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Rank 3 Groups and Biplanes

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1. INTRODUCTION

Let G be a primitive rank 3 permutation group on a set X in which $\Gamma(x)$ is a nontrivial G_x-orbit, with $n = |X|$, $v = |\Gamma(x)|$. Tsuzuku [27] showed that, if G_x acts as the symmetric group on $\Gamma(x)$, then $(v, n) = (2, 5)$, $(3, 10)$, $(5, 16)$, or $(7, 50)$; he determined the possible groups in each case. Bannai [2] obtained essentially the same result under the assumption G_x is 4-transitive on $\Gamma(x)$. (Of course the cases $(2, 5)$ and $(3, 10)$ do not then arise.) Cameron [6, 7] showed that, if G_x is 3-transitive on $\Gamma(x)$, and if $(v, n) \neq (3, 10)$ or $(7, 50)$, then either

(a) $n = \frac{1}{2}(v^2 + v + 2)$, or

(b) $v = (s + 1)(s^2 + 5s + 5)$, $n = (s + 1)^2(s + 4)^2$, for some non-negative integer s.

In case (b), the known examples have $s = 0$, $v = 5$, $n = 16$ ($G = V_{16} \cdot S_5$ or $V_{16} \cdot A_5$) and $s = 1$, $v = 22$, $n = 100$ ($G = HS$ or $HS \cdot Z_2$). Non-existence has been shown for a variety of values of s, including $2 \leqslant s \leqslant 103$, but the question is not yet settled. In this paper we will determine all groups that occur under case (a).

THEOREM 1. *Let G be a primitive rank 3 permutation group of degree $n = \frac{1}{2}(v^2 + v + 2)$ with subdegrees $1, v, \frac{1}{2}v(v - 1)$. Suppose that the constituent of G_x of degree v is 3-transitive. Then either*

(i) $v = 5, n = 16, G = V_{16} \cdot S_5$, *or* $V_{16} \cdot A_5$; *or*

(ii) $v = 10$, $n = 56$, $G = PSL(3, 4) \cdot V_4$, *or either of two of its three subgroups of index* 2.

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The proof of Theorem 1 depends upon results of Cameron [7] on biplanes (symmetric designs with $\lambda = 2$) and Kantor [17, 18] on Steiner systems. If, however, one applies a very deep group theoretic classification theorem (concerning $3'$-groups), the 3-transitivity assumption in Theorem 1 can be removed for many values of k:

THEOREM 2. *Let G be a primitive rank 3 group of degree $n = \frac{1}{2}(v^2 + v + 2)$ with subdegrees $1, v, \frac{1}{2}v(v-1)$. If $v \not\equiv 1 \pmod 3$, then $v = 5$, $n = 16$, and $G = V_{16} \cdot S_5$, $V_{16} \cdot A_5$, or $V_{16} \cdot (Z_5 \cdot Z_4)$.*

As remarked in Cameron [7], these theorems are equivalent to results about biplanes:

COROLLARY 1. *Let $(X, \mathscr{B})$ be a biplane on n points, with $v + 1$ points in any block, admitting a null polarity $\perp$. Suppose that G is a group of automorphisms commuting with $\perp$ such that G is transitive on $\mathscr{B}$. For $B \in \mathscr{B}$, assume that G_B is 2-transitive on $B - \{B^\perp\}$, and even 3-transitive if $v \equiv 1 \pmod 3$. Then $v = 5$ or 10, and in each case the biplane is unique.*

The designs are described in [7].

This corollary and the corollary to [8, Theorem 4] both solve special cases of the following problem. Which biplanes admit a group G fixing a block B and a point $x \in B$, and acting 3-transitively on $B - \{x\}$? Our methods give some additional information on this problem: in Section 6 we show that, in such a biplane with $|B| > 11$, the stabilizer of three points of $B - \{x\}$ fixes exactly one further point.

From Theorem 1 we obtain the following contribution to a problem of D. G. Higman [14]:

COROLLARY 2. *Let G be a primitive rank 3 group of degree n with a prime subdegree p, and suppose that an element of order p in G is conjugate to its inverse. Then one of the following occurs:*

(i) $p = 2$, $n = 5$, $G = Z_5 \cdot Z_2$;

(ii) $p = 3$, $n = 10$, $G = S_5$, *or* A_5;

(iii) $p = 5$, $n = 16$, $G = V_{16} \cdot S_5$, $V_{16} \cdot A_5$, *or* $V_{16} \cdot (Z_5 \cdot Z_4)$;

(iv) $p = 7$, $n = 50$, $G = P\Sigma U(3, 5)$.

Proof. We may assume $p > 2$. Since an element of order p fixes a unique point, it is conjugate to its inverse in the stabilizer of that point. Also, n is even (since otherwise G has odd degree and odd subdegrees, and hence odd order).

Suppose G_x is soluble. By Higman [14], if (ii) does not occur, then G has a regular normal subgroup N, an elementary abelian 2-group. An involution in G_x then fixes at least $n^{1/2}$ points; but it fixes at most one point in each orbit of a Sylow p-subgroup, so $p \leqslant n^{1/2} + 1$. (Note that G_x acts faithfully on each orbit.) Also, since G is primitive, $n \leqslant p^2 + 1$; so $n = 2^{2d}$, and G has subdegrees 1, $2^d + 1$, and $(2^d + 1)(2^d - 2)$. Then $|G_x| = |N_G(P)|$ divides $(2^d + 1)\, 2^d$, and $(2^d + 1)(2^d - 2)$ divides $|G_x|$. So $d = 2$, $p = 5$, $n = 16$.

If G_x is insoluble, then it acts 3-transitively on its orbit of length p [21]. Case (b) above cannot occur with $s > 0$, since then v is composite. So Theorem 1 applies.

Notation. If G is a permutation group on a set X, then Fix(G) denotes the set of fixed points of G in X. If $Y \subset X$, then G_Y and $G(Y)$ are the setwise and pointwise stabilizers of Y; $G_{\{x\}} = G_x$. If Y is a fixed set of G, then G^Y denotes the permutation group induced on Y by G.

Z_n and V_n are the cyclic and elementary Abelian groups of order n; S_n and A_n the symmetric and alternating groups of degree n. Notation for the classical projective groups is standard. HS is the Higman–Sims simple group. $A \cdot B$ is a split extension of A by B.

2. Preliminaries

Suppose G is a group satisfying the hypotheses of the theorem. Then the graph Γ associated with the suborbit of length v is regular of valency v, contains no triangles, and has the property that if two vertices are not adjacent then exactly two vertices are adjacent to both. We will call a graph with these properties a *B-graph*. Given a biplane with a null polarity, we can construct a B-graph by calling two points adjacent whenever they are distinct and conjugate under the polarity; conversely, any B-graph arises in this way (see [7]). (We will be primarily dealing with B-graphs, instead of the equivalent biplanes with null polarities, partly so as to reserve the term "block" for a different use later.)

In any B-graph Γ, select a vertex ∞ (fixed throughout the discussion). Let $\Gamma(\infty)$ be the set of vertices adjacent to ∞. Any vertex not adjacent to ∞ is adjacent to two members of $\Gamma(\infty)$; and, given $x, y \in \Gamma(\infty)$, there is one vertex other than ∞ adjacent to both. So we can label the set $\Delta(\infty)$ of vertices not adjacent to ∞ by the 2-subsets of $\Gamma(\infty)$.

If G is a group of automorphisms of Γ, and H is a subgroup of G_∞, then Γ_H will denote the connected component containing ∞ of the restriction of Γ to Fix(H) (or the vertex set of this graph), and $M_G(H)$ denotes the setwise stabilizer of Γ_H in $N_G(H)$. Thus $N_G(H)_\infty \leqslant M_G(H) \leqslant N_G(H)$.

LEMMA 2.1. (i) *Γ_H is regular.*

(ii) *If H has no subgroup of index 2 then Γ_H is a B-graph and coincides with* Fix(H).

(iii) *If $N_G(H)_x$ is transitive on $\Gamma_H(x)$ for all $x \in \Gamma_H$, then either $M_G(H)$ is transitive on Γ_H, or Γ_H is bipartite and the $M_G(H)$-orbits are the bipartite blocks.*

Proof. (i) If H fixes $x_1, \ldots, x_k \in \Gamma(\infty)$, then the vertices of $\Gamma_H(x_1)$ are ∞ and the vertices of $\Delta(\infty)$ adjacent to x_1 and x_i, $2 \leqslant i \leqslant k$. So two adjacent vertices of Γ_H have the same valency. The result follows by connectedness.

(ii) Clearly Γ_H contains no triangle. If y and z are nonadjacent vertices in $\mathrm{Fix}(H)$, then by hypothesis H fixes pointwise the two vertices adjacent to both.

(iii) $M_G(H)$ is transitive on the edges of Γ_H.

Next, we require a technique first used by Graham Higman to show that a Moore graph of valency 57 admits no even involutions (see also [23, 24]). Let Γ be a strongly regular graph, and Δ its complement. Let C, D be the intersection matrices of Γ and Δ (see [13]). Then I, C, D span a three-dimensional commutative algebra over $\mathbb{C}$, so they are simultaneously diagonalizable. Let (l, m, n) be an eigenvector, with eigenvalues c and d at C and D, respectively. For any automorphism g of Γ, let

$$\begin{aligned}
\mathrm{fix}(g) &= |\{x \in X \mid x^g = x\}| = |\,\mathrm{Fix}(g)|,\\
\alpha(g) &= |\{x \in X \mid x^g \in \Gamma(x)\}|,\\
\beta(g) &= |\{x \in X \mid x^g \in \Delta(x)\}|,\\
\chi(g) &= (l\,\mathrm{fix}(g) + m\alpha(g) + n\beta(g))/(l + mc + nd).
\end{aligned}$$

Then the function $g \to \chi(g)$ is a character of the automorphism group of Γ (not necessarily irreducible), so $\chi(g)$ is an algebraic integer. Note that Γ has $\mathrm{fix}(g) + \alpha(g) + \beta(g)$ points.

LEMMA 2.2. *Let Γ be a B-graph of valency v. Then*

(i) *$v = u^2 + 1$ for some integer u not divisible by 4;*

(ii) *an automorphism of order 3 fixes at least two points; if ∞ is one of these, and exactly w points of $\Gamma(\infty)$ are fixed, then u divides $\frac{1}{2}(w - 1)(w + 2)$; if $w = 1$ then u is odd; and*

(iii) *if E is an automorphism group of odd order with $|F(E)| > 2$, then $F(E)$ is a B-graph of valency w with $w \leqslant u$.*

Proof. (i) In this case we find

$$C = \begin{pmatrix} 0 & 1 & 0 \\ v & 0 & 2 \\ 0 & v-1 & v-2 \end{pmatrix}, \qquad c = u - 1, \qquad d = -u$$

$(l, m, n) = (u(u^2 + 1), u(u - 1), -2)$ (where $v = u^2 + 1$), and

$$\chi(g) = ((u + 1)\,\mathrm{fix}(g) + \alpha(g) - (u^2 + u + 2))/2u.$$

Put $g = 1$; then $\mathrm{fix}(1) = \frac{1}{2}(v^2 + v + 2)$, $\alpha(1) = 0$, and so

$$\chi(1) = \tfrac{1}{4}(u^2 + 1)(u^2 + u + 2).$$

Since $\chi(1)$ is an integer, u is an integer not divisible by 4.

(ii) Now let g be an element of order 3. Then $\alpha(g) = 0$, since if x and x^g are adjacent then $\{x, x^g, x^{g^2}\}$ is a triangle in Γ. Thus u divides $\mathrm{fix}(g) - 2$ If $\mathrm{fix}(g) \leqslant 1$ then $u = 1$ or 2, and $v = 2$ or 5. Thus, $\mathrm{fix}(g) \geqslant 2$. By Lemma 2.1(ii), $\mathrm{fix}(g) - 2 = \frac{1}{2}(w - 1)(w + 2)$; while if $w = 1$ then $2u$ divides $(u + 1)2 - (u^2 + u + 2) = -u(u - 1)$.

(iii) This follows from Lemma 2.1(ii) and Kantor [15, Lemma 9.5].

Next we need some information on the way $PSL(2, q)$ can act on a B-graph.

PROPOSITION 2.3 [1]. *Let $\mathscr{D}$ be a biplane, B a block of $\mathscr{D}$, and $x \in B$. Suppose a subgroup G of* Aut($\mathscr{D}$) *fixes x and B, and acts as $PSL(2, k - 2)$ (in its usual 2-transitive representation) on $B - \{x\}$, where $|B| = k$. Then $\mathscr{D}$ is uniquely determined by k, and $k = 4, 5, 6,$ or 11.*

This was proved by Aschbacher by a detailed calculation within $PSL(2, k - 2)$. For a description of the designs that occur, see [7]. The result should be useful in attacking the problem mentioned in the Introduction; but to prove Theorem 1 we require only a simple corollary, which we prove directly.

COROLLARY 2.4. *Let G be a rank 3 group on X with subdegrees 1, v, $\frac{1}{2}v(v - 1)$ $(v > 2)$. Suppose that, for $x \in X$, G_x acts on its orbit of length v as a subgroup of $P\Gamma L(2, v - 1)$ containing $PSL(2, v - 1)$. Then $v = 5$ or 10.*

Proof. G acts on a B-graph (and hence G_x acts on a biplane in the manner of Proposition 2.3). Let $t \in G_x$ be an involution fixing w points of $\Gamma(x)$. Then t fixes $1 + w + \frac{1}{2}w(w - 1) + \frac{1}{2}(v - w) = \frac{1}{2}(w^2 + v + 2)$ points of X. Since the function $w \to \frac{1}{2}(w^2 + v + 2)$ is one-to-one, t fixes exactly w points adjacent to any one of its fixed points. We can choose t so that $w = 0$, 1 or 2. Then $t^G \cap G_x$ is a conjugacy class in G_x; so $C_G(t)$ is transitive on the $\frac{1}{2}(w^2 + v + 2) = \frac{1}{2}(v + 2)$, $\frac{1}{2}(v + 3)$, or $\frac{1}{2}(v + 6)$ fixed points of t. Now $|C_G(t)| = |\mathrm{Fix}(t)|\,|C_{G_x}(t)|$ divides $|G| = \frac{1}{2}(v^2 + v + 2)\,|G_x|$. Using Lemma 2.2(i) and the fact that $v - 1$ is a prime power, we find that $v = 5, 10,$ 26, or 50. If $v = 26$ or 50, then an element of order 3 in G_x fixes exactly two points adjacent to x, contradicting Lemma 2.2(ii).

LEMMA 2.5. *Let $(X, \mathscr{B})$ be a biplane, $X' \subset X$, $\mathscr{B}' \subset \mathscr{B}$, and $B \in \mathscr{B}'$. Then $(X', \mathscr{B}')$ is a subbiplane if the following hold: $|B \cap X'| \geqslant 3$; if $x \in B \cap X'$ and $x \neq y \in X'$ then both blocks containing $\{x, y\}$ are in $\mathscr{B}'$; and if C, $C' \in \mathscr{B}' - \{B\}$ satisfy $|B \cap C \cap C'| = 1$, then $C \cap C' \subset X'$.*

Proof. If $l = |B \cap X'|$ then each point of X' is on l blocks of $\mathscr{B}'$, and dually. Also $|X'| = |\mathscr{B}'| = 1 + \frac{1}{2}l(l-1)$. Now count the triples (x, C, C') with $x \in X' \cap C \cap C'$ and $C, C' \in \mathscr{B}'$, and find that $C \cap C' \subset X'$ for any such C, C'.

3. INITIAL REDUCTION

Let G, acting on X, be a counterexample to Theorem 1. Let ∞ be a point of X, $H = G_\infty$, and $\Gamma(\infty) = \{x_1, ..., x_v\}$. Suppose $H_{x_1x_2x_3}$ fixes f points of $\Gamma(\infty)$ altogether. Then $N_H(H_{x_1x_2x_3})$ is sharply 3-transitive on these f points. By Zassenhaus [30] and Corollary 2.4, H is not sharply 3-transitive on $\Gamma(\infty)$; so $f < v$. In [7, Theorem 6], Cameron obtained some restrictions on f, and structural information about the graph Γ when f is small. In this section we will strengthen these restrictions.

A Steiner system $S(3, K, v)$, where K is a set of integers greater than 2, is a collection of subsets (called blocks) of a set of v points, such that the cardinality of any block lies in K, and any three points lie in a unique block. If $K = \{k\}$, we write $S(3, k, v)$. A subsystem of an $S(3, K, v)$ is a set Y of points such that the block through any three points of Y lies entirely in Y; if Y contains a block, it evidently determines an $S(3, K, |Y|)$.

We will require the following consequence of a theorem of Kantor [17].

PROPOSITION 3.1. *Let $\mathscr{S} = S(3, k, v)$, $k > 3$, admit a group G of automorphisms with the property that the stabilizer of any three points has order 2, fixes pointwise the block B containing the three points, and acts semiregularly outside B. Then $v = (k-1)^2 + 1$, $\mathscr{S}$ is an inversive plane of order $k - 1$, and $G = PGL(2, (k-1)^2) \cdot Z_2$.*

THEOREM 3.2. *Let G, Γ, H, f be as in the first paragraph. If $f > 4$, then $f = 5$ or 10, and the graph Γ_K, $K = H_{x_1x_2x_3}$, is a B-graph.*

Proof. The translates under H of the set of fixed points of K in $\Gamma(\infty)$ form a Steiner system $S(3, f, v)$, and the set of fixed points of any subgroup of K is a subsystem. Assume $f > 4$. Let y be a vertex of Γ_K at distance 2 from ∞ (in Γ_K). Suppose there is a vertex z of Γ_K adjacent to y but not at distance 1 or 2 from ∞. Then, if a and b are the vertices adjacent to ∞ and z in Γ, $L = K_a$ is a subgroup of index 2 in K. So $K \leqslant N_H(L)$, and K acts on the set $\Gamma_L(\infty)$ of fixed points of L in $\Gamma(\infty)$ as a group of order 2 fixing pointwise the f points of the block containing x_1, x_2, and x_3 (and fixing no further point). Since L has index 2 in the stabilizer of any three points of $\Gamma_L(\infty)$, Proposition 3.1 shows that $N_H(L)$ acts on $\Gamma_L(\infty)$ as $PGL(2, (f-1)^2) \cdot Z_2$, in particular, 3-transitively. Then $N_H(L)_y$ is transitive on $\Gamma_L(y) - \{x, x'\}$, where x and x' are adjacent to ∞ and y. Since one point in this set (namely, z) is at distance 2 from ∞ in Γ_L, the same is true of every point. Moreover, if $\infty' \in \Gamma_L$, then $M_G(L)_{\infty'}$ acts on $\Gamma_L(\infty')$ as $PGL(2, (f-1)^2) \cdot Z_2$. So $M_G(L)$ has rank 3 on Γ_L, and Γ_L is a B-graph. Now Corollary 2.4 implies $(f-1)^2 = 4$ or 9, contradicting the assumption $f > 4$.

Thus any point of Γ_K is at distance at most 2 from ∞ in Γ_K; and $N_H(K)$ acts on $\Gamma_K(\infty)$ as a sharply 3-transitive subgroup of $P\Gamma L(2, f-1)$. Exactly the same argument shows $f = 5$ or 10 and Γ_K is a B-graph.

In the cases $f = 3$ and $f = 4$, the argument also proves parts (ii)–(iv) of [7, Theorem 6] (that is, any 3-claw in Γ lies in a unique B-graph with valency 5 or 10, respectively). We will require this information later.

4. The Case $f = 3$ or 5

Given a Steiner system $S(3, 5, v)$, a regular graph Γ of valency v can be constructed as follows. The vertices are the subsets of the point set of cardinality 0, 1, or 2; vertices P_1 and P_2 are adjacent whenever either

(i) $P_1 \subset P_2$, $|P_2| = |P_1| + 1$; or

(ii) $|P_1| = |P_2| = 2$, $P_1 \cap P_2 = \varnothing$, $P_1 \cup P_2 \subset C$ for some block C.

Γ is a B-graph if and only if the Steiner system has the properties

(a) there do not exist three blocks B_1, B_2, B_3 with $|B_i \cap B_j| = 2$ for $i \neq j$ and $B_1 \cap B_2 \cap B_3 = \varnothing$;

(b) given four points x_1, x_2, x_3, x_4 not contained in a block, there are just two point-pairs $\{y, z\}$ such that $\{x_1, x_2, y, z\}$ and $\{x_3, x_4, y, z\}$ are subsets of blocks.

We shall call an $S(3, 5, v)$ a *B-system* if (a) and (b) hold. (A B-system is a point-pair-schematic system with $k = 5$, in the sense of Cameron [9].)

Lemma 4.1. *A subsystem of a B-system is a B-system.*

Proof. Let $\mathscr{S}$ be a B-system and $\mathscr{S}'$ a subsystem. Clearly condition (a) holds in $\mathscr{S}'$, since it holds in $\mathscr{S}$. Regarding (b), an easy calculation using (a) shows that the average number of pairs $\{y, z\}$ in $\mathscr{S}'$ (over all $x_1, \ldots, x_4$) is equal to 2; but there are at most two pairs for any $x_1, \ldots, x_4$, since there are exactly two in $\mathscr{S}$.

LEMMA 4.2. *Let $S(3, 5, v)$ be a Steiner system (with $v > 5$) admitting an automorphism group G such that any block is the fixed point set of an element of order 3, and the fixed point set of any element of order 3 is a block. Then $v \equiv 5 \pmod{18}$.*

Proof. Let t be an element of order 3, and $B = \text{Fix}(\langle t \rangle)$ (so B is a block). If C is a $\langle t \rangle$-orbit outside B, then $|C| = 3$ and C is contained in a unique block B', with $|B \cap B'| = 2$. Thus there is a map θ from $\langle t \rangle$-orbits outside B to 2-subsets of B. Now t normalizes $G(B')$, and so centralizes an element s of order 3 in $G(B')$. Then $\langle s, t \rangle$ is a group of order 9, with two fixed points, and four orbits of length 3 (corresponding to blocks fixed pointwise by subgroups of order 3); all other orbits have length 9. So $v \equiv 5 \pmod 9$.

Note also that $\langle s \rangle$ acts transitively on $B - \theta(C)$. So any 2-subset of B which belongs to the image of θ is fixed by an element of order 3 in $C_G(t)$. There are three possibilities for the image of θ: a single pair; all four pairs containing some point; or all 10 2-subsets of B. Note that $C_G(t)$ acts transitively on $\text{Im}(\theta)$, so each pair occurs equally often (namely, $(v - 5)/3$, $(v - 5)/12$, or $(v - 5)/30$ times) as the image of a $\langle t \rangle$-orbit.

Assume v is even. Then the first possibility must always occur; that is, an element of order 3 fixes every block containing some point-pair. With s and t as before, $\langle s, t \rangle$ has no orbits of length 9; so $v = 14$. Then the number of blocks of S is $14 \cdot 13 \cdot 12/5 \cdot 4 \cdot 3$, which is not an integer. We conclude that v is odd, that is, $v \equiv 5 \pmod{18}$.

Remark. It is not hard to show that G_B induces A_5 or S_5 on B for each B.

THEOREM 4.3. *No B-system with $v > 5$ admits a 3-transitive automorphism group.*

Proof. Let S be a B-system with a 3-transitive group G. Since 3 divides $v - 2$, and an element of order 3 in G cannot fix just two points (Lemma 2.2(ii)), there are two possibilities:

(i) some element of order 3 fixes four points not contained in a block;

(ii) every element of order 3 fixes a block pointwise, and any block is the fixed point set of such an element.

If (i) holds, let P be a 3-group maximal with respect to fixing four points not contained in a block, and $X' = \text{Fix}(P)$. Then X' varries a B-system (Lemma 4.1). If P is a Sylow 3-subgroup of the stabilizer of three points, then $N_G(P)$ is 3-transitive on X', and an element of order 3 in $N_G(P)^{X'}$ fixes just two points, contradicting Lemma 2.2(ii). So, given any block B contained in X', P is properly contained in a 3-group fixing B pointwise, and so $N_G(P)^{X'}$ contains an element of order 3 whose fixed point set is B. By the maximality of P, no element of order 3 in $N_G(P)^{X'}$ fixes four points not contained in a block. So by restricting attention to X' if necessary, we may assume that (ii) holds.

Then, Theorem 4.2 implies $v \equiv 5 \pmod{18}$, but Lemma 2.2(ii) shows that $v = u^2 + 1$ where u divides $\frac{1}{2}(5 - 1)(5 + 2)$, so $u = 7$ or 14, $v = 50$ or 197, a contradiction.

THEOREM 4.4. *With the hypotheses of Theorem* 3.2, $f \neq 3$ *or* 5.

Proof. By Theorem 3.2 in the case $f = 5$, and by [7, Theorem 6(ii), (iii)] in the case $f = 3$, there is a Steiner system $S(3, 5, v)$ on $\Gamma(\infty)$, which is a B-system whose associated B-graph is Γ. By Theorem 4.3, this is not possible.

Another consequence of Theorem 4.3 is the following.

COROLLARY 4.5. *There is no Steiner system* $S(3, 5, v)$ *with* $v > 5$ *admitting an automorphism group transitive on ordered quadruples of points not contained in a block.*

Proof. Easy counting arguments (see [9]) show that a Steiner system admitting such a group must be a B-system. (This result was first proved in Cameron [31], using a different method.)

5. THE CASE $f = 4$ OR 10

In this section we need another theorem of Kantor [18]:

PROPOSITION 5.1. *Let* $\mathcal{S} = S(3, K, v)$, *where K is the set of even integers. Suppose that* $\mathcal{S}$ *admits an automorphism group G with the properties*:

(i) *if an involution fixes more than two points, then its fixed point set is a block*;

(ii) *for any block B, there is an involution fixing B pointwise.*

Then all blocks have the same size k, and one of the following occurs:

(a) $v = (k - 1)^2 + 1$, $\mathcal{S}$ *is an inversive plane*, $G \geqslant PGL(2, v - 1) \cdot Z_2$;

(b) $k = 4$, $v = 8$, $\mathcal{S}$ *is* $AG(3, 2)$, *G contains the setwise stabilizer of a plane in* $V_8 \cdot GL(3, 2)$;

(c) $k = 4$, $v = 16$, $\mathcal{S}$ *is* $AG(4, 2)$, $G = V_{16} \cdot A_7$.

Let G, acting on X, be a counterexample to Theorem 1, with $f = 4$ or 10. by Theorem 3.2 in the case $f = 10$, and [17, Theorem 6(iv)] in the case $f = 4$, there are Steiner systems $S(3, 4, v)$ and $S(3, 10, v)$ on $\Gamma(\infty)$ (whose blocks we will call 4-*blocks* and 10-*blocks*, respectively) such that any 10-block, together with the 4-blocks it contains, forms an inversive plane admitting a 3-transitive subgroup of $P\Gamma L(2, 9)$. Moreover, if S is a subsystem of the system $S(3, 10, v)$, and S' is the set of points of $\Delta(\infty)$ indexed by 2-subsets of S, then $\{\infty\} \cup S \cup S'$ is a B-graph.

Let $H = G_\infty$, and let T be a 2-subgroup of H of maximal order with respect to fixing four points of $\Gamma(\infty)$ not contained in a 10-block; note that T may be 1. Set $A = \Gamma_T(\infty)$, the set of fixed points of T in $\Gamma(\infty)$. (Γ_T was defined in Section 2.)

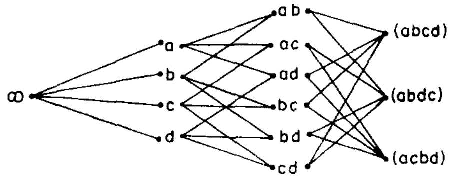
Suppose first that T is not a Sylow 2-subgroup of the stabilizer of three points of $\Gamma(\infty)$. Then, given any three points of A, $N_H(T)^A$ contains an involution fixing those three points, whose set of fixed points in A is a 4-block or a 10-block. Call any such block a T-block.

LEMMA 5.2. *No T-block properly contains another.*

Proof. Suppose t and u are involutions in $K = N_H(T)^A$ such that $\mathrm{Fix}(t) = F$ is a 4-block, $\mathrm{Fix}(u) = B$ is a 10-block, and $F \subseteq B$. Then B is the unique 10-block containing F. Choose $x \in F$, $y \in A - B$. Then t normalizes K_{xyy^t}, and so it centralizes an involution $s \in K_{xyy^t}$. Let $S = \mathrm{Fix}(s)$; $|S| = 4$ or 10. Then t fixes S, so $|S \cap F| \leqslant 2$; we must have $|S \cap F| = 2$, since $S \not\subseteq B$. Now $\langle s, t\rangle$ fixes F, and so it fixes B and centralizes an involution in $K(B)$; without loss of generality we can suppose this involution is u. Now $\langle t, u\rangle$ acts on S, fixing the two points of $S \cap F$. Also tu is an involution fixing F pointwise, so $\mathrm{Fix}(tu) \subseteq B$, and $\langle t, u\rangle$ is semiregular on $S - (S \cap F)$. This implies that $|S| = 10$ and $K_S{}^S \subseteq P\Gamma L(2, 9)$. But $P\Gamma L(2, 9)$ contains no Klein group fixing two points and semiregular on the remaining points.

It follows that the T-blocks form a Steiner system $\mathscr{S} = (3, \{4, 10\}, v^*)$ on A, satisfying the hypotheses of Proposition 5.1. We conclude that S is $AG(3, 2)$, $AG(4, 2)$, or the Miquelian inversive plane of order 3 or 9. In the last case, Γ_T is a B-graph (since A contains a 10-block), and K contains $PSL(2, 81)$, contradicting Corollary 2.4; so this case cannot occur. In any case, Γ_T is regular by Lemma 2.1(i). Also, T has maximal order among the 2-subgroups of G_a ($a \in \Gamma_T$) fixing four points of $\Gamma(a)$ not contained in a 10-block of the $S(3, 10, v)$ induced on $\Gamma(a)$; thus, all $\Gamma_T(a)$ carry isomorphic Steiner systems.

In the other three cases, we must examine in detail the structure of the graph Γ_T. We take the three possibilities in turn. First note that in the graph Γ, any 3-claw generates a unique graph on 14 points, which can be drawn thus:



Here, $\{a, b, c, d\}$ is a 4-block, and the symbols on the right denote the three possible 4-gons on the set $\{a, b, c, d\}$; this graph is, in fact, the incidence graph of the complement of the 7-point projective plane (see [7, p. 88]). Any 3-claw of this graph (in $\Gamma(a)$, say) determines this graph, and hence also a 4-block (in $\Gamma(a)$). In the future, we will omit the edges of such a graph. Note that if a vertex of Γ_T at distance 3 from ∞ receives two different labels, then these labels are disjoint since Γ is a B-graph; similarly, if two such vertices are adjacent then their labels must be disjoint.

Case 1. $\mathscr{S} = AG(3, 2)$. Let $A = \{a, b, c, d, e, f, g, h\}$. The list of blocks of $\mathscr{S}$ follows.

abcd	*aceg*	*bceh*
abef	*acfh*	*bcfg*
abgh	*adeh*	*bdeg*
cdef	*adfg*	*bdfh*
cdgh		
efgh		

Now $\{\infty, ef, eg, eh\}$ is a 4-block in $\Gamma(e)$; so $\{ae, be, ce, de\}$ is another, and we have the graph

		$(aebf)$	x
	ae	$(aecg)$	
	be	$(aedh)$	y
e	ce	$(bech)$	
	de	$(bedg)$	z
		$(cedf)$	

Now z is uniquely determined as the second vertex adjacent to $(aecg)$ and $(bech)$, and is also adjacent to $(aedh)$ and $(bedg)$. Repeating the argument with other vertices in place of e, we find that z is also adjacent to $(bfdh)$, $(bfcg)$, $(afdg)$, and $(afch)$.

It cannot occur that two symbols such as $(aecg)$ and $(bfdh)$ index the same vertex v; for if so, then (applying a suitable elation in K) another pair such as $(aegc)$ and $(bfhd)$ would index the same vertex v', and the four vertices ae, cg, bf, dh would be adjacent to v and v'. Thus any vertex at distance 3 from ∞ is uniquely represented by its label, and there are 42 such vertices, each adjacent to four vertices at distance 2 from ∞. Each vertex such as $(aebf)$ is adjacent to four vertices at distance 4 from ∞. Any vertex at distance 4 from ∞ (such as z) is adjacent to 8 vertices at distance 3 from ∞; so there are 21 such vertices, and the Γ_T terminates and is bipartite.

Let Γ' be the graph on a bipartite block of Γ_T, in which two vertices are adjacent whenever they lie at distance 2 in Γ_T. Then Γ' has 50 vertices and has valency 28. Vertices of Γ' adjacent to ∞ are indexed by lines of $AG(3, 2)$, two of them being adjacent if the corresponding lines are coplanar (intersecting or parallel). Similar arguments show that Γ' is strongly regular, with intersection matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 28 & 15 & 16 \\ 0 & 12 & 12 \end{pmatrix}.$$

In the notation of Section 2, $c = -4$, $d = 3$, $(l, m, n) = (7, -1, 1)$,

$$\chi(g) = (7 \operatorname{fix}(g) - \alpha(g) + \beta(g))/14.$$

Let t be an element of K acting on $AG(3, 2)$ as a translation. It is readily checked that t permutes the eight vertices adjacent to z (in Γ_T), so t fixes z. Also, t fixes four lines of $AG(3, 2)$, and maps any other line to a parallel line. So $\operatorname{fix}(t) = 26$, $\alpha(t) = 24$, $\beta(t) = 0$. But $(7 \cdot 26 - 24)/14$ is not an integer.

Case 2. $S = AG(4, 2)$. Take a point $e \in A$. The 4-blocks containing e, together with the sets $\{a, b, c, d\}$ for which $\{ea, eb, ec, ed\}$ is a 4-block in $\Gamma(e)$, form a system $S(3, 4, 16)$ isomorphic to $AG(4, 2)$. Since $AG(4, 2)$ is uniquely determined by its contraction $PG(3, 2)$, this system is identical with the original system on A; that is, $\{ea, eb, ec, ed\}$ is a 4-block in $\Gamma(e)$ if and only if $\{a, b, c, d\}$ is a 4-block in $\Gamma(\infty)$.

Consider the following "closure" operation on subgraphs of a B-graph: a subgraph is "closed" if, whenever it contains the vertices and edges of a path (x, y, z) of length 2, it also contains the vertices and edges of the other path of length 2 from x to z. Let Γ^* be the closure of the graph consisting of ∞ and the vertices of an affine 3-space in A together with the edges joining them. Since Γ_T is closed, $\Gamma^* \subseteq \Gamma_T$; since the 14-graph is closed, the arguments used in Case 1 apply here to Γ^* (in place of Γ_T). Thus we obtain the same graph Γ', and the same contradiction, as before.

Case 3. $\mathscr{S} = I(3)$, the inversive plane of order 3, and $K = P\Gamma L(2, 9)$. The blocks of $I(3)$ are:

abcd	*adhi*	*bchi*	*cdef*	*degh*
abef	*adej*	*bceg*	*cdgi*	*dfij*
abgh	*adfg*	*bcfj*	*cdhj*	*efhi*
abij	*aegi*	*bdgj*	*ceij*	*efgj*
acgj	*afhj*	*bdfh*	*cfgh*	*ghij*
acfi		*bdei*		
aceh		*behj*		
		bfgi		

Now $I(3)$ is determined by its contraction $AG(2, 3)$, given a group isomorphic to the stabilizer of a point x in $PSL(2, 9)$: the blocks not containing x are all orbits of length 4 of subgroups of $PSL(2, 9)_x$. Thus $\{ae, be, ce, de\}$ is a 4-block in $\Gamma(e)$, and we have the 14-graph:

		$(aebf)$	
	ea	$(aech)$	x
e	eb	$(aedj)$	
	ec	$(becg)$	y
	ed	$(bedi)$	
		$(cedf)$	z

Assume first that any point at distance 3 from ∞ has a unique label. There are 90 such labels, permuted transitively by K. Then z is adjacent to $(aech)$, $(aedj)$, $(becg)$, and $(bedi)$. Using the fact that Γ is a B-graph, it is straightforward to check that K_z has index 2 in K_f, and z is adjacent to the five additional vertices $(abgh)$, $(abij)$, $(cdig)$, $(cdjh)$, and $(ghij)$. Similarly, x is adjacent to $(aebf)$, $(aedj)$, $(becg)$, $(cedf)$, $(agdf)$, $(ajcg)$, $(bfcj)$, and $(bgdj)$, and $K_x = K_{\{h,i\}}$. Thus, $|z^K| = 20$ and $|x^K| = 45$. Since $PGL(2, 9)$ is transitive on nonincident point–block pairs of $\mathscr{S}$, and has an element fixing z while interchanging x and y, it follows that $z^K \cup x^K$ consists of all points of Γ_T at distance 4 from ∞. Moreover, all edges from points at distance 3 from ∞ are now accounted for, so Γ_T contains no circuits of length 7; hence no two points of $z^K \cup x^K$ can be adjacent, as they are at distance 3 from some point of A. Note that K_x is transitive on the set $\{h, i\}$ of points of $\Gamma_T(\infty)$ at distance 5 from x, and hence also on the points of $\Gamma_T(x)$ at distance 5 from ∞. Consequently, the $20 + 90$ further edges leaving $z^K \cup x^K$ terminate at a set of $2 + 10$ points at distance 5 from ∞, the 10 corresponding to edges leaving x^K and the 2 each sending 10 edges back to z^K. Let $S \in \mathrm{Syl}_2 N_H(T)_{ab}$, so $|S^A| = 16\,|T|$ and S fixes ∞' and some point $q \in \Gamma_T$ at distance 4 from ∞. Let $\{s, t\} = \Gamma(\infty) \cap \Gamma(q)$ and $\{u, v\} = \Gamma(\infty) \cap \Gamma(\infty')$. ($\infty' \notin \Gamma(\infty)$ as $\infty' \notin \Gamma_T(\infty)$.) Then S_{stuv} is a 2-group strictly larger than T, so its fixed points lie in a 10-block B in $\Gamma(\infty)$. B is the unique 10-block containing u, v, and a. Repeating the argument with any point of $A - \{a, b\}$ replacing b, we see that $A \subseteq B$, a contradiction.

Since two labels for a point at distance 3 from ∞ must be disjoint, we see that any such point has exactly two labels, and there are 45 such points, each joined to 8 points at distance 2 from ∞. Now we find easily (proceeding as above) that there are 10 points at distance 4 from ∞, and a unique point ∞' at distance 5; the graph Γ_T is a "double cover" of the 56-graph occurring in Theorem 1(iii). Now let S be as above, let $q \in \Gamma_T$ be a fixed point of S at distance 3 from ∞, and obtain the same contradiction as before.

Now we must deal with the case when T *is a Sylow 2-subgroup of the stabilizer of three points of* $\Gamma(\infty)$. Then, with $A = \Gamma_T(\infty)$ and $K = N_H(T)^A$, K is a 3-transitive group in which the stabilizer of three points has odd order. From theorems of Bender [3, 4] and Hering [12], we conclude that $K \trianglerighteq PSL(2, q)$ for some q, or $K = A_6$.

If A contains a 10-block, then the 3-transitivity of K ensures that Γ_T is a B-graph, and Corollary 2.4 is contradicted. So A contains no 10-block. Note that A carries a Steiner system $S(3, 4, v')$; so K contains $PSL(2, 3^d)$ for some d. (Clearly $K \neq A_6$; if the stabilizer of a 3-set in a 3-transitive group containing $PSL(2, q)$ has a fixed point, then q is divisible by 3.)

If d is even, then there is a subsystem $S(3, 4, 10) = \mathscr{S}'$ admitting a 3-transitive subgroup of $P\Gamma L(2, 9)$. By the argument used previously in Case 3, we construct a subgraph which is a double covering of the 56-graph and is fixed pointwise by T. (This graph Γ^* is not Γ_T, but the "closure" of the 10-claw on $\{\infty\} \cup \mathscr{S}'$.) Let ∞' be the vertex "opposite" ∞. Take $a, b \in \mathscr{S}'$ and let $P \in \mathrm{Syl}_2 L_{ab}$, for $L = N_H(T)_{\mathscr{S}'}$ (so $| P^A | = 8 | T |$). As before, $\infty' \notin \Gamma(\infty)$. Let $\{u, v\} = \Gamma(\infty) \cap \Gamma(\infty')$. Then P_a fixes a, b, and x, where $| P_u | \geqslant 4 | T |$, and this contradicts the fact that T is the stabilizer of three points of $\Gamma(\infty)$.

So d is odd, and $K \geqslant PGL(2, 3^d)$. Let B be a 10-block meeting A in four points (so $A \cap B$ is a 4-block). The T-orbits in $B - (A \cap B)$ are the pairs indexing (as in Section 2) the three points at distance 3 from ∞ in the 14-graph generated by $A \cap B$. Let $\{x, y\}$ be one such orbit, and $S = T_x$. Then S is normal in T, so T induces an involution t on $Y = \mathrm{Fix}(S) \cap \Gamma(\infty)$. Moreover, given any three points of Y, S is normal in a Sylow 2-subgroup of their pointwise stabilizer, so there is an involution in $L = N_H(S)^Y$ fixing those three points.

Thus, for any $a \in A$, t fixes $\{x, y, a\}$, and so centralizes an involution $s \in L_{xya}$. Then s fixes A and fixes exactly two points a, b of A, where $\{a, b, x, y\}$ is a 4-block. Thus $(L_{xyA})^A$ is a subgroup of $P\Gamma L(2, 3^d)$ with the property that the stabilizer of any point fixes another point and has even order. Also, $(L_{xyA})^A \cap PGL(2, 3^d)$ contains a dihedral group of order 8 (fixing $A \cap B$). From Dickson's list of subgroups of $PSL(2, q)$ [10, Chap. 12] we deduce that $(L_{xyA})^A$ contains a dihedral group of order $2(3^d + 1)$, transitive on A. The images of $A \cap B$ under this group are pairwise disjoint and cover A.

In the graph Γ_T, the vertex indexed by $\{x, y\}$ is joined to $3^d + 1$ vertices at distance 2 from ∞ (four indexed by pairs in each image of $A \cap B$ under L_{xyA}). So this graph has diameter 3, and is the incidence graph of a biplane. This biplane admits a group $M_H(T)$ of automorphisms and correlations transitive on points and blocks; the stabilizer of a block acts on it as a subgroup of $P\Gamma L(2, 3^d)$ containing $PGL(2, 3^d)$. But this cannot occur [7, proof of Theorem 3].

Thus we have proved

THEOREM 5.3. *With the hypotheses of Theorem* 3.2, $f \neq 4$ *or* 10.

Theorems 3.2, 4.4, and 5.3 complete the proof of Theorem 1.

6. AUTOMORPHISM GROUPS OF BIPLANES

Throughout this section, $\mathscr{D} = (X, \mathscr{B})$ is a biplane, and G is an automorphism group of $\mathscr{D}$ fixing a block B and a point $x \in B$ and 3-transitive on $B - \{x\}$. Let $k = |B|$. We assume that $\mathscr{D}$ is not one of the known examples (these have $k = 4$, 6, and 11). Suppose the stabilizer of three points of $B - \{x\}$ fixes f points of $B - \{x\}$ altogether.

THEOREM 6.1. *With the above hypotheses,*

(i) $f = 4$;

(ii) *there is a null polarity* $\perp$ *of* $\mathscr{D}$ *commuting with G such that* $B^{\perp} = x$.

Proof. Let $x = x_0$, $B = \{x_0, x_1, \ldots, x_{k-1}\}$. Let B_{ij} be the block (different from B) incident with x_i and x_j. Since $G_{x_i x_j}$ fixes only the points x_0, x_i, x_j of B, there is a point y_{0ij} incident with B_{0i}, B_{0j}, and B_{ij}. All points and blocks of $\mathscr{D}$ are now labeled.

Given three points x_i, x_j, x_k other than x_0, the two blocks incident with y_{0ij} and x_k meet B again in two points x_l and x_m which are fixed or interchanged by $G_{x_i x_j x_k} = H$. (By the 3-transitivity of G, the same alternative holds for any three points.) If H fixes x_l, then the connected component containing B (in the incidence graph defined on $X \cup \mathscr{B}$) of the set of fixed points of H is a biplane (by Lemma 2.5), and $N_G(H)$ is sharply 3-transitive on $\mathrm{Fix}(H) \cap (B - \{x\})$. By Zassenhaus [30] and Proposition 2.3, $f = 5$ or 10. (The cases $f = 3$, 4 cannot arise here, because the points x_i, x_j, x_k, x_l, x_m are fixed by H.) By Kantor [17], the resulting Steiner system $S(3, 5, k - 1)$ or $S(3, 10, k - 1)$ has a subsystem S admitting $PSL(2, 4^d)$ or $PSL(2, 9^d)$ for some $d > 1$. We claim that $\{x_0\} \cup S$ generates a subbiplane with block size $1 + |S|$; and this will contradict Proposition 2.3. By the 3-transitivity of $G_S{}^S$. If x_u, x_v, $x_w \in S$ $(x_u \neq x_v)$, then these points lie in a block of S, so y_{0uv} and x_w lie in the corresponding biplane $B(6)$ or $B(11)$ (see [7]). Hence, Lemma 2.5 applies, and the claim is proved.

So we may assume H interchanges x_l and x_m. Let $K = H_{x_l}$, $Y = \mathrm{Fix}(K) \cap (B - \{x\})$. There is a Steiner system $S(3, f, v^*)$ on Y admitting a group $N_G(K)$ satisfying the hypotheses of Proposition 3.1 if $f > 3$. From that result, and Hering [12] in the case $f = 3$, it follows that $N_G(K)^Y$ is 3-transitive (so a connected component of $\mathrm{Fix}(K)$ is a biplane) and contains

$PSL(2, (f-1)^2)$. By Proposition 2.3, $f = 3$ or 4, and any four points x_0, x_i, x_j, x_k lie in a unique biplane $B(6)$ or $B(11)$.

Let $\perp$ be the correspondence $B \leftrightarrow x_0$, $B_{0i} \leftrightarrow x_i$, $B_{ij} \leftrightarrow y_{0ij}$. We show $\perp$ is a null polarity. The only nontrivial step involves showing that if y_{0ij} and B_{kl} are incident, then so are y_{0kl} and B_{ij}. But if y_{0ij} and B_{kl} are incident, then they are contained in the $B(6)$ or $B(11)$ generated by $\{x_0, x_i, x_j, x_k\}$, and the restriction of $\perp$ to this biplane is known to be a polarity. Now Theorem 4.3 shows $f \neq 3$.

COROLLARY 6.2 [8]. *Let $\mathscr{D}$ be a biplane admitting an automorphism group fixing a block B and 4-transitive on B. Then $|B| = 4$ or 6.*

Proof. If $k > 11$, then any four points of B generate $B(11)$ by Theorem 6.1; this is also true if $k = 11$. But the stabilizer of a block in Aut($B(11)$) is not transitive on that block.

For related results, see [15; 16 8E(10 − 13)].

7. 2-TRANSITIVITY

We now turn to the proof of Theorem 2. Let G be provide a counterexample with minimal v. Once again, G is acting on a B-graph. Let ∞, $\Gamma(\infty)$, $\Delta(\infty)$, and u be as in Section 2. Set $H = G_\infty$.

We claim that H is 2-transitive on $\Gamma(\infty)$. For it is certainly transitive on the $\frac{1}{2}(u^2+1)u^2$ 2-sets of $\Gamma(\infty)$. Hence $|H|$ is even, and this implies our claim.

Now consider the possibility $3 \nmid |G|$. H has a unique minimal normal subgroup M, and M is simple or elementary Abelian [5, p. 202]. Suppose M is simple. Then $M \simeq Sz(2^e)$ for some $e \geqslant 3$ [11, 28] and it follows easily that $2^{2e}+1 = v = u^2+1$, whereas $4 \nmid u$ (Lemma 2.2). Thus M is elementary Abelian of order $p^e = u^2+1$ for some prime p. Then $e = 1$, $v = p$, and Corollary 2 provides a contradiction.

Thus, $3 \mid |G|$. Let $P \leqslant H$ be a 3-group maximal with respect to $|\mathrm{Fix}(P)| > 16$ (possibly $P = 1$), and $Q \geqslant P$ a 3-group maximal with respect to $|\mathrm{Fix}(Q)| \geqslant 3$. By Lemma 2.2(ii), $Q \neq 1$. Both $F = \mathrm{Fix}(P)$ and $F' = \mathrm{Fix}(Q)$ are B-graphs; let w and w' denote their valencies.

We claim that $w' = 2$ or 5. For, $N_H(Q)$ is 2-transitive on $F'(\infty) = F' \cap \Gamma(\infty)$. By Lemma 2.1, $N_G(Q)^{F'}$ satisfies the hypotheses of Theorem 2, provided $w' > 2$. Thus, the claim follows from the minimality of v.

In particular, $P < Q$. Set $K = N_H(P)^F$. We now describe several properties of K and $Y = F \cap \Gamma(\infty)$.

(a) $|Y| = 50$ or 197. For, let $P \lhd Q_1 \leqslant Q$ with $|Q_1 : P| = 3$. Then $\mathrm{Fix}(Q_1)$ is a B-graph contained in F, and $|Q_1{}^F| = 3$. Our choice of P forces $|\mathrm{Fix}(Q_1)| \leqslant 16$. Then $|\mathrm{Fix}(Q_1)| = 4$ or 16 by Lemma 2.2(i). Now Lemma

2.2(ii) implies that $(w-1)^{1/2}$ divides $\frac{1}{2}(2-1)(2+2)$ or $\frac{1}{2}(5-1)(5+2)$, so $w = 50$ or 197.

(b) If $x, y \in Y$, then $3 \mid |K_{xy}|$. For P is properly contained in a Sylow 3-subgroup of H_{xy}.

(c) If $g \in K$ has order 3, then g fixes exactly five points of Y. This is proved as in (a).

(d) If $g, g' \in K$ have order 3, and fix three points of Y in common, then they fix precisely the same five points. For, the fixed points of g correspond to a sub-B-graph of Γ of valence 5. Such a B-graph is generated by any 3-claw [7, p. 88].

(e) Let $E \leqslant K$ with $|E|$ odd, and assume that E fixes at least three points of Y. Then the fixed points of E on F form a B-graph (Lemma 2.1), and hence E fixes precisely $e^2 + 1$ points of Y for some integer e, where $1 + (e^2 + 1)^2 \leqslant |Y|$ (Lemma 2.2). Thus, if $|Y| = 50$ then $e = 2$.

In the next section, we will show that a permutation group K on a set Y, satisfying (a)–(e), must be 2-transitive. Assuming this, we deduce that $N_H(P)$ is 2-transitive on $F(\infty)$, and hence is transitive on $F \cap \Delta(\infty)$. It follows that $N_G(P)^F$ has rank 3. This contradicts the minimality of v, and hence proves Theorem 2.

Remarks. The case $v \equiv 1 \pmod 3$ seems much harder than the above, as the proof of Theorem 5.3 indicates. Lemma 2.2(ii) is no longer useful in this situation: the example $B(11)$ shows that all elements of order 3 can fix exactly two points, in which case Lemma 2.2(ii) provides no restriction at all.

Note also that the proof of Theorem 2 primarily used H, not G. Only when H was a Frobenius group of prime degree was G employed. Unfortunately, we have not been able to handle B-graphs admitting such a group H.

8. A Technical Result

The following grotesque result can be regarded as an unrefined refinement of parts of the proof of Lemma 4.2 and Theorem 4.3.

PROPOSITION 8.1. *Let G be a permutation group on a set X. Then G is 2-transitive if the following conditions all hold*:

(a) $|X| = 50$ *or* 197;

(b) *if* $x, y \in X$, *then* $3 \mid |G_{xy}|$;

(c) *if* $g \in G$ *has order* 3, *then* $|\mathrm{Fix}(g)| = 5$;

(d) *if* $|g| = |g'| = 3$ *and* $|\text{Fix}(g) \cap \text{Fix}(g')| \geqslant 3$, *then* $\text{Fix}(g) = \text{Fix}(g')$;

(e) *if* $E \leqslant G$ *with* $|E|$ *odd and* $|\text{Fix}(E)| \geqslant 3$, *then* $|\text{Fix}(E)| = e^2 + 1$ *for some integer* e; *moreover*, $e = 2$ *if* $|X| = 50$.

Proof. Suppose G is not 2-transitive. Let $\mathscr{T} = \{g \in G \mid |g| = 3\}$. We may assume $G = \langle \mathscr{T} \rangle$. Let g denote any element of $\mathscr{T}$, and let $T \in \text{Syl}_3\, G$. Set $n = |X|$. That the case $n = 197$ is much easier than $n = 50$ is due to the following.

LEMMA 8.2. *If* $n = 197$ *then* $|T| = 3$.

Proof. Suppose $|T| \geqslant 9$. Since $n \equiv 8 \pmod 9$, T fixes two points x, y, and also two 3-sets Σ_1, Σ_2 in $X - \{x, y\}$. By (c) and (d), T is elementary Abelian of order 9; moreover, its four subgroups of order 3 have pairwise disjoint fixed point sets on $X - \{x, y\}$. Thus, T is semiregular on a set of size $197 - 2 - 4 \cdot 3$, which is not a multiple of 9.

LEMMA 8.3. *If* G *is transitive then* $n = 50$. *If* G *is intransitive, then its orbit structure is one of the following*:

(I) *two orbits* Γ, Δ, *with* $|\Gamma| + 2 \equiv |\Delta| \equiv 2 \pmod 3$;

(II) *two orbits* Δ_1, Δ_2, *with* $|\Delta_i| \equiv 1 \pmod 3$;

(III) *three orbits* $\Gamma, \Delta_1, \Delta_2$, *with* $|\Gamma| + 1 \equiv |\Delta_i| \equiv 1 \pmod 3$.

Proof. If there are two orbits Γ_1, Γ_2 with $|\Gamma_1| \equiv |\Gamma_2| \equiv 0 \pmod 3$, choose $x \in \Gamma_1$, $y \in \Gamma_2$ and contradict (b) and (c). Thus, at most one orbit has length divisible by 3. By (b), (c), and (d), G fixes at most two points. If Δ_1, Δ_2, and Δ_3 are orbits with $|\Delta_i| \equiv 1 \pmod 3$ for $i = 1, 2, 3$, and if $|\Delta_1| > 1$, then choose $x, y \in \Delta_1$, $x \neq y$, in order to contradict (b) and (c). Thus, there are at most two orbits of length $\equiv 1 \pmod 3$. Similarly, there is at most one orbit of length $\equiv 2 \pmod 3$. Since $n \equiv 2 \pmod 3$, this proves the lemma, except for the first assertion. But if G is transitive and $n = 197$, then G is 2-transitive by Burnside's theorem [5, p. 341].

LEMMA 8.4. $n = 50$.

Proof. Suppose $n = 197$. By (b) and Lemmas 8.2 and 8.3, G is intransitive and has no orbits of length divisible by 3. By Lemma 8.3, (II) must hold. If $|\Delta_i| > 1$, choose $x, y \in \Delta_i$, $x \neq y$, let $g_i \in \mathscr{T} \cap G_{xy}$, and note that $|\text{Fix}(g_i) \cap \Delta_{3-i}| = 1$. Since $|T| = 3$, it follows that we may assume $|\Delta_2| = 1$. By (b) and (c), Δ_1 inherits the structure of an $S(2, 4, 196)$, whose blocks are the fixed point sets on Δ_1 of members of $\mathscr{T}$. Let $g \in \mathscr{T}$. Since $r = 195/3 = 65 \equiv 2 \pmod 3$, g fixes a block other than $\text{Fix}(g) - \Delta_2$. This again produces the contradiction $|T| > 3$.

LEMMA 8.5. (i) *Each element of G of odd prime order fixes* 0, 1, 2, *or* 5 *points of X.*

(ii) *G is a $\{2, 3, 5, 7\}$-group.*

Proof. (i) is clear by (e), and (ii) follows immediately from (i).

LEMMA 8.6. *Suppose Δ is an orbit of G, and Σ is a nontrivial imprimitivity set of G^{Δ}. Then the following hold:*

(i) $3 \nmid |\Sigma|$;

(ii) $|\Sigma| \neq 2$;

(iii) $|\Sigma| \neq 5$;

(iv) $|\Sigma| \neq 10$;

(v) $|\Sigma| \neq 7$.

Proof. (i) Suppose $3 \mid |\Sigma|$, and let $x \in \Sigma$, $y \in \Delta - \Sigma$. Then (b) contradicts (c).

(ii) Suppose $|\Sigma| = 2$. Then each $g \in \mathscr{T}$ fixes an even number of points of Δ, so $|\Delta| \not\equiv 0 \pmod 3$. Assume $|\Delta| \equiv 1 \pmod 3$. Then $|\operatorname{Fix}(g) \cap \Delta| = 4$ for each $g \in \mathscr{T}$, and hence $|X - \Delta| = 1$. However, 49 is odd.

Assume $|\Delta| \equiv 2 \pmod 3$. Then each $g \in \mathscr{T}$ must fix just two points of Δ. However, if x and y are chosen as in (i), this again yields a contradiction.

(iii) Suppose $|\Sigma| = 5$. Choosing x and y as above, we find first that $|\Delta| \equiv 1 \pmod 3$, and then that $|X - \Delta| \leqslant 1$. Thus, $\Delta = X$, $|\Sigma^G| = 10$, and $g \in \mathscr{T} \cap G_{xy}$ fixes at least $2 \cdot 4$ points.

(iv) If $|\Sigma| = 10$, then $G_{\Sigma}{}^{\Sigma}$ is primitive by (ii) and (iii). Hence, $G_{\Sigma}{}^{\Sigma} \geqslant A_{10}$ by (b) and Sims [26], and this contradicts (d).

(v) If $|\Sigma| = 7$, then $G_{\Sigma}{}^{\Sigma} \geqslant A_7$ by (b), and this contradicts (d).

LEMMA 8.7. *Case* I *does not occur.*

Proof. Suppose it does. Recall that $|\Delta| > 2$ since $G = \langle \mathscr{T} \rangle$. By (b) and (c), $|T| \geqslant 9$.

By Lemma 8.5, $|\Gamma| = 15, 18, 30, 36, 42$, or 45.

If $|\Gamma| = 15$ then G^{Γ} is primitive of order divisible by 7. By Sims [26], property (d) contradicts the known properties of the possible groups G^{Γ}.

If $|\Gamma| = 18$ then G^{Γ} is primitive (Lemma 8.6) and hence $17 \mid |G|$ by [26].

If $|\Gamma| = 30$ then G^{Δ} is imprimitive by (c) and [26]. Let Σ^G be a nontrivial imprimitivity system of G^{Δ}. Then $|\Sigma| = 4$ and $O_2(G)^{\Gamma} = 1$ by Lemma 8.6, and these yield a contradiction.

If $|\Gamma| = 36$, then G^{Δ} is imprimitive by [26], and hence has a system Σ^G with $|\Sigma| = 7$. But clearly $9 \mid |G^{\Gamma}|$, whereas $9 \nmid |G_{\Sigma}{}^{\Sigma}|$ and $9 \nmid |G(\Delta)|$.

If $|\Gamma| = 42$, $7 \mid |G^{\Delta}|$. By (d), $G^{\Delta} = A_8$ or S_8. By Lemma 8.6, $G(\Delta) = 1$ and $O_2(G) = 1$. Thus, $G = G^{\Delta}$ is $PSL(2, 7)$ or $PGL(2, 7)$. Now G^{Γ} and $\mathscr{T}^{\Gamma}$ yield a contradiction.

Finally, if $|\Gamma| = 45$ then $|\Delta| = 5$. Now $\mathscr{T}^{\Delta}$ implies that $G^{\Delta} \geqslant A_5$. However, $27 \mid |G^{\Gamma}|$, so $|T(\Delta)| = 9$ and $|T| = 27$. Since $N_G(T(\Delta))^{\Delta} \geqslant A_5$, $C_G(T(\Delta))^{\Delta} \geqslant A_5$. Thus, T is Abelian. Now $T(\Delta)$ fixes $\mathrm{Fix}(g) \cap \Gamma$ for any $g \in T$ having $|\mathrm{Fix}(g) \cap \Gamma| = 3$, and this implies the lemma.

LEMMA 8.8. *If Δ is an orbit with $|\Delta| \equiv 1 \pmod 3$, then $|\Delta| = 1, 4, 16, 25, 28$, or 40.*

Proof. By Lemma 8.5 only the cases $|\Delta| = 7, 10$ need to be eliminated. But these are handled precisely as in Lemma 8.6.

LEMMA 8.9. *Case* (II) *cannot occur.*

Proof. By Lemma 8.8, in this case we may assume $|\Delta_1| = 1$ or 25.

Suppose $|\Delta_1| = 25$, so $|\Delta_2| = 25$. By Lemma 8.6 and Wielandt [29], either G^{Δ_i} has a regular normal subgroup, or else it is 2-transitive. Since $g \in \mathscr{T}$ exists fixing two points of Δ_i, in the former case $\mathrm{Fix}(g) \subseteq \Delta_i$, which is impossible. Thus, G^{Δ_i} is 2-transitive for $i = 1, 2$. Let $x \in \Delta_1$. If G_x is transitive on Δ_2 then some nontrivial 5-element fixes more than five points of Δ_1, contradicting Lemma 8.5. Thus, the two 2-transitive representations have the same character. But this is impossible as $g \in \mathscr{T}$ fixes different numbers of points.

Thus, we may assume $|\Delta_1| = 1$ and $|\Delta_2| = 49$. Then by (c), Δ_2 inherits the structure of an $S(2, 4, 49)$, whose blocks are the fixed point sets on Δ_2 of elements of order 3. As above, G^{Δ_2} is primitive and hence 2-transitive. If $x \in \Delta_2$, then G_x is transitive on the $(49 - 1)/(4 - 1) = 16$ blocks through x.

By O'Nan [22] and Shult [25], $Z(O_2(G_x)) = 1$ and $Z(O_3(G_x)) = 1$. Thus, G_x has a faithful transitive representation of degree 16, which must have a (unique) imprimitivity system $\{\Sigma_1, \Sigma_2\}$ with $|\Sigma_i| = 8$ (by [26]). Moreover, some block on x is fixed pointwise by an element of order 7, and this contradicts Lemma 8.5.

LEMMA 8.10. *Case* (III) *does not occur.*

Proof. Let Γ, Δ_1, Δ_2 be as in (III). First note that $|\Gamma| > 3$. For if $|\Gamma| = 3$ then (d) and $G = \langle \mathscr{T} \rangle$ would imply that G fixes two points, which is not the case.

Now suppose $|\Delta_1| = 1$. By (d), Γ inherits the structure of a Steiner triple system $\mathscr{S}$. Moreover, each $v \in \Delta_2$ determines a partition of $\mathscr{S}$ into $|\Gamma|/3$ pairwise disjoint blocks, while each block determines a unique point of Δ_2 (all by (b)–(d)). Thus, $|\Delta_2| = (|\Gamma| - 1)/2$, so $|\Gamma| = 33$, which contradicts Lemma 8.5.

Thus, $|\Delta_1| > 1$ and $|\Delta_2| > 1$. Also $|\Delta_i| > 4$ $(i = 1, 2)$. For if $|\Delta_1| = 4$, then by (b) and (c), $T(\Delta_1)$ would be nontrivial and semiregular on Δ_2. Thus, by Lemma 8.8, $|\Delta_i| \geqslant 16$ for $i = 1, 2$.

If $|\Delta_1| = 28$ then $|\Gamma| \geqslant 6$ forces $|\Delta_2| = 16$ and $|\Gamma| = 6$, whereas this contradicts Lemma 8.5 (applied to an element of order 7). Thus, $|\Delta_i| \in \{16, 25\}$ for $i = 1, 2$.

If $|\Delta_1| = |\Delta_2| = 16$, then $|\Gamma| = 18$. By Lemma 8.6, G^Γ is primitive. Hence, by [26], $17 \mid |G^\Gamma|$, and this contradicts Lemma 8.5.

Thus, we may assume $|\Delta_1| = 16$, $|\Delta_2| = 25$, and $|\Gamma| = 9$. But now G^Γ is primitive by Lemma 8.6, so (b) and [26] imply that $G^\Gamma \geqslant A_9$. This contradicts (d), and proves the lemma.

At this stage, Lemma 8.3 shows that G *is transitive*. Since $G = \langle \mathscr{T} \rangle$, Lemma 8.6 implies that G is primitive. Let $x \in X$. We may assume $T \leqslant G_x$.

LEMMA 8.11. *G_x^{X-x} has precisely $s + 1$ orbits, $s \geqslant 1$, which can be labeled $\Delta, \Gamma_1, \ldots, \Gamma_s$ so that $|\Delta| \equiv 1 \pmod 3$, $|\Delta| > 1$, $|\Gamma_i| \equiv 0 \pmod 3$, and $|\Gamma_i| > 3$ for $1 \leqslant i \leqslant s$. Moreover, $|T| \geqslant 9$.*

Proof. Recall that G_x^{X-x} is intransitive. Also, it has no orbits of length 1 or 2, and hence (by (b)–(d)) none of length 3. Now, as in the proof of Lemma 8.3 since $49 \equiv 1 \pmod 3$, G_x^{X-x} has at most one orbit of length $\equiv 1 \pmod 3$ and none of length $\equiv 2 \pmod 3$. This implies the first assertion. The second follows from (b).

LEMMA 8.12. *$|\Delta| = 4$ or 10.*

Proof. By Lemma 8.5, the only other possibilities are $|\Delta| = 7, 25, 28$, or 40.

If $|\Delta| = 25$ and $P \in \mathrm{Syl}_5\, G_x$, then P fixes five points of $X - \Delta$ and cannot act semiregularly on the remaining 20. This contradicts (e).

If $|\Delta| = 40$, then $s = 1$ and $|\Gamma_1| = 9$. However, no rank 3 group exists having these parameters.

If $|\Delta| = 7$, then $7 \mid |\Gamma_i|$ for each i by Lemma 8.5. Thus, $s = 1$ and $|\Gamma_1| = 42$. Now $G \trianglerighteq PSU(3, 5)$ [12], and $G_x^\Delta \trianglerighteq A_7$. This contradicts (d).

Suppose $|\Delta| = 28$. As above, $s = 1$ and $|\Gamma_1| = 21$. In the standard rank 3 notation, necessarily $\lambda = 8$ and $\mu = 9$ for the Γ_1-graph.

The irreducible constituents of the permutation character have degrees 1, 24, 25. Thus G is of $2p$-type, in the terminology of Scott [23]. His results include the nonexistence of such a group.

LEMMA 8.13. $|\Delta| = 10$.

Proof. Suppose $|\Delta| = 4$. By a result of Manning [19], we may assume $|\Gamma_1| \mid 12$. Then $s > 1$, so $T(\Delta) \neq 1$. Thus, $|\Gamma_1| = 12$.

Now T fixes 3-sets of both Δ and Γ_1, so $|T| = 9$ and hence $s \leqslant 3$ by (b). Then $s = 3$ by Lemma 8.5. Also $9 \nmid |\Gamma_i|, i = 2, 3$. Since $|\Gamma_2| + |\Gamma_3| = 33$ it follows that $\{|\Gamma_2|, |\Gamma_3|\} = \{12, 21\}$, and this contradicts Lemma 8.5.

We can now complete the proof of Proposition 8.1. Some $g \in \mathscr{T} \cap G$ fixes three points of Γ_1 and one of Δ. Thus, $G_x{}^\Delta$ is primitive. By [26] and (d), $G_x{}^\Delta \trianglerighteq PSL(2, 9)$. By Manning [19], G_x is faithful on Δ and we may assume $|\Gamma_1| \mid 10 \cdot 9$.

By [6], $|\Gamma_1| > 20$. $|\Gamma_1| \neq 30$, as then $|X - (\{x\} \cup \Delta \cup \Gamma_1| = q$ while $|\Gamma_i| \neq 3, 6, 9$. This contradiction completes the proof.

REFERENCES

1. M. ASCHBACHER, "Collineation Groups of Symmetric Block Designs," Thesis, University of Wisconsin, 1969.
2. E. BANNAI, On rank 3 groups with a multiply transitive constituent, *J. Math. Soc. Japan* **24** (1972), 252–254.
3. H. BENDER, Endliche zweifach transitive Permutationsgruppen, deren Involutionen keine Fixpunkte haben, *Math. Z.* **104** (1968), 175–204.
4. H. BENDER, Transitive Gruppen gerader Ordnung, in denen jede Involutionen genau einen Fixpunkte haben, *J. Algebra* **17** (1971), 527–554.
5. W. BURNSIDE, "Theory of Groups of Finite Order," Dover, New York, 1955.
6. P. J. CAMERON, Permutation groups with multiply transitive suborbits, *J. London Math. Soc.* **25** (1972), 427–440.
7. P. J. CAMERON, Biplanes, *Math. Z.* **131** (1973), 85–101.
8. P. J. CAMERON, Characterisations of some Steiner systems, parallelisms and biplanes, *Math. Z.* **136** (1974), 31–39.
9. P. J. CAMERON, Two remarks on Steiner systems, *Geometriae Dedicata*, to appear.
10. L. E. DICKSON, "Linear Groups," Dover, New York, 1958.
11. G. GLAUBERMAN, A characterization of the Goldschmidt groups, to appear.
12. C. HERING, Zweifach transitive Permutationsgruppen, in denen zwei die maximale Anzahl von Fixpunkte von Involutionen ist, *Math. Z.* **104** (1968), 150–174.
13. D. G. HIGMAN, Intersection matrices for finite permutation groups, *J. Algebra* **6** (1967), 22–24.
14. D. G. HIGMAN, Primitive rank 3 groups with a prime subdegree, *Math. Z.* **91** (1966), 70–86.
15. W. M. KANTOR, 2-transitive symmetric designs, *Trans. Amer. Math. Soc.* **146** (1969), 1–28.
16. W. M. KANTOR, 2-transitive designs, *in* "Combinatorics" (M. Hall, Jr. and J. H. van Lint, Eds.), pp. 365–418, Reidel, Dordrecht, 1975.
17. W. M. KANTOR, Plane geometries associated with certain 2-transitive groups, to appear.
18. W. M. KANTOR, Axial automorphisms of designs, to appear.
19. W. A. MANNING, A theorem concerning simply transitive primitive groups, *Bull. Amer. Math. Soc.* **35** (1929), 330–332.

20. P. M. NEUMANN, Primitive permutation groups of degree $3p$, unpublished.
21. P. M. NEUMANN, Transitive permutation groups of prime degree, *J. London Math. Soc.* **5** (1972), 202–208.
22. M. E. O'NAN, A characterization of $L_n(q)$ as a permutation group, *Math. Z.* **127** (1972), 301–314.
23. L. L. SCOTT, "Uniprimitive Groups of Degree kp," Thesis, Yale University, 1968.
24. L. L. SCOTT, Uniprimitive permutation groups, *in* "Theory of Finite Groups," Benjamin, New York, 1969.
25. E. E. SHULT, On the fusion of an involution in its centralizer, unpublished.
26. C. C. SIMS, Computational methods in the study of permutation groups, *in* "Computational Problems in Abstract Algebra," pp. 169–183, Pergamon, London, 1970.
27. T. TSUZUKU, On primitive extensions of rank 3 of symmetric groups, *Nagoya Math. J.* **27** (1966), 171–177.
28. J. G. THOMPSON, unpublished.
29. H. WIELANDT, "Permutation Groups through Invariant Relations and Invariant Functions." Lecture notes, Ohio State University, 1971.
30. H. ZASSENHAUS, Kennzeichnung endlicher linearer Gruppen als Permutationsgruppen, *Abh. Math. Sem. Univ. Hamburg* **11** (1935), 17–40.
31. P. J. CAMERON, On basis-transitive Steiner systems, *J. London Math. Soc.* (2), **13** (1976), 393–399.