\langle t \rangle = B/\langle t \rangle \times Z(S)/\langle t \rangle$ with $B > \langle t \rangle$ invariant under H. Here, $|B| = 8$, and H acts nontrivially on B, so B is quaternion. Now $u \in S = BZ(S)$ implies that t is a square in $C_S(u)$, which is not the case.

LEMMA 9.3. *There is no subspace $\Delta \approx \mathscr{D}(k)$ such that $G_\Delta{}^\Delta \geqslant PSL(2, 2k)$.*

Proof. Assume Δ exists. Then S has an elementary abelian subgroup X of order $2k \geqslant 8$ all of whose involutions are conjugate to t. If $Q > \langle t \rangle$, it has a subgroup $\langle g \rangle$ of order 4 normal in S. If $Q = \langle t \rangle$, then once again $S = EZ(S)$ with E extraspecial, so S has a normal subgroup $\langle g \rangle$ of order 4.

In either case, $g^2 = t$. For each $u \in X - \langle t \rangle$, we know that $g \notin C_G(u)$, so $g^u = g^{-1}$. Since $|X| \geqslant 8$, this is absurd.

LEMMA 9.4. $F_L = 1$.

Proof. Deny! By (9.2), $F(L) = F_L \neq 1$. Let p be a prime dividing $|F(L)|$, P a Sylow p-subgroup of $G(L)$, and Δ the set of fixed points of P. Clearly, $N_{PF}(P) > P$, so $N_F(P)$ moves L by (9.2). Thus, $|\Delta| > k$.

Since $t \in Z^*(G(L))$ by (7.5) and [9], $N_{G(L)}(P)$ contains an involution. Clearly, this is also true for any line $L' \subset \Delta$. Thus, the minimality of G forces $|\Delta| = k^2$ or $k(2k - 1)$. By (9.3), $|\Delta| = k^2$.

By (9.2) and (6.5), $|N_G(P)^\Delta| = k^2(k + 1)2$. Then $N_G(P)$ has a subgroup Y of index 2 such that $t \notin Y$ and Y/X is a Frobenius group of order $k^2(k + 1)$ whose kernel is an elementary abelian 2-group; here, $X = Y \cap G(\Delta)$.

Let $A = O_p(Z(F))$. Then $N_{PA}(P) > P$ and $N_A(P)$ moves L (by (9.2)). Thus, $N_A(P) \leqslant Y$, $N_A(P) \nleqslant X$, and X normalizes $N_A(P)$. By (6.1ii), $C_G(X)$ has an elementary abelian 2-subgroup R of order k^2 regular on Δ.

Let $U = O_p(G_x)$. Then $U_y \leqslant P$, so $R \leqslant C_G(U_y)$. Let Γ be the set of fixed points of U_y. Then $\Gamma \supseteq \Delta$. Since $U \lhd G_x$, (7.2i) implies that $N_G(U_y)_x$ is transitive on the fixed lines of U_y through x. Also, $R_L{}^L \leqslant C_G(U_y)_L^L$ is transitive, while t normalizes U_y. It follows that $N_G(U_y)^\Gamma$ satisfies the hypotheses of Theorem 3, so $\Gamma = \Delta$ by (9.3).

Let L' be one of the $(v/k) - (|\Gamma|/k)$ lines not in Γ fixed by t. Then t centralizes an involution $u \in G(L')$. Since $C_G(U_y)_L$ is transitive on L, $U_y = O_p(G_y)_x$. Thus, $U_y \lhd G_L$, so u normalizes U_y and hence acts on Γ. However, $L' \cap \Gamma = \varnothing$, so we can choose R so that u normalizes R and $u^\Gamma \in R^\Gamma$. Thus, $u \in R$, so $C_G(u)^{L'}$ has an elementary abelian subgroup $R^{L'}$ of order $\geqslant k^2/2 > k$, which is absurd.

COROLLARY 9.5.

(i) *F is abelian and is regular on the lines on x.*

(ii) *t is the unique involution in $G(L)$.*

Proof. (9.4) and (7.8).

LEMMA 9.6.

(i) *$G(L)$ is semiregular off of L.*

(ii) *$G(L)$ is fixed-point-free on F so $|G(L)|$ divides $|F| - 1 = r - 1$.*

(iii) *If $u \in t^G \cap (C_G(t) - \{t\})$ then $C_{G(L)}(u) = \langle t \rangle$.*

Proof. In view of the proof of (8.9), we need only prove (i).

We will imitate the proof of (8.8). Let $1 \neq W \leqslant G(L)$, and suppose that the set Δ of fixed points of W has more than k points. By the minimality of G, (9.5(ii)), and (9.3), $|\Delta| = k^2$. Also, $|N_G(W)^{\Delta}(L)| = 2$ by (9.2) and (6.5), while $|C_F(W)| = k + 1$ by (2.2). Consequently, the following condition holds:

(*) If $1 \neq W \leqslant G(L)$ and $C_F(W) \neq 1$, then $N_{G(L)}(W) = \langle t \rangle O(N_{G(L)}(W))$, and $C_F(W) = C_F(U)$ whenever $1 \neq U \leqslant O(N_{G(L)}(W))$.

Write $F_0 = C_F(W)$ and $H_0 = C_{G(L)}(F_0) \neq 1$. By (*), $H_0 \geqslant O(N_{G(L)}(U))$ whenever $H_0 \geqslant U \neq 1$; choose $1 \neq H \leqslant H_0$ minimal with respect to this property. Then H is Hall in $G(L)$, and $H \cap H^g = 1$ whenever $g \in G(L) - N_{G(L)}(H)$. Note that, by (*), $N_{G(L)}(H) = \langle t \rangle \times H$.

We may now assume that W is Sylow in H. In view of the proof of (8.8), some $u \in t^G \cap (C_G(t) - \{t\})$ does not normalize any conjugate of W in $G(L)$.

In particular, $|Q| > 2$, so that $G(L) > \langle t \rangle H$. Since $\langle t \rangle H \cap \langle t \rangle H^g = \langle t \rangle$ whenever $g \in G(L) - \langle t \rangle H$, it follows that $G(L)/\langle t \rangle$ is a Frobenius group with complement $\langle t \rangle H/\langle t \rangle$; let $K/\langle t \rangle$ be its kernel. By Thompson's theorem [10, p. 337], K is nilpotent. Here, $O_2(K) = Q$ has order $\geqslant 4$. Since H is fixed-point-free on $Q/\langle t \rangle$, it follows that Q is quaternion of order 8 and $|H| = |W| = 3$.

Thus, $G(L)$ is solvable. There are an odd number of Hall $\{2, 3\}$-subgroups of $G(L)$, so u normalizes one of them. Thus, we may assume u normalizes HQ. Then u permutes the 4 Sylow 3-subgroups of HQ, and $C_Q(u) = \langle t \rangle$. Thus, either u centralizes H, or $\langle u \rangle HQ/\langle t \rangle \approx S_3$. In either case, u normalizes some conjugate of H. However, we have already noted that this leads to a contradiction.

LEMMA 9.7. *In the notation of* (7.7), *the following statements hold after suitably relabeling the* β_i's.

(i) $r - 1 = \frac{1}{2}k\,|G(L)|$.

(ii) $\delta = k/2$.

(iii) $\beta_1 = 1$, $\beta_i = 2$ *for* $2 \leqslant i \leqslant \delta$.

(iv) $G(L)$ *is cyclic.*

(v) u_i *inverts* $G(L)$ *if* $2 \leqslant i \leqslant \delta$; u_1 *inverts* $O(G(L))$.

(vi) $|Q| > 2$.

Proof. By (9.5) and (9.6), we can apply (7.7). By (7.7ii), $r - 1 = k\,|G(L)|$ or $\frac{1}{2}k\,|G(L)|$.

Suppose first that $r - 1 = k\,|G(L)|$, so $\delta = k - 1$ and each $\beta_i = 2$. By (7.7(iii)), each $u \in t^G \cap (C_G(t) - \{t\})$ inverts $G(L)$. Hence, $C_G(G(L))^L$ must be transitive. Thus, either $C_G(G(L))$ has the unique involution t, or $G(L) = \langle t \rangle$.

If $C_G(G(L))$ has just one involution, then so does $T = C_S(G(L))$. But T^L is transitive, so necessarily T is a generalized quaternion group and $k = 4$. Now $S = QT$ and $[Q, T] = 1$. This leads to a contradiction precisely as in the proof of (9.2).

If $G(L) = \langle t \rangle$ then G_L has order $2k$, and (by (7.1)) has $1 + (v - k)/k = 2k - 1$ involutions. This contradicts (8.0).

Thus, we must have $r - 1 = \frac{1}{2}k \mid G(L)|$ in (7.7), so $2(k - 1) \geqslant 2\delta \geqslant \sum \beta_i = k - 1$. Hence, $\delta \geqslant k/2$. Moreover, since $(v - k)/k = (k - 1)(r - 1)/k$ is even (by (9.1)), so is $\frac{1}{2} \mid G(L)|$, and hence (vi) holds.

Suppose $\delta = k/2$, so (ii) holds. Then $\sum \beta_i = k - 1$ and $0 \leqslant \beta_i \leqslant 2$ imply that we may assume (iii) holds. By (7.7(iii)), u_i inverts $G(L)$ if $2 \leqslant i \leqslant \delta$. Hence, (iv) holds by (9.6(ii)). Finally, (v) holds by (9.6(iii)).

Thus, we must show $\delta > k/2$ is impossible. In this case, $\sum \beta_i = k - 1$ implies that some $\beta_j = 2$. By (7.7(iii)) and (9.6(iii)), $G(L)$ is cyclic. In particular, $Q \lhd G_L$.

Since $|Q| > 2$, Q has an element g of order 4, and then $\langle g \rangle \lhd G_L$. Each u_i inverts g. Thus, $\delta > k/2$ implies that $| C_G(g)^L | > k/2$. Then $C_G(g)$ is transitive on L, so there is a 2-element $h \in C_G(g)$ with $h^L = u_j^L$. However, $u_j G(L)$ consists entirely of involutions conjugate to t (as $\beta_j = 2$), so $h \in t^G$. Since h centralizes g, this is impossible.

COROLLARY 9.8. *With the notion of* (9.7), *fix* $u_1 \in t^G$ *such that* u_1^L *induces the first involution of* G_L^L. *For* $2 \leqslant i \leqslant \delta = k/2$, *let* u_i *denote any involution inducing the i-th involution of* G_L^L. *Then the following hold.*

(i) $T = \langle Q, u_1 u_i \mid 2 \leqslant i \leqslant \delta \rangle$ *has index* 2 *in* S.

(ii) *If* $g \in T - Q$ *then* u_i *can be found so that* $g = u_1 u_i$.

(iii) *If* $g \in S - \langle t \rangle$ *is an involution such that* $g \notin t^G$ *and* $gt \in t^G$, *then* $g \in u_1 Q$.

Proof.

(i) In the notation of the last paragraph of the proof of (9.7), $| C_G(g)^L | \leqslant k/2$. This proves (i).

(ii) Since $(u_1 g)^L \notin T^L$, we have $(u_1 g)^L = u_i^L$ for some u_i, $i \neq 1$. Since $u_i Q \subset t^G$, we can choose u_i as in (ii).

(iii) $u_i Q \subset t^G$ for $i > 1$.

Conclusion of the Proof of Theorem 3

We first show that $C_S(Q)$ is cyclic or generalized quaternion. For, suppose $g \in C_S(Q) - \langle t \rangle$ is an involution. By (9.7(vi)) and (9.8), $g \notin t^G$ and $g \in T$. As in (9.8(ii)) we can write $g = u_1 u_i$ for some u_i. Then $g \in C_G(u_i)$. By

(9.8(iii)), applied to u_1 instead of t, we know that g is in a uniquely determined coset of $G(L_1)$ in G_{L_1} (where $u_1 \in G(L_1)$). But the same argument shows that tg is also in this coset, whereas $t \notin G(L_1)$. This proves our assertion.

By (9.6(iii)), $\langle u_1 \rangle Q$ is dihedral or quasidihedral. Suppose it is dihedral. Then, by (9.7(v)), $T = C_S(Q)$ has index 2 in S. Also, $u_1 Q$ contains an involution not in t^G (since $\beta_1 = 1$). By (2.4), $G = \langle t^G \rangle$ has a normal subgroup of index 2, and this is impossible since t is a square.

In particular, $|Q| \geqslant 8$.

Next, $k = 4$. For suppose $k > 4$. By (9.8(ii)), $u_2 u_3 = u_1 u_i$ for some i. However, $u_2 u_3$ centralizes Q, while $u_1 u_i$ does not.

By (9.7(ii)) and (2.4) (with $S_0 = \langle u_2 \rangle Q$), t is the only involution in $\langle u_1 u_2 \rangle Q$. If $\langle u_1 u_2 \rangle Q$ is cyclic, then $S = \langle u_1, u_2 \rangle$ is dihedral, whereas $\langle u_1 \rangle Q$ is not.

Thus, $a^{u_1 u_2} = a^{-1}$, where $Q = \langle a \rangle$. But we already know $a^{u_1 u_2} = at$. Thus, $|Q| = 4$.

This contradiction completes the proof of Theorem 3.

10. Appendix

Harada's results [12] on 2-transitive groups can be formulated more precisely as follows.

THEOREM. *Let G be a finite group 2-transitive on a set S of v points. Suppose that the stabilizer of two points fixes exactly k points, where $2 < k < v$. If all involutions in G fix at most k points, then one of the following holds for the associated design $\mathscr{D}$.*

(i) $v = k^2$, *$\mathscr{D}$ is $AG(2, k)$, and G contains the translation group.*

(ii) $v = 28$, $k = 4$, *$\mathscr{D}$ is $\mathscr{D}(4)$, and $G \approx P\Gamma L(2, 8)$.*

(iii) $k = 3$, *$\mathscr{D}$ is $PG(2, 2)$, and $G \approx PSL(3, 2)$.*

(iv) $k = 3$, *$\mathscr{D}$ is $PG(3, 2)$, and $G \approx A_7$.*

(v) *k is odd, $v = k^3$, $\mathscr{D}$ is $AG(3, k)$, G has a regular normal subgroup R, and the stabilizer of a point has a normal subgroup $SL(3, k)$ acting on R as usual.*

(vi) *k is odd, $v = k^2$, $\mathscr{D}$ is an affine translation plane, and G contains the translation group.*

Remarks. If k is even in (i), then $G_x \trianglerighteq SL(2, k)$. But if k is odd in (i), this need not hold. Only one nondesarguesian example of (vi) is known, of order $k = 9$: the so-called exceptional nearfield plane (see [7, p. 229]).

Proof. When k is even, this is just Theorem 4 of Harada [12]. Suppose k is odd. If $O(G) = 1$, the theorem is a straightforward consequence of Harada's theorem and [1]. Thus, assume $O(G) \neq 1$, so G has a regular normal subgroup R. By (2.2), an involution t fixes $|C_R(t)|$ points. As in (3.3ii), there is a Klein group $\langle t, u\rangle$ with t and u axial. By (2.3), $v = k^2 | C_R(tu)|$. If tu fixes only one point, then (vi) holds (see the proof of (3.4)). We may thus assume that all involutions are axial and $v = k^3$. Thus, $r = k^2 + k + 1$.

Suppose $1 \neq g \in G$, Δ is the set of fixed points of g, and $|\Delta| > k$. By (2.2), $|\Delta| = |C_R(g)|$. The argument at the end of (3.0), together with (3.3ii), provides us with a Klein group $\langle t_1, u_1\rangle \leqslant C_R(g)$ with t_1^{Δ} and u_1^{Δ} axial. The argument of the preceding paragraph shows that $|\Delta| = k^2$ and Δ is an affine plane.

Fix x. If $t \in G_x$ let t^θ be the set of lines on x fixed by t other than its axis. Then $|t^\theta| = (r-1)/k = k+1$ as in (3.3v), since $C_R(t)$ is transitive on A_t.

Suppose t and u are distinct involutions in G_x. We claim that either $|t^\theta \cap u^\theta| \leqslant 1$ or $t^\theta = u^\theta$. For, suppose $|t^\theta \cap u^\theta| \geqslant 2$. As in (3.2iv), tu fixes more than k points. Let Δ be the set of fixed points of tu, so $|\Delta| = k^2$. Assume first that $A_t \neq A_u$. Since $t^\Delta = u^\Delta$, it follows that t and u each fix only $x \in \Delta$, and hence fix all $k+1$ lines of Δ through x. Thus, $t^\theta = u^\theta$ in this case. Now consider the possibility $A_t = A_u \subset \Delta$. Then $t^\theta \cap u^\theta$ has a unique line contained in Δ. Consequently, $t^\theta \cap u^\theta$ has a line $L_0 \not\subset \Delta$. Set $R_0 = |C_R(L_0)|$, so $|R_0| = k$ (by (2.2)). Also, $C_R(tu) \cap R_0 = 1$ since $L_0 \cap \Delta = \{x\}$. Thus, $R = C_R(tu) \times R_0$. Now t and u invert R_0, so tu centralizes R_0, which is absurd. This proves our claim.

There are $r = k^2 + k + 1$ lines on x, and $k+1$ lines in each t^θ. As in (2.5), we obtain a projective plane $\mathscr{P}$. Moreover, G_x is transitive on the lines of $\mathscr{P}$, and t fixes the "line" t^θ pointwise. Hence, $\mathscr{P}$ is desarguesian and G_x induces at least $PSL(3, k)$ on $\mathscr{P}$ (see Dembowski [7], p. 196). In particular, G_x is 2-transitive on the lines of $\mathscr{D}$ through x.

Moreover, it is now easy to see that some nontrivial element g of G fixes more than k points. This provides us with an affine subplane of $\mathscr{D}$. Now the 2-transitivity of G_x and Buekenhout's theorem (6.2) imply that $\mathscr{D}$ is $AG(3, k)$ except when $k = 3$, in which case $\mathscr{D}$ is $AG(3, 3)$ by Hall [11].

References

1. J. L. Alperin, D. Gorenstein, and R. Brauer, Finite simple groups of 2-rank two, *Scripta Math.* **29** (1974), 191–214.
2. R. Baer, Polarities in finite projective planes, *Bull. Amer. Math. Soc.* **52** (1946), 77–93.

3. H. Bender, Endliche zweifach transitive Permutationsgruppen, deren Involutionen keine Fixpunkte haben, *Math. Z.* **104** (1968), 175–204.
4. H. Bender, Transitive Gruppen gerader Ordnung, in denen jede Involution genau einen Punkt festlasst, *J. Algebra* **17** (1971), 527–554.
5. R. Brauer, Some applications of the theory of blocks of characters of finite groups II, *J. Algebra* 1 (1964), 307–334.
6. F. Buekenhout, Une caractérisation des espaces affins basée sur la notion de droite, *Math. Z.* 111 (1969), 367–371.
7. P. Dembowski, "Finite geometries," Springer, Berlin-Heidelberg-New York, 1968.
8. W. Feit and J. G. Thompson, Solvability of groups of odd order, *Pacific J. Math.* **13** (1963), 771–1029.
9. G. Glauberman, Central elements in core-free groups, *J. Algebra* **4** (1966), 403–420.
10. D. Gorenstein, "Finite groups," Harper, New York 1968.
11. M. Hall, Jr., Automorphisms of Steiner triple systems. *IBM J. Res. Dev.* **4** (1960), 460–472.
12. K. Harada, On some doubly transitive groups. *J. Algebra* **17** (1971), 437–450.
13. C. Hering (to appear).
14. D. G. Higman and J. E. McLaughlin, Geometric ABA-groups, *Illinois J. Math.* **5** (1961), 382–397.
15. B. Huppert, Zweifach transitive, auflösbare Permutationsgruppen, *Math. Z.* **68** (1957), 126–150.
16. W. M. Kantor, On 2-transitive groups in which the stabilizer of two points fixes additional points, *J. Lond. Math. Soc.* **5** (1972), 114–122.
17. W. M. Kantor, On 2-transitive collineation groups of finite projective spaces, *Pacific J. Math.* **48** (1973), 119–131.
18. H. Nagao, On multiply transitive groups IV, *Osaka J. Math.* **2** (1965), 327–341.
19. T. G. Ostrom and A. Wagner, On projective and affine planes with transitive collineation groups, *Math. Z.* **71** (1959), 186–199.
20. H. Wielandt, Beziehungen zwischen den Fixpunktzahlen von Automorphismengruppen einer endlichen Gruppe, *Math. Z.* **73** (1960), 146–158.
21. H. Wielandt, "Finite Permutation Groups," Academic Press, New York, 1964.