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2-Transitive and Antiflag Transitive Collineation Groups of Finite Projective Spaces

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1. INTRODUCTION

An unpublished result of Perin [20] states that a subgroup of $\Gamma L(n, q)$, $n \geqslant 3$, which induces a primitive rank 3 group of even order on the set of points of $PG(n-1, q)$, necessarily preserves a symplectic polarity. (Such groups are known, if $q \neq 2$, by another theorem of Perin [19].) The present paper extends both Perin's result and his method, in order to deal with some familiar problems concerning collineation groups of finite projective spaces; among these, 2-transitive collineation groups [25], and the case $q = 2$ of Perin's theorem [19].

An *antiflag* is an ordered pair consisting of a hyperplane and a point not on it; if the underlying vector space is endowed with a symplectic, unitary or orthogonal geometry, both the point and the pole of the hyperplane are assumed to be isotropic or singular. Our main results are the following four theorems.

THEOREM I. *If $G \leqslant \Gamma L(n, q)$, $n \geqslant 3$, and G is 2-transitive on the set of points of $PG(n-1, q)$, then either $G \geqslant SL(n, q)$, or G is A_7 inside $SL(4, 2)$.*

THEOREM II. *If $G \leqslant \Gamma(n, q)$ and G is transitive on antiflags and primitive but not 2-transitive on points, then G preserves a symplectic polarity, and one of the following holds:*

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(i) $G \geq SP(n, q)$;

(ii) *G is A_6 inside $Sp(4, 2)$; or*

(iii) *$G \geq G_2(q)$, q even, and G acts on the generalized hexagon associated with $G_2(q)$, which is itself embedded naturally in $PG(5, q)$.*

THEOREM III. *If $G \leqslant \Gamma L(n, q)$ and G is transitive on antiflags and imprimitive on points, then $q = 2$, $G \leqslant \Gamma L(\frac{1}{2}n, 4)$, and $G \geq SL(\frac{1}{2}n, 4)$, $Sp(\frac{1}{2}n, 4)$, or $G_2(4)$ (with $n = 12$). In each case, G is embedded naturally in $GL(n, 2)$.*

THEOREM IV. *If $G \leqslant \Gamma Sp(n, q)$, $\Gamma O^{\pm}(n, q)$ or $\Gamma U(n, q)$, for a classical geometry of rank at least 3, and G is transitive on antiflags, then one of the following holds (and the embedding of G is the natural one):*

(i) *$G \geq Sp(n, q)$, $\Omega^{\pm}(n, q)$, resp. $SU(n, q)$;*

(ii) *$G \geq G_2(q)$ inside $\Gamma O(7, q)$ (or $\Gamma Sp(6, q)$, q even);*

(iii) *$\Omega(7, q) \trianglelefteq G/Z(G) < P\Gamma O^-(8, q)$, with $G/Z(G)$ conjugate in* Aut *$(P\Omega^-(8, q))$ to a group fixing a nonsingular* 1*-space;*

(iv) *$Sp(n, 4) \trianglelefteq G < Sp(2n, 2)$ (or $\Omega(n + 1, 4) \trianglelefteq G \leqslant O(2n - 1, 2)$ for n even);*

(v) *$G_2(4) \trianglelefteq G \leqslant Sp(12, 2) \cong \Omega(13, 2)$;*

(vi) *$SU(m, 2) \trianglelefteq G \leqslant O^{\epsilon}(2m, 2)$, where $\epsilon = (-)^m$.*

Theorem I solves a problem posed by Hall and Wagner [25], which has been studied by Higman [8, 10], Perin [19], Kantor [13] and Kornya [15]. An independent and alternative approach to this theorem is given by Orchel [16]; we are grateful to Orchel for sending us a copy of his thesis.

If G is 2-transitive, then G is antiflag transitive; and also $G_H{}^H$ is antiflag transitive for each hyperplane H. This elementary fact allows us to use induction. (Indeed, Theorems I–III are proved simultaneously by induction in Part I of this paper.) Another problem, solved in Theorem II and IV, is that of primitive rank 3 subgroups of classical groups. This was posed by Higman and McLaughlin [11], and solved by Perin [19] and Kantor and Liebler [14] except in the case of $Sp(2n, 2) \cong \Omega(2n + 1, 2)$. Here, induction is made possible by the fact that the stabilizer of a point x is antiflag transitive on $x^{\perp}/x$.

The striking occurrence of $G_2(q)$ in these theorems is related to a crucial element of our approach. This case is obtained from a general embedding theorem for metrically regular graphs (3.1), in which the Feit–Higman theorem [7] on generalized polygons arises unexpectedly but naturally. Other familiar geometric objects and theorems come into play later on: the characterizations of projective spaces due to Veblen and Young [24] and Ostrom and Wagner [18], as well as translation planes, arise in Theorem III, while Tits' classification of polar spaces [23] and the triality automorphism of $P\Omega^+(8, q)$ are used for Theorem IV.

All the proofs require familiarity with the geometry of the classical groups. On the other hand, group-theoretic classification theorems have been entirely avoided. Moreover, knowledge of $G_2(q)$ is not assumed for Theorem I, and what is required for Theorems II–IV is contained in the Appendix, where we have given a new and elementary proof of the existence of the generalized hexagons of type $G_2(q)$.

This paper began as an attempt to extend Perin's result [20] to rank 4 subgroups of classical groups. As in Perin [19], one case with $q = 2$ is left open:

THEOREM V. *Suppose $G \leqslant \Gamma Sp(n, q)(n \geqslant 6)$, $\Gamma O^{\pm}(n, q)(n \geqslant 7)$, or $\Gamma U(n, q)$ $(n \geqslant 5)$. If G induces a primitive rank 4 group on the set of isotropic or singular points, then one of the following holds:*

(i) *$G \trianglerighteq G_2(q)$ is embedded naturally in $\Gamma O(7, q)$ (or $\Gamma Sp(6, q)$, q even);*

(ii) *$G \trianglerighteq \Omega(7, q)$, q even, or $2.\Omega(7, q)$, q odd, each embedded irreducibly in $\Gamma O^+(8, q)$; or*

(iii) *$G \leqslant O^{\pm}(2m, 2)$, and G is transitive on the pairs (x, L) with L a totally singular line and x a point of L.*

The examples (ii) (and (iii) in Theorem IV) are obtained by applying the triality automorphism to the more natural $\Omega(7, q)$ inside $P\Omega^+(8, q)$. As for (iii), examples are A_7 and S_7 inside $O^-(6, 2)$.

Other results in a similar spirit are given in Section 8, as corollaries to Theorem I.

Some further results are of interest independent of their application to the above Theorems. A general result on embedding metrically regular graphs in projective spaces is proved in Section 3; this is crucial for all the theorems. Theorem 10.3 characterizes nonsingular quadrics of dimension $2n - 1$ contained in an $O^+(2n, q)$ quadric for $n \geqslant 3$. In Section 12, parameter restrictions are obtained for rank 4 subgroups of rank 3 groups (and their combinatorial analogous). Finally, the Appendix gives an elementary construction and characterization of the $G_2(q)$ hexagon.

The paper falls into two parts. The first (Sections 2–8) deals with antiflag transitive collineation groups of projective spaces (Theorems I–III); we note that Sections 3 and 5, on the primitive, not 2-transitive case, are virtually self-contained. The second part (Sections 9–14) contains the proofs of Theorems IV and V, concerning polar spaces.

I. THEOREMS I–III

2. Preliminaries

A *point* (*hyperplane*) of a vector space V is a subspace of dimension 1 (codimension 1). If V is n-dimensional over $GF(q)$, the set of points (equipped with the structure of projective geometry) is denoted by $PG(n-1, q)$; but in this paper, its dimension will always be n. The notation $SL(V) = SL(n, q)$, $GL(n, q)$, and $\Gamma L(n, q)$ is standard.

If, in addition, V is equipped with a symplectic, unitary or orthogonal geometry, then $\Gamma Sp(n, q)$, $\Gamma U(n, q)$ and $\Gamma O^{\pm}(n, q)$ denote the groups of semilinear maps preserving the geometry projectively. For example, $\Gamma O^{\pm}(n, q)$ consists of all invertible semilinear maps g such that $Q(v^g) = cQ(v^\sigma)$ for all $v \in V$, where Q is the quadratic form defining the geometry, c is a scalar, and σ a field automorphism. The groups $Sp(n, q)$, $SU(n, q)$ and $\Omega^{\pm}(n, q)$ are defined as usual. We will occasionally require the fact that $Sp(2n, q) \cong \Omega(2n-1, q)$ when q is even. (Explicitly, if V is the natural $Sp(2n, q)$-module, then there is a $2n-1$-dimensional orthogonal space $\tilde{V}$ such that $\tilde{V}/\text{rad}\,\tilde{V} = V$, with the natural map $\tilde{V} \to V$ inducing a bijection between singular and isotropic points.) The reader is referred to Dieudonne [6] for further information concerning these groups.

Points will be denoted x, y, z, lines L, L', and hyperplanes H, H'. We will generally identify a subspace Δ of V with its set of points; $|\Delta|$ denotes the number of points, and $x \in \Delta$ will be used instead of $x \subseteq \Delta$. Similarly, for subspaces Δ and Σ, $\Delta - \Sigma$ denotes the set of points in Δ but not Σ. The dimension dim Δ of a subspace Δ denotes its vector space dimension.

If Δ is any subset of V, then G_Δ and $C_G(\Delta)$ are respectively the setwise and vector-wise stabilizers of Δ in the semilinear group G; $G_{\Delta\Sigma} = G_\Delta \cap G_\Sigma$. Moreover, $G_\Delta{}^\Delta = G_\Delta/C_G(\Delta)$ is the semilinear group induced on Δ if Δ is a subspace. Similarly, if $x \in H$, then $G_{xH}^{H/x}$ is the group induced by G_{xH} on the space H/x.

The *rank* of a transitive permutation group is the total number of orbits of the stabilizer of a point.

The remainder of this section lists further definitions and results required in the proofs of Theorems I–V.

Theorem 2.1 (Ostrom-Wagner [18], Ostrom [17]). *If a projective plane P of prime power order q admits a collineation group G transitive on non-incident point-line pairs, then P is desarguesian and $G \geqslant PSL(3, q)$.*

Of course, (2.1) is true without the prime power assumption, but we will only need the stated case, which is much easier to prove. The next result is needed for (2.1), and is also used elsewhere in our argument.

THEOREM 2.2 [4, pp. 122, 130–134]. *Let $\mathcal{A}$ be an affine translation plane of order q, L a line, $x \in L$, and E the group of elations with axis L. Then*

(i) *E is semiregular on the set of lines different from L on x; and*

(ii) *if $|E| = q$ for each L, then $\mathcal{A}$ is desarguesian.*

Additional, more elementary results concerning translation planes will also be required; the reader is referred to Dembowski [4, Chap. 4] for further information concerning perspectivities and Baer involutions.

Consider next a geometry $\mathcal{G}$ of points, with certain subsets called "lines", such that any two points are on at most one line, each line has at least three points, and each point is on at least three lines. Call $\mathcal{P}$ and $\mathcal{L}$ the sets of points and lines. If $a, b \in \mathcal{P} \cup \mathcal{L}$, the distance $\partial(a, b)$ between them is the smallest number k for which there is a sequence $a = a_0, a_1, \ldots, a_y = b$, with each $a_i \in \mathcal{P} \cup \mathcal{L}$ and a_i incident with a_{i-1} for $i = 0, \ldots, k - 1$. Such a sequence is called a "path" from a to b. Now $\mathcal{G}$ is a *generalized n-gon* ($n \geqslant 3$) if

(i) whenever $\partial(a, b) < n$, there is a unique shortest path from a to b;

(ii) for all a and b, $\partial(a, b) \leqslant n$; and

(iii) there exist a and b with $\partial(a, b) = n$.

A generalized n-gon has *parameters s, t* if each line has exactly $s + 1$ points and each point is on exactly $t + 1$ lines.

THEOREM 2.3 (Feit–Higman [7]). *Generalized n-gons can exist only for $n = 3, 4, 6$ or 8; those with $n = 8$ cannot have parameters s, s.*

Generalized quadrangles enter our considerations as the geometries of points and lines in low-dimensional symplectic, unitary, and orthogonal geometries. Generalized hexagons are much less familiar; the ones we need are discussed in the Appendix (see also Sections 3, 5 below).

Generalized n-gons are special cases of *metrically regular* graphs. Let Γ be a connected graph defined on a set X of vertices. If $x, y \in X$, let $d(x, y)$ denote the distance between them. Let d be the diameter, and $\Gamma_i(x)$ the set of points at distance i from x, for $0 \leqslant i \leqslant d$. Then Γ is metrically regular if

(i) $|\Gamma_i(x)|$ depends only on i, not on x; and

(ii) if $d(x, y) = i$, the numbers of points at distance 1 from x and distance $i - 1$ (resp. i, $i + 1$) from y depend only on i, and not on x and y.

(Condition (i) follows from (ii) here).

If $\mathcal{G}$ is a geometry as previously defined, its *point graph* Γ is obtained by joining two points of $\mathcal{G}$ by an edge precisely when they are distinct and collinear. This graph may be metrically regular; for example, it is so when $\mathcal{G}$ is a generalized

n-gon. (Here the distances d and ∂ in graph and geometry are related by $d(x, y) = \frac{1}{2}\partial(x, y)$ for $x, y \in \mathscr{P}$.)

If q is a power p^e of p (where p, as always, is a prime), and $k \geqslant 2$, a *primitive divisor* of $q^k - 1$ is a prime $r \mid q^k - 1$ such that $r \nmid p^i - 1$ for $1 < p^i < q^k$. Note that $r \equiv 1 (\mathrm{mod}\ ek)$, by Fermat's theorem.

THEOREM 2.4 (Zsigmondy [28]). *If $q > 1$ is a power of p and $k > 1$, then $q^k - 1$ has a primitive divisor unless either*

(i) *$k = 2$ and q is a Mersenne prime, or*

(ii) $q^k = 64$.

3. EMBEDDING METRICALLY REGULAR GRAPHS IN PROJECTIVE SPACES

In this section we will prove a general result concerning certain embeddings in projective spaces. Let $\mathscr{G}$ be a geometry, with point set Ω and point graph Γ. For $x \in \Omega$, let $W_i(x)$ be the set of points distant at most i from x. We assume the following axioms (for all $x \in \Omega$):

(a) Ω is a set of points spanning $PG(n-1, q)$;

(b) each line L of $\mathscr{G}$ (or *$\mathscr{G}$-line*) is a line of $PG(n-1, q)$;

(c) Ω is the union of the set of $\mathscr{G}$-lines;

(d) Γ is metrically regular with diameter $d \geqslant 2$;

(e) $W_1(x)$ is a subspace of $PG(n-1, q)$;

(f) $W_i(x) = \Omega \cap U_i(x)$ for some subspace $U_i(x)$; and

(g) $|W_2(x)| = (q^h - 1)/(q - 1)$ for some integer h.

Note that (a)–(d) are among the embedding hypotheses in Buekenhout–Lefèvre [1].

In (3.1) and (3.2) we will determine all geometries satisfying (a)–(g). For Theorem I, a complete classification is not required; the weaker result (3.1) suffices.

THEOREM 3.1. *If $\mathscr{G}$ satisfies* (a)–(g), *then either*

(a) *$d = 2$ and $\mathscr{G}$ consists of the totally isotropic points and lines of a symplectic polarity $x \leftrightarrow W_1(x)$; or*

(b) *$d = 3$ and $\mathscr{G}$ is a generalized hexagon with parameters q, q. (Moreover, if $W_2(x)$ and $W_3(x)$ are subspaces for all x, then $n = 6$ and $x \leftrightarrow W_2(x)$ is a symplectic polarity.)*

Proof. Set $m = \dim W_1(x)$ (recalling from Section 2 that "dim" means vector space dimension). If $d(x, y) = i > 1$, let

$$e_i = \dim W_1(x) \cap W_{i-1}(y),$$
$$f_i = \dim W_1(x) \cap W_i(y).$$

(Note that both $W_1(x) \cap W_{i-1}(y)$ and $W_1(x) \cap W_i(y)$ are subspaces. For, if $W_j(y) = \Omega \cap U_j(y)$, then $W_1(x) \cap W_j(y) = W_1(x) \cap \Omega \cap U_j(y) = W_1(x) \cap U_j(y)$.) These dimensions depend only on i, not x or y. For, if $\Gamma_i(x) = W_i(x) - W_{i-1}(x)$ is the set of points at distance i from x, then

$$|\Gamma_1(x) \cap \Gamma_{i-1}(y)| = (q^{e_i} - 1)/(q - 1)$$
$$|\Gamma_1(x) \cap \Gamma_i(y)| = (q^{f_i} - 1)/(q - 1) - (q^{e_i} - 1)/(q - 1) - 1,$$

and

$$|\Gamma_1(x) \cap \Gamma_{i+1}(y)| = (q^m - 1)/(q - 1) - (q^{f_i} - 1)/(q - 1)$$

(provided also that $i < d$). By (g), $|\Gamma_2(x)| = (q^h - q^m)/(q - 1)$.

Counting pairs (y, z) with $d(x, y) = 1 = d(y, z)$ and $d(x, z) = 2$ yields

$$|\Gamma_1(x)|\,|\Gamma_2(x) \cap \Gamma_1(y)| = |\Gamma_2(x)|\,|\Gamma_1(x) \cap \Gamma_1(z)|,$$

whence $(q^m - q)(q^m - q^{f_1}) = (q^h - q^m)(q^{e_2} - 1)$. Equating powers of q yields $1 - f_1 = m$. There are then two possibilities:

(i) $m - 1 = e_2$, $1 = m - f_1 = h - m$; or

(ii) $m - 1 = h - m$, $1 = m - f_1 = e_2$.

Suppose (i) holds. Each point is on exactly $(q^{m-1} - 1)/(q - 1) = (q^{e_2} - 1)/(q - 1)$ $\mathscr{G}$-lines. Thus, if $d(x, z) = 2$, each of the $\mathscr{G}$-lines on z contains a point of the e_2-space $W_1(x) \cap W_1(z)$. Consequently, the graph has diameter $d = 2$. Moreover, Ω is a subspace. (For if x and y are distinct points of Ω but $\langle x, y\rangle$ is not a $\mathscr{G}$-line, then there is a point $z \in W_1(x) \cap W_1(y)$; then x and y are in the subspace $W_1(z)$, all of whose points are in Ω.) Now (a) yields $h = n$, so $m = n - 1$ and $W_1(x)$ is a hyperplane. Since $y \in W_1(x)$ implies that $x \in W_1(y)$, it follows that $x \leftrightarrow W_1(x)$ is a symplectic polarity, so (3.1.i) holds.

From now on, assume that case (ii) occurs. Since $e_2 = 1$ there is a unique point joined to two given points at distance 2. The restriction of the relation "joined or equal" to $\Gamma_1(x)$ is thus an equivalence relation, so $\Gamma_1(x)$ is a disjoint union of complete graphs, each of size $(q^{f_1} - q^{e_1})/(q - 1) = q(q^{m-2} - 1)/(q - 1)$. Since $|\Gamma_1(x)| = q(q^{m-1} - 1)/(q - 1)$, this implies that $m - 2 \mid m - 1$, whence $m = 3$. Then $f_1 = m - 1 = 2$ (and of course $e_2 = 1$).

We next determine the sequences $\{e_i\}$, $\{f_i\}$. Both are nondecreasing: if $d(x, y) = i$, $d(y, z) = 1$ and $d(x, z) = i + 1 < d$, then $W_1(x) \cap W_{i-1}(y) \subseteq W_1(x) \cap W_i(z)$ and $W_1(x) \cap W_i(y) \subseteq W_1(x) \cap W_{i+1}(z)$. Also, $e_i < f_i$ since

$|\Gamma_1(x) \cap \Gamma_i(y)| \geqslant 0$. If $f_i = 3$ for some i, then $\Gamma_1(x) \subseteq W_i(y)$, and so $i = d$; and conversely $f_d = \dim(W_1(x) \cap W_d(y)) = \dim W_1(x) = 3$. Thus, $e_i = 1$ and $f_i = 2$ for $i < d$, while $f_d = 3$ and $e_d = 1$ or 2.

We will show that $\mathscr{G}$ is generalized $(2d + 1)$-gon or $2d$-gon (with parameters q, q) according as $e_d = 1$ or $e_d = 2$. Thus, we must verify axioms (i)–(iii) given in Section 2, where ∂ was defined. For convenience, we separate the two cases.

Case $e_d = 1$. Since $e_i = 1$ for all $i \geqslant 1$, there is a unique shortest path joining any two points. Also, a $\mathscr{G}$-line L contains a unique point nearest x, unless $L \subseteq \Gamma_d(x)$. (For, if $y \in L$ with $d(x, y) = i < d$ minimal, and $u \in W_1(y) \cap W_{i-1}(x)$, then $\langle y, u\rangle = W_1(y) \cap W_i(x) \neq L$.) Thus, there is a unique shortest path between x and L if $\partial(x, L) < 2d + 1$.

Let L and L' be two $\mathscr{G}$-lines. Then there is a unique shortest path between L and L', except possibly if $L' \subseteq \Gamma_d(x)$ for some $x \in L$. (Two shortest paths could not start at the same point of L; but this would yield points of L and L' with more than one shortest path between them.) Suppose $L' \subseteq \Gamma_d(x)$. Then there is a unique shortest path from x to each of the $q + 1$ points of L', no two such paths using the same line through x (since this would produce a point y with $\partial(y, L') < 2d$ and two shortest paths from y to L'). Then these paths use all $q + 1$ $\mathscr{G}$-lines through x, and hence L must occur among them. Thus, $\partial(L, L') = 2d$ and a unique shortest path again exists. Consequently, axioms (i) and (ii) hold. Since $f_d = 3$ and $e_d = 1$, so does axiom (iii).

Case $e_d = 2$. This time, there is a unique shortest path from x to x' unless $x' \in \Gamma_d(x)$. As above, any $\mathscr{G}$-line L contains a unique point closest to x, and there is a unique shortest path from x to L. Finally, let L and L' be $\mathscr{G}$-lines with $\partial(L, L') < 2d$. Then only one shortest path can exist between L and L': two such paths would produce either two shortest paths from a point of L to L', or two shortest paths between points of L and L'. Thus, axioms (i)–(iii) again hold.

Since $e_2 = 1$, we have $d \geqslant 3$. The Feit–Higman Theorem (2.3) now shows that $d = 3$ and $e_3 = 2$.

It remains to prove the parenthical remark in (3.1.ii). Here $\Omega = W_3(x)$ is a subspace, and $\dim W_2(x) = \dim W_3(x) - 1$, so $x \leftrightarrow W_2(x)$ is a symplectic polarity. Since $|\Omega| = (q^6 - 1)/(q - 1)$, we have $n = 6$, as required.

THEOREM 3.2. *Suppose the hypotheses and conclusion* (ii) *of* (3.1.) *hold. Then*

(i) *If* $n = 6$, *then* q *is even*; *and*

(ii) *otherwise* $n = 7$ *and* Ω *is the set of singular points of a geometry of type* $O(7, q)$.

In either case the embedding of $\mathscr{G}$ *is unique.*

We defer the proof to the Appendix.

4. A Reformulation of Antiflag Transitivity

Sometimes the following criterion for antiflag transitivity is convenient.

Lemma 4.1. *A subgroup G of $\Gamma L(n, q)$ is antiflag transitive if and only if $G_L{}^L$ is 2-transitive for every line L.*

Proof. Suppose G_x has s orbits of hyperplanes on x, t orbits of hyperplanes not on x, and $s' + 1$ point-orbits in all. Then $s + t = s' + 1$, and G_x has s orbits of lines through x. Each such line-orbit defines at least one point-orbit other than $\{x\}$. Thus $t - 1 = s' - s \geqslant 0$, with equality if and only if G_{xL}^{L-x} is transitive for every line L through x, as required.

From Dickson's list of subgroups of $SL(2, q)$ [5, chap. 12], it is seen that only when $q = 4$ is there a 2-transitive subgroup H of $\Gamma L(2, q)$ for which $H \cap GL(2, q)$ is not 2-transitive. We deduce the following.

Corollary 4.2. *If $q \neq 4$ and $G \leqslant \Gamma L(n, q)$ is antiflag transitive then so is $G \cap GL(n, q)$.*

Remark. Subgroups of $\Gamma L(n, q)$ not in $GL(n, q)$ will arise in the inductive part of our proof, as the examples occurring in Theorem III indicate.

5. The Heart of Theorem II

Suppose $G \leqslant \Gamma L(n, q)$ is antiflag transitive but not 2-transitive on the points of V. The following lemma incorporates Perin's main idea [20].

Lemma 5.1. *If x is a point, then there is a subspace $W(x)$ (different from x and V) containing x, such that G_x fixes $W(x)$ and is transitive on $V - W(x)$.*

Proof. A Sylow p-subgroup of G fixes a hyperplane H and a point $x \in H$, and is transitive on $V - H$. Then

$$W(x) = \bigcap \{H^g \mid g \in G_x\}$$

is a G_x-invariant subspace; G_x is transitive on the pairs (H^g, y) for $g \in G_x$, $y \notin H^g$, and hence is transitive on $V - W(x)$.

Theorem 5.2. *Suppose $G \leqslant \Gamma L(n, q)$ is primitive but not 2-transitive on points, and is antiflag transitive. Then G preserves a symplectic polarity, and either*

(i) *G has rank 3 on points; or*

(ii) *G has rank 4 on points, $G \leqslant \Gamma Sp(6, q)$, and G acts on a generalized hexagon with parameters q, q.*

The proof involves an iteration of (5.1), followed by (3.1). Let $d-1$ denote the rank of G in its action on points.

LEMMA 5.3. *There are subspaces*

$$x = W_0(x) \subset W_1(x) \subset W_2(x) \subset \cdots \subset W_{d-1}(x) \subset W_d(x) = V$$

with the properties

(a) *G_x fixes $W_i(x)$ and is transitive on $W_i(x) - W_{i-1}(x)$ for $1 \leqslant i \leqslant d$;*

(b) *if $y \in W_1(x)$ and $0 \leqslant i \leqslant d-1$, then $W_i(y) \subseteq W_{i+1}(x)$;*

(c) *$W_i(x^g) = W_i(x)^g$ for all $g \in G$; and*

(d) *$d > 1$.*

Proof. Set $W_d(x) = V$ and $W_{d-1}(x) = W(x)$ (cf. (5.1)). Then $d > 1$ by (5.1). Since $W_d(x) - W_{d-1}(x)$ is the largest orbit of G_x, certainly $W_{d-1}(x^g) = W_{d-1}(x)^g$.

Now proceed by "backwards induction". Suppose $W_{i+1}(x),..., W_d(x)$ have been defined. Set $m_j = \dim W_j(x)$ for $j = i-1,..., d$. A Sylow p-subgroup P of G_x fixes a line L on x; necessarily $L \subseteq W_{i+1}(x)$. Let $y \in L - x$. Since all P-orbits on $V - W_{i+1}(x)$ have length at least $q^{m_{i+1}}$, all P_y-orbits on $W_{i+1}(y) - W_{i+1}(x)$ have length at least $q^{m_{i+1}-1}$. (By primitivity, $W_{i+1}(y) \neq W_{i+1}(x)$.) It follows that $W_{i+1}(x) \cap W_{i+1}(y)$ is a hyperplane of $W_{i+1}(x)$, and that G_{xy} is transitive on $W_{i+1}(y) - W_{i+1}(x)$. Then

$$W_i(y) = \bigcap \{W_{i+1}(x)^g \mid g \in G_y\}$$

is a subspace of $W_{i+1}(y)$, and G_y is transitive on $W_{i+1}(y) - W_i(y)$. Then (c) holds, since G_x has only one orbit of size $| W_{i+1}(x) - W_i(x)|$.

This process terminates when $W_0(x) = x$. Then $W_1(x) - x$ consists of all points y for which $\langle x, y\rangle$ is fixed by some Sylow p-subgroup of G. Now (b) follows from the construction. Thus, all parts of (5.3) are proved.

Let $\mathscr{G}$ be the geometry with line set $\{\langle x, y\rangle \mid y \in W_1(x)\}$, and Γ its point graph. By (5.3b) and induction on i, we see that $W_i(x)$ is the set of points at distance at most i from x (relative to the metric d in Γ). Consequently, Γ is metrically regular, and (3.1) applies. Since $W_2(x)$ and $W_3(x)$ are subspaces, the theorem follows.

By (3.2), the generalized hexagon in (5.2ii) must be the one associated with $G_2(q)$. However, as stated in Section 1, we will make the proof of Theorem I, and most of Theorems II and III, independent of the existence and uniqueness of the $G_2(q)$ hexagon. The required information is easily proved (frequently in the spirit of other of our arguments), and is collected in the following lemma.

LEMMA 5.4. *If G is as in (5.2ii), then the following statements hold*:

(a) *G has exactly two orbits of totally isotropic lines*;

(b) *G has exactly two orbits of totally isotropic planes*;

(c) *there is a totally isotropic plane E such that $G_E{}^E \geqslant SL(3, q)$*;

(d) *there is an element $t \in G$ with $t^p = 1$ and* $\dim C_V(t) = 4$;

(e) $|G \cap GL(6, q)| = q^6(q^6 - 1)(q^2 - 1)d$, *where $d \mid q - 1$ if $q \neq 4$, $d \mid 6$ if $q = 4$*;

(f) *if r is a prime divisor of $q + 1$ and R a Sylow r-subgroup of G_x, then* $\dim C_V(R) = 2$ *and $N_G(R)$ is 2-transitive on $C_V(R)$*; *and*

(g) *G_x has no element of prime order greater than $q + 1$.*

Proof. Since G_x has three point-orbits other than $\{x\}$, (a) is clear (cf. (4.1)). Clearly, $W_1(x)^G$ is an orbit of $(q^6 - 1)/(q - 1)$ totally isotropic planes. Let E be any of the remaining

$$(q^3 + 1)(q^2 + 1)(q + 1) - (q^6 - 1)/(q - 1) = q^3(q^3 + 1)$$

totally isotropic planes. If L is any $\mathscr{G}$-line, the $q + 1$ totally isotropic planes on L are all of the form $W_1(x)$ for $x \in L$. It follows that E contains no $\mathscr{G}$-lines, and for $y, z \in E$, $d(y, z) = 0$ or 2. Let $M = \langle y, z\rangle$ and $x = W_1(y) \cap W_1(z)$. If $P \in \mathrm{Syl}_p(G_x)$, then there are q choices for E on M (any totally isotropic plane except $\langle x, y, z\rangle$), while inside $W_1(x)$, there are q^2 choices for M. Thus $|P: P_{ME}| \leqslant q^3$, so each orbit of P_{ME} on $V - x^\perp$ has length at least q^3. Since $E - M$ is fixed by P_{ME}, we have $|P: P_{ME}| = q^3$, and P_{ME} is transitive on $E - M$. This proves (b). Moreover, since M is any line of E, (c) follows from (2.2).

Let $X \leqslant G_E$ induce all $(z, \langle w, z\rangle)$-elations (transvections) of E, where $w \in E - M$. Then X fixes M, and hence also the unique point x joined to all of M by $\mathscr{G}$-lines, as well as the unique point x' joined to all of $\langle w, z\rangle$ by $\mathscr{G}$-lines. Since we may assume that X is a p-group, $C_V(X) \supseteq \langle z, x'\rangle^\perp$; then $C_V(t) = \langle z, x'\rangle^\perp$ for all $t \in X - \{1\}$. (Note that G cannot contain nontrivial transvections of $Sp(6, q)$, since $W_2(x) \cap W_1(u) \neq W_2(x) \cap W_1(u')$ for $u, u' \in \Gamma_3(x)$, $u \neq u'$.)

Clearly, $|G| = (q^3 + 1)\, q^3 \mid G_E|$. But $|Sp(6, q)_E| = q^6 \mid GL(3, q)|$. If $g \in C_G(E)$ is a p-element then, proceeding as above, we find that g fixes a basis for V. Thus, $|G_E \cap GL(6, q)|$ divides $|GL(3, q)|$, and (e) holds.

Since G_{xE} is transitive on the $q + 1$ $\mathscr{G}$-lines through x, the group R in (f) cannot fix any point of $x^\perp - x$. Recall that $R \leqslant GL(6, q)$. Since R fixes a point of $V - x^\perp$, $\dim C_V(R) \geqslant 2$. But $C_V(R) \cap x^\perp = x$, so $\dim C_V(R) = 2$. The last part of (f) follows from antiflag transitivity.

Finally, (g) follows from (e), or more simply thus. If there were such an element $g \in G_x$, then g would fix all $q + 1$ $\mathscr{G}$-lines through x and all their points, and hence all points of $\mathscr{G}$, by connectedness.

Remarks. 1. Only (5.4 d,e,f,g) are needed for Theorem I.

2. $G \cap Sp(6, q)$ is generated by the conjugates of the group X appearing in the above proof.

3. If $G \leqslant \Gamma L(n, q)$ is antiflag transitive and primitive on points, then it is primitive on hyperplanes. For, if G preserves a symplectic polarity, then its actions on points and hyperplanes are isomorphic; otherwise, by (5.2), G is 2-transitive on points, and so also on hyperplanes. We will see later (7.1) that a stronger result can be obtained by elementary arguments independent of (3.1).

6. The Primitive Case

We now begin the inductive part of the proof of Theorems I–III. In order to avoid identifying $G_2(q)$ during the proof of Theorem I (cf. Section 1), we restrate the theorems in slightly weaker form.

Theorem 6.1. *Let $G \leqslant \Gamma L(n, q)$, $n \geqslant 2$, be antiflag transitive. Then one of the following holds:*

(i) $G \trianglerighteq SL(n, q)$;

(ii) G is A_7 *inside* $SL(4, 2)$;

(iii) $G \trianglerighteq Sp(n, q)$;

(iv) *G is A_6 inside* $SL(4, 2)$;

(v) $|G| = 20$, $G < \Gamma L(2, 4)$;

(vi) $G \leqslant \Gamma Sp(6, q) < \Gamma L(6, q)$, *$q$ is even, and G acts as a rank* 4 *group on the points of a generalized hexagon with parameters q, q, whose points and lines consist of all points and certain totally isotropic lines for* $\Gamma Sp(6, q)$;

(vii) $G \trianglerighteq SL(\frac{1}{2}n, 4)$, *embedded naturally in* $SL(n, 2)$;

(viii) $G \trianglerighteq Sp(\frac{1}{2}n, 4)$, *embedded naturally in* $SL(n, 2)$; *or*

(ix) *G is a subgroup of $\Gamma Sp(6, 4)$, itself embedded naturally in $SL(12, 2)$, such that G acts on a generalized hexagon in $PG(5, 4)$ as in* (vi).

Note that 2-transitive subgroups of $\Gamma L(n, q)$ are automatically antiflag transitive (Wagner [25], or (4.1)).

The theorem will be proved by induction on n in Sections 6, 7. The case $n = 2$ is omitted, while (2.2) handles $n = 3$. We therefore assume $n \geqslant 4$. By (4.2), if $q \neq 4$ we may assume that $G \leqslant GL(n, q)$.

In this section we will consider only *primitive* groups G. Then either (5.2) applies, or G is 2-transitive. In either case, induction or known results almost always produce sufficiently large groups of transvections for G to be identified.

PROPOSITION 6.2. *If* (5.2i) *holds then either* $G \trianglerighteq Sp(n, q)$ *or* G *is* A_6 *inside* $Sp(4, 2)$.

Proof. Let x and y be distinct points of the (totally isotropic) line L. There is a Sylow p-subgroup P of G fixing x and L, and transitive on $V - x^\perp$. Then all orbits of P_y on $V - x^\perp$ have length at least q^{n-1}/q, so P_y is transitive on $y^\perp - x^\perp$. Since G_y is already transitive on $y^\perp/y$, it is thus antiflag transitive there.

By our inductive hypothesis concerning (6.1), $K = G_y^{y^\perp/y}$ satisfies one of the following conditions:

(α) $K \trianglerighteq Sp(n-2, q)$;

(β) $K = A_6$, $n - 2 = 4$, $q = 2$;

(γ) K acts on a generalized hexagon as in (6.1v), $n - 2 = 6$;

(δ) $Sp(\frac{1}{2}n - 1, 4) \trianglelefteq K \leqslant \Gamma Sp(\frac{1}{2}n - 1, 4)$; or

(ϵ) $K < \Gamma Sp(6, 4)$ acts on a generalized hexagon over $GF(4)$ as in (6.1 v), $q = 2$, $n - 2 = 12$.

At this stage it is easiest to quote Perin [19] when $q > 2$. In fact, we will use his method to handle all these cases when $q = 2$. Since $Sp(n, 2)$ is generated by transvections, it suffices to show that G contains a nontrivial transvection.

Set $S = Sp(n, 2)$ and $P = O_2(S_y) = C_S(y^\perp/y)$. Then $|P| = 2^{n-1}$, and P is S_y-isomorphic to the natural representation space of $S_y/P \cong Sp(n-2, 2) \cong \Omega(n-1, 2)$ of degree $n - 1$; the radical of the orthogonal space corresponds to the group T of all transvections in P. (Explicitly, view S as $\Omega(n+1, 2)$, acting on an $n + 1$ — space $\tilde{V}$. Let e and f be non-perpendicular singular vectors, with $\langle e\rangle + \operatorname{rad} \tilde{V} = y$. Then P consists of all transformations $e \to e$, $f \to f + c$, $u \to u + (u, c)e$ for some $c \in \langle e, f\rangle^\perp$ and all $u \in \langle e, f\rangle^\perp$.) Now suppose that $G \cap P \neq 1$ and $G \cap T = 1$. Since G_y is a transitive on $y^\perp/y$, it is transitive on $P/T - \{1\}$. Thus $|G \cap P| = 2^{n-2}$, and Gy fixes a nondegenerate hyperplane of $\tilde{V}$, so $K \leqslant O^\pm(n-2, 2)$. But no subgroup of the latter group can be transitive on $y^\perp/y$.

Consequently, if we can show that $G \cap P \neq 1$ we will have $G \cap T \neq 1$, and hence $G = S$.

First, suppose $n = 6$, so $K \leqslant Sp(4, 2)$. Certainly, $2^5 \mid |G|$, and $2^5 \nmid |K|$, so $G \cap P \neq 1$.

Now let $n \geqslant 8$, and consider (α), (γ), (δ) and (ϵ). For these cases, set $i = (n-2) - 4$, 2, $(n-2) - 4$, resp. 4.

Let r be a primitive divisor of $2^i - 1$ (see (2.4); use $r = 7$ if $i = 6$), and $R \in \operatorname{Syl}_r(G_{xL})$ for $x \in L$. By (5.4f), $\dim_{x^\perp/x}(R) = 2$ resp. 4 in cases (δ) resp. (ϵ), and hence dim $C_V(R) = 4$ resp. 6. Similarly, in cases (α) and (δ) dim $C_V(R) = 6$, except that dim $C_V(R) = 4$ when $n = 8$ (and $r = 3$) in (α).

Since R is completely reducible by Maschke's theorem. $W = C_V(R)$ is a nonsingular subspace.

Now R is a Sylow subgroup of the stabilizer of two distinct perpendicular points, and also of two non-perpendicular points unless $n = 8$ and $r = 3$. Thus, $N_G(R)$ induces a rank 3 group on W. (If $r = 3$, $n = 8$ and dim $W = 4$, then $N_G(R)^W$ is a subgroup of $Sp(4, 2) \simeq S_6$ transitive on ordered pairs of distinct perpendicular points.) Then $N_G(R)^W$ is $Sp(6, 2)$, $Sp(4, 2)$ or A_5.

Also, $N_G(R)^{W^-}$ is a subgroup of $\Gamma L(1, 2^i)$, $\Gamma L(2, 2^2)$ (if $i = 2$), $\Gamma L(2, 2^3)$ (when $n = 12$ and $i = 6$ in (α)) or $\Gamma L(2, 2^4)$ (when $i = 4$ in (ϵ)).

It follows that $N_G(R)$ has a subgroup N inducing the identity on W^- and $Sp(6, 2)$ or A_6 on W. In either case, N_y has an involution centralizing $(y^- \cap W)/y$. Thus, $G \cap P \neq 1$, so $G = Sp(n, 2)$, as required.

PROPOSITION 6.3. *If $G \leqslant \Gamma L(n, q)$ $(n \geqslant 3)$ is 2-transitive on points, then either $G \geqslant SL(n, q)$ or G is A_7 inside $SL(4, 2)$.*

Proof. In view of Wagner [25], we may assume that $n \geqslant 6$. We recall the following additional facts from Wagner [25]: G is 2-transitive on hyperplanes, and if H is a hyperplane, then $G_H{}^H$ is antiflag transitive.

Once again, we will run through the possibilities provided by induction for $G_H{}^H$ and, dually, $G_x^{V/x}$. If either is 2-transitive, then G is flag-transitive, and the result follows from Higman [8]; so suppose not.

Suppose $G_x^{V/x}$ is contained in $\Gamma Sp(n - 1, q)$. If x' is a second point, then $G_{xx'}$ fixes a hyperplane H on x and x'. So there is a G-orbit of length $v(v - 1)$ of ordered triples (x, x', H) with $x, x' \in H$, $x \neq x'$. (Here $v = (q^n - 1)/(q - 1)$; k will denote $(q^{n-1} - 1)/(q - 1)$.) Then G_{xH}^H has an orbit of length $v(v - 1)/vk = q$, where $x \in H$. By (4.1), this orbit, together with x, forms a line Δ. Clearly $G_{xH} < G_{\Delta H}$, so $G_H{}^H$ is imprimitive. (Conversely, if G_{xH} fixes a line Δ with $x \in \Delta \subseteq H$, then $\Delta/x \leftrightarrow H/x$ is a symplectic polarity of V/x preserved by G_x.)

Thus we may assume that $q = 2$ and $K = G_H{}^H$ is imprimitive. Then $n - 1$ is even, $n \geqslant 7$, and $K \leqslant \Gamma L(\frac{1}{2}(n - 1), 4)$ behaves in one of the following ways:

(α) $K \trianglerighteq SL(\frac{1}{2}(n - 1), 4)$;

(β) $K \trianglerighteq Sp(\frac{1}{2}(n - 1), 4)$; or

(γ) $K < \Gamma Sp(6, 4)$ acts on a generalized hexagon over $GF(4)$ as in (6.1v), $n - 1 = 12$.

Let i be $(n - 1) - 2$ in (α), $(n - 1) - 4$ in (β) and 4 in (γ). Let r be a primitive divisor of $2^i - 1$ (use $r = 7$ if $i = 6$), and $R \in \mathrm{Syl}_r(G_{xH})$ for $x \in H$. Then dim $C_V(R)$ is $1 + 2$ in (α) and $1 + 4$ in (β) and (γ) (using (5.4f) in case (γ)). Moreover, $N_G(R)$ is 2-transitive on $C_V(R)$, while $N_G(R)_H$ is impromitive on

$C_H(R)$. Since $N_G(R)$ induces $SL(3, 2)$ or $SL(5, 2)$ on $C_V(R)$ by induction, we have the contradiction which implies the proposition.

Now (5.2), (3.2), (6.2) *and* (6.3) *complete the inductive step in* (6.1) *when G is primitive on points.*

Having dealt with the primitive case, we record an elementary corollary for use in the next section.

LEMMA 6.4. *Suppose G is as in* (6.1) *and is primitive on points. If $F \leqslant G$ with F antiflag transitive and $|G:F|$ a power of p, then F is also primitive on points.*

Proof. Let $P \in \mathrm{Syl}_p(G_x)$. Then P fixes a unique line on x. (In case (6.1vi), apply (4.1) to a line L of $W_1(x)$ not on x.) Clearly $G = PF$ and $P \cap F \in \mathrm{Syl}_p(F_x)$.

In each instance of (6.1), $P \cap F$ also fixes a unique line on x. (For (6.1vii–ix) this just says that $P \cap F$ fixes a unique point over $GF(4)$.) If F were imprimitive, then F_x, $P \cap F$, and jence $G_x = PF_x$ would fix L, contradicting (4.1) and the primitivity of G.

7. THE IMPRIMITIVE CASE; COMPLETION OF THE PROOF

Continuing our proof of (6.1), we now turn to the case of an antiflag transitive subgroup G of $\Gamma L(n, q)$ which is *imprimitive on points*. The method here is entirely different from that of Sections 5, 6; we build a new projective space on which G continues to act antiflag transitively.

If Δ is a nontrivial imprimitivity block for the action of G on points, then Δ is the set of points of a subspace. (For, G_Δ is transitive on the hyperplanes of $\langle\Delta\rangle$, hence on its points, and thus Δ must contain all points of $\langle\Delta\rangle$.) We usually identify Δ with $\langle\Delta\rangle$. Let $\delta = \dim \Delta$. By Remark 3 at the end of Section 5, G is also imprimitive on hyperplanes, and a block of imprimitivity consists of all hyperplanes containing a subspace Σ. The next result (independent of the aforementioned Remark) shows that there is a close connection between blocks of points and hyperplanes. It is due to Orchel [16], and simplifies and improves a result in an earlier version of this paper.

LEMMA 7.1 (Orchel). *Let Δ be a block of imprimitivity for G acting on points, and $\delta = \dim \Delta$. For any hyperplane H, let Σ be the union of the members of Δ^G contained in H. Then Σ is a subspace of dimension $n - \delta$, and the set of hyperplanes containing Σ is a block of imprimitivity for G acting on hyperplanes.*

Proof. We have $|\Delta^G| = (q^n - 1)/(q^\delta - 1)$. Set $H \cap \Delta^G = \{\Delta' \in \Delta^G \mid \Delta' \subseteq H\}$. If $\Delta' \in \Delta^G$, $\Delta' \nsubseteq H$, then $|\Delta' - H| = q^{\delta-1}$; so there are $q^{n-\delta}$ such subspaces. Thus $|H \cap \Delta^G| = (q^{n-\delta} - 1)/(q^\delta - 1)$. The union Σ of the members of $H \cap \Delta^G$ has cardinality $(q^{n-\delta} - 1)/(q - 1)$.

Let P be a Sylow p-subgroup of G_H. Then P is transitive on $V - H$, and hence on $\Delta^G - (H \cap \Delta^G)$. Let Σ' be a subspace of H of dimension $n - \delta$ fixed by P. If $\Sigma' \cap \Delta' \neq 0$ for one (and hence all) $\Delta' \in \Delta^G - (H \cap \Delta^G)$, then $|\Sigma'| \geqslant q^{n-\delta}$, which is false; so $\Sigma' \subseteq \Sigma$, and comparing cardinalities shows that $\Sigma' = \Sigma$.

Now, if H' is any hyperplane containing Σ, then Σ is the union of the members of $\Delta^G \cap H'$; and G_Σ is transitive on the set of such hyperplanes H'. This proves the lemma.

Notation. Let Δ be a minimal proper block of imprimitivity, and define Σ as in (7.1). Set $W \cap \Delta^G = \{\Delta' \in \Delta^G \mid \Delta' \subseteq W\}$ for any subspace W. We have shown that $\Sigma \cap \Delta^G$ partitions Σ. Let $\mathscr{L}$ be the lattice of all intersections of members of Σ^G.

LEMMA 7.2. *If $n > 2\delta$ then $\mathscr{L}$ is the lattice of subspaces of a projective space $PG(n/\delta - 1, q^\delta)$ on which G acts as an antiflag transitive collineation group.*

Proof. If $W \in \mathscr{L}$, $W \subseteq \Sigma_1 \in \Sigma^G$, and $W \cap \Delta_1 \neq 0$ for $\Delta_1 \in \Delta^G$, then $\Delta_1 \subseteq \Sigma_1$, by (7.1). Thus, $W \cap \Delta^G$ partitions W. If $W = \langle \Delta_1, ..., \Delta_k \rangle$ with $\Delta_i \subset \Delta^G$ and k minimal, then dim $W = k\delta$ and $|W \cap \Delta^G| = (q^{k\delta} - 1)/(q^\delta - 1)$. Call W a Point, Line, or Plane if $k = 1$, 2 or 3, respectively. Then two Points are on a unique Line (containing $q^\delta - 1$ Points), and three Points not on a Line are in a unique Plane (containing $q^{2\delta} + q^\delta - 1$ Points). The Veblen and Young axioms [24] imply that $\mathscr{L}$ is a projective space.

By (7.1), $H \cap \Delta^G = \Sigma \cap \Delta^G$, and G_H is transitive on the $q^{n-\delta}$ Points not in Σ. Thus, G acts antiflag transitively on $\mathscr{L}$, and the lemma follows from (2.1).

DEFINITION. Let $\mathcal{C}\!\ell$ denote the set of all cosets of members of $\mathscr{L}$. (Since $0 \in \mathscr{L}$, all vectors of V are in $\mathcal{C}\!\ell$.)

LEMMA 7.3. *If $n > 2\delta$ then $\mathcal{C}\!\ell$ is the lattice of subspaces of $AG(n/\delta, q^\delta)$.*

Proof. From $\mathcal{C}\!\ell \cup \mathscr{L}$ by attaching $\mathscr{L}$ "at infinity" as follows: adjoin $U \in \mathscr{L}$ to $W + v$ if $U \subseteq W \in \mathscr{L}$. Thus, $\mathcal{C}\!\ell \cup \mathscr{L}$ will have two types of "points" (vectors and members of Δ^G), and two types of "lines" (cosets of members of Δ^G), and two types of "lines" (cosets of members of Δ^G, and Lines of $\mathscr{L}$). If $\langle \Delta, \Delta' \rangle$ is a Line of $\mathscr{L}$, then it and any vector determine a translation plane of order q^δ in a standard manner [4, p. 133]; $\langle \Delta, \Delta' \rangle$ plays the role of line at infinity. By (7.2), $\mathcal{C}\!\ell \cup \mathscr{L}$ satisfies the Veblen and Young axioms, and hence is $PG(n/\delta, q^\delta)$. This proves the lemma.

LEMMA 7.4. *If $n = 2\delta$ then $\mathcal{C}\!\ell$ is $AG(2, q^\delta)$.*

Proof. As above, $\mathcal{C}\!\ell$ is an affine translation plane. But here Δ^G is merely its line at infinity, so proving that $\mathcal{C}\!\ell$ is desarguesian will be more difficult.

Let $x \in \Delta$ and $P \in \mathrm{Syl}_p(G_x)$. The group $E = C_p(\Delta)$ consists of all elations of $\mathcal{A}$ with axis Δ; it is semiregular on the set $\Delta^G - \{\Delta\}$ of linest $\neq \Delta$ of $\mathcal{A}$ through the point 0 of $\mathcal{A}$, and $\mathcal{A}$ is desarguesian if $|E| = q^\delta$, by (2.2). We may thus assume that $|E| < q^\delta$.

Since $|H \cap \Delta^G| = 1$, G_x is transitive on $\Delta^G - \{\Delta\}$. Thus, G_Δ is transitive on the pairs (x, Δ') with $x \in \Delta$ and $\Delta' \in \Delta^G - \{\Delta\}$, so $G_{\Delta\Delta'}^{\Delta}$ is transitive. But $P_{\Delta'}$ is transitive on $\Delta' - H$ if P fixes $H \supset \Delta$. Thus, $G_{\Delta\Delta'}^{\Delta}$ is even antiflag transitive. Moreover, $G_\Delta = P \cdot G_{\Delta\Delta'}$ since P is transitive on $\Delta^G - \{\Delta\}$. Then $G_\Delta{}^\Delta = P^\Delta G_{\Delta\Delta'}^{\Delta}$; since $G_\Delta{}^\Delta$ is primitive by the minimality of Δ, $G_{\Delta\Delta'}^{\Delta}$ is primitive by (6.4). We claim that $C_G(\Delta)_{\Delta'} = 1$. For, $C_G(\Delta) \trianglelefteq G_\Delta$, where G_Δ is transitive on $\Delta^G - \{\Delta\}$, while $C_G(\Delta)_{\Delta'}$ consists of homologies of $\mathcal{A}$ with axis Δ. Thus, if $C_G(\Delta)_{\Delta'} \neq 1$, then this holds for every $\Delta' \in \Delta^G - \{\Delta\}$. Then in the action of $C_G(\Delta)$ on $\Delta^G - \{\Delta\}$, the stabilizer of two points is trivial, but the stabilizer of any point is nontrivial. This implies that $C_G(\Delta)$ acts as a transitive Frobenius group on $\Delta^G - \{\Delta\}$, with kernel E of order q^δ, contrary to assumption.

It follows that $C_G(\Delta) = E$, and $|G_\Delta{}^\Delta : G_{\Delta,\Delta'}^{\Delta}| = q^\delta / |E|$.

Suppose q is odd. By (4.2) we may assume that $G \leqslant GL(n, q)$. By induction, both $G_{\Delta,\Delta'}^{\Delta}$ and $G_\Delta{}^\Delta$ have normal subgroups $SL(\delta, q)$ or $Sp(\delta, q)$ or a group as in (5.4). It follows that $G_\Delta{}^\Delta = G_{\Delta,\Delta'}^{\Delta}$ (cf. (5.4)), and $|E| = q^\delta$, a contradiction.

Consequently, q is even. If $t \in G_{\Delta,\Delta'}$ is an involution, then $\dim C_\Delta(t) = \frac{1}{2}\delta$ and $|C_E(t)| \leqslant q^{1/2\delta}$ (since $C_E(t)$ acts on the Baer subplane for t). Induction for $G_{\Delta,\Delta'}^{\Delta}$, together with this restriction on involutions and (5.4), imply that either $(\alpha)\delta = 2$, or $(\beta)\delta = 4$, $q = 2$.

(α) The argument used for q odd applies, unless $q = 4$, $G_\Delta{}^\Delta \geqslant SL(2, 4).2$ and $G_{\Delta,\Delta'}^{\Delta} \geqslant SL(2, 4)$. Here $4^2/|E| = q^\delta/|E| = 2$, so an $SL(2, 4)$ inside $G_{\Delta,\Delta'}$ will centralize E. Choosing t in this $SL(2, 4)$ yields a contradiction to $|C_E(t)| \leqslant 4$.

(β) In this case, $|G_\Delta{}^\Delta : G_{\Delta,\Delta'}^{\Delta}| = 8$ or 2, corresponding to (6.1ii, iv), so $|E| = 2$ or 8. Since $G_{\Delta,\Delta'}^{\Delta} \geqslant A_6$, the arguments in ($\alpha$) yield $|E| = 2$. Then $G_\Delta{}^\Delta \cong A_8$ acts on the $q^\delta/|E| = 8$ nontrivial E-orbits on Δ^G. If $\Delta' \in \Delta^G - \{\Delta\}$, then $G_{\Delta,\Delta'}$ fixes the unique Δ'' for which $\Delta'^E = \{\Delta', \Delta''\}$. It follows that G preserves a Steiner triple system on Δ^G, which is impossible since $|\Delta^G| = 17$.

This completes the proof of (7.4).

Proof of 6.1. We may temporarily assume that G contains the group S of all scalar transformations of V. Then $VS = V \rtimes S$ is a collineation group of $\mathcal{A}$, with V its translation subgroup. Each minimal imprimitivity block of VS in its action on the vectors of V is a coset of a 1-space of V. Thus, the structure of V as a $GF(q)$-space is deducible from $\mathcal{A}$ in a unique manner. The group G^+ of all collineations of $\mathcal{A}$ induced by elements of $\Gamma L(n, q)$ must then be $\Gamma L(n/\delta, q^\delta)$.

In particular, $(G_\Delta{}^+)^\Delta$ is the semidirect product of $GF(q^\delta)^*$ with $\mathrm{Aut}(GF(q^\delta))$, and $|G_\Delta^\Delta| = (q^\delta - 1)\delta e/(q-1)$, where $q = p^e$. Since this group is antiflag transitive, $q^{\delta-1}$ divides δe, whence $q = \delta = 2$.

Finally, G acts primitively on the set Δ^G of points of $\mathscr{L}$. For otherwise, there must be an imprimitivity block $\Gamma \supset \Delta$ of dimension 4, corresponding to an imprimitivity block for the action of G on Δ^G. This contradicts the previous paragraph, since $\mathscr{L}$ is a projective space over $GF(4)$, not $GF(2)$. This completes the proof of (6.1).

Remarks. 1. The argument used for q odd applies in almost all cases provided that the groups in (6.1vi) have been identified. The only obstacles occur when $q = 4$, or $q = 2, \delta = 4$.

2. Examples of (6.1 vii–ix) actually occur. Consider $G = \Gamma Sp(\frac{1}{2}n, 4) < GL(n, 2)$, for example. Clearly, G is transitive on $V - \{0\}$. Let $P \in \mathrm{Syl}_2(G)$. Then P fixes a hyperplane F over $GF(4)$, and is transitive on the $GF(4)$-points Y outside F. Over $GF(2)$ we have $\dim F = n - 2$, and P fixes a hyperplane $H \supset F$. Clearly $|Y - H| = 2$. Let $t \in P$ acts as an involutory field automorphism. Then $|C_V(t)| = 2^{n/2}$, and we may assume that t fixes Y. Thus $\langle t \rangle$ is transitive on $Y - H$, and G is antiflag transitive.

By (6.1), the proof of Theorem I is complete. Moreover, for Theorems II and III, we have only to identify the groups occurring in (6.1 vii)—the hexagon is already known, by (5.2) and (3.2). It is known that the group of automorphisms of the hexagon $\mathscr{G}$ induced by elements of $Sp(6, q)$ is $G_2(q)$; this is implicit in Tits [22] and explicit in Tits [23]. We prove, independently of this, that $G \cap Sp(6, q) = G_2(q)$. Since $G_2(q) \leqslant \mathrm{Aut}(\mathscr{G})$, this follows from the fact that $|GS \cap GL(6, q)| = |G_2(q)S|$ and $G_2(q) \cap S = 1$ (where S denotes the group of scalar transformations of V). But this is shown in (A.6 iii).

8. Corollaries

In this section we give some consequences of Theorems I–III.

The *affine group* $A\Gamma L(n, q)$ is defined as the group

$$\{v \to v^g + c \mid g \in \Gamma L(n, q), c \in V\} \cong T \rtimes \Gamma L(n, q)$$

of all collineations of the affine space $AG(n, q)$ based on V, an n-space over $GF(q)$. (T denotes the translation group.)

Proposition 8.1. *Let $G \leqslant A\Gamma L(n, q)$, $n \geqslant 3$, be transitive on ordered non-collinear triples of points of $AG(n, q)$. Then $G = T \rtimes G_0$, where T is the translation group, and $G_0 \trianglerighteq SL(n, q)$ or G_0 is A_7 (with $n = 4$, $q = 2$).*

Proof. The hypothesis implies that G_0 (the stabilizer of 0) is one of the groups of Theorem *I*; it remains only to show that G contains T. If not, then $G \cap T = 1$ (since G_0 is transitive on nonzero vectors), and so $|G| \leqslant |\Gamma L(n, q)|$. But then $|G: G_0| = q^n$ contradicts $|\Gamma L(n, q): G_0| \leqslant (q-1)e$ (resp. $|\Gamma L(n, q): G_0| = 8$) if $G_0 \geqslant SL(n, q)$, $q = p^e$ (resp. $G_0 = A_7$).

COROLLARY 8.2. *The only* 3-*transitive proper subgroups of* $AGL(n, 2)$ *are* $V_{16} \rtimes A_7$ *when* $n = 4$.

This corollary improves various results in the literature (for example [3, Theorem 1]); and also Jordan's theorem (Wielandt [26, (9.9)]):

COROLLARY 8.3. *A normal subgroup N of a* 3-*transitive group G is* 2-*transitive, unless it is elementary abelian of order* 2^n *and either* $G = N \rtimes GL(n, 2)$ *or* $n = 4$ *and* $G = N \rtimes A_7$.

From results of Perin [19] and Kantor [12], we deduce the following.

PROPOSITION 8.4. *Suppose* $G \leqslant \Gamma L(n, q)$ *is transitive on the j-subspaces of* $PG(n-1, q)$ *for some j with* $2 \leqslant j \leqslant n-2$. *Then G is transitive on the i-subspaces for all i with* $1 \leqslant i \leqslant n-1$, *and one of the following occurs*:

(i) $G \trianglerighteq SL(n, q)$;

(ii) *G is* A_7 *inside* $GL(4, 2)$; *or*

(iii) *G is* $\Gamma L(1, 2^5)$ *inside* $GL(5, 2)$.

Remark. A "t-(v, k, λ) design in a finite vector space" is a collection of k-subspaces or "blocks" in a v-space, any t-space being contained in precisely λ blocks. No nontrivial examples are known with $t \geqslant 2$; and (8.4) shows that none can be constructed by the analogue of the familiar construction of t-designs from t-homogeneous groups (Dembowski [4, (2.4.4)]).

To motivate the next result, we sketch the deduction of Perin's Theorem [20] (mentioned in Section 1) from Theorem II. Suppose $G \leqslant \Gamma L(n, q)$, $n \geqslant 4$, and suppose G acts as a primitive rank 3 group of even order on the points of $PG(n-1, q)$. For a point x, G_x has three orbits on points, and hence three orbits on hyperplanes. If G is antiflag transitive, then $G \leqslant \Gamma Sp(n, q)$ by Theorem II (and indeed G is known). Otherwise, G_x is transitive on the hyperplanes through x, and so also on the lines through x, in contradiction to Kantor [12].

PROPOSITION 8.5. *Suppose* $G \leqslant \Gamma L(n, q)$, $n \geqslant 4$, *and G acts as a primitive rank* 4 *group on the points of* $PG(n-1, q)$. *Then either* $q = 2, 3, 4$ *or* 9, *or* $G \trianglerighteq G_2(q)$, *q even, embedded naturally in* $\Gamma Sp(6, q)$.

Proof. By Theorem II, we may assume that G is not antiflag transitive; by the previous argument and Kantor [12], we may assume it is not transitive on incident point-hyperplane pairs. Thus, of the four G_x-orbits on hyperplanes, two consist of hyperplanes containing x. Then G_x has two orbits on lines containing x. There are thus two G-orbits on lines, with G_x transitive on the lines of each orbit which pass through x. Consequently, $G_L{}^L$ is transitive for each line L.

Since G_x has three orbits on points different from x, it follows that, for suitable L and M chosen from different line-orbits, G_{xL}^{L-x} is transitive while $G_{x\,M}^{M-x}$ has two orbits. Thus, $G_L{}^L$ is 2-transitive while $G_M{}^M$ has rank 3. But, using Dickson's list of subgroups of $PSL(2, q)$ [5, chap. 12], we see that $P\Gamma L(2, q)$ has a rank 3 subgroup only if $q = 2, 3, 4$ or 9.

PROPOSITION 8.6. *Let G be an irreducible subgroup of $P\Gamma L(n, q)$, $n \leqslant 4$. Suppose G_x is transitive on the lines through x, for some point x. Then G is 2-transitive on points (and Theorem I applies).*

Proof. By Kantor [12], it is enough to show that G is transitive on points. So let $X = x^G$ and assume X is not the set of all points. If L is a line and $L \cap X \neq 0$, then $l = |L \cap X|$ is independent of L, and $1 < l < q - 1$. If $\dim W = m$ and $W \cap X \neq 0$, then $|W \cap X| = 1 - (l - 1)(q^{m-1} - 1)/(q - 1)$.

It follows that there is an $(n - 2)$-space U disjoint from X (for otherwise the hyperplane sections of X would be the blocks of a symmetric design). The hyperplanes containing U partition X into sets of cardinality $1 - (l - 1)(q^{n-2} - 1)/(q - 1) = k$; so k divides $1 + (l - 1)(q^{n-1} - 1)/(q - 1) = |X|$, whence k divides q^{n-2}. Since $k > (q^{n-2} - 1)/(q - 1) > q$ we have $0 \equiv k \equiv l$ (mod q), whence $l = q$. But then the complement of X contains one or all points of each line, and so is a hyperplane fixed by G, contradicting irreducibility.

II. THEOREMS IV AND V

9. THE GEOMETRY OF PRIMITIVE ANTIFLAG TRANSITIVE GROUPS

The proof of Theorem IV occupies Sections 9–11. The present section contains notation and the analogue of (5.3). The primitive case is concluded in Section 10; there the method is different from that of Section 6. Unlike Theorems I–III, the primitive case here does not depend on the imprimitive one. Finally, Section 11 corresponds to Section 7.

The symplectic case is covered by Theorems II and III; so *we will exclude the case $G \leqslant \Gamma Sp(2n, q)$ for the remainder of the proof.* Also, in view of the isomorphism between the $Sp(2n, q)$ and $O(2n + 1, q)$ geometries when q is even, *we will also exclude the case $G \leqslant \Gamma O(2n - 1, q)$, q even.* Thus, the geometry is associated with a nondegenerate sesquilinear form.

In the proof, Ω denotes the set of totally isotropic or totally singular (abbreviated t.i. or t.s.) points of the appropriate classical geometry, defined on a vector space V over $GF(q)$. (This assumption involves a slight change of notation in the unitary case: G will be a subgroup of $\Gamma U(n, q^{1/2})$.) By convention, we make no reference to 1-spaces outside Ω without explicit mention. Thus, if S is a subset of Ω, then $S^\perp$ is the set of points of Ω collinear with (i.e. perpendicular to) every point of S. The subspace 0 plays the role of $\varnothing$, so $0^\perp = \Omega$. A t.i. or t.s. subspace W is maximal if and only if $W^\perp = W$. The dimension of a t.i. or t.s. subspace is its vector space dimension, and the *rank* r of the geometry is the maximal such dimension.

We begin with two preliminary lemmas.

LEMMA 9.1. *There do not exist subspaces T, W with $T \cup T^\perp = W^\perp$ and $T, T^\perp \neq W^\perp$.*

Proof. If $T \cup T^\perp = W^\perp$ then $T \cap T^\perp = (W^\perp)^\perp = W$. Let $t_1 \in T - W$ and $t_2 \in T^\perp - W$, and observe that a point of $\langle t_1, t_2 \rangle - \{t_1, t_2\}$ is not in $T \cup T^\perp$.

LEMMA 9.2. *Suppose T, W are* t.i. *or* t.s. *subspaces with* $\dim T = i - 1$, $\dim W = i$, *and* $T \subset W$. *Then* $|T^\perp - W^\perp| = q^{2r-i-c}$, *where* $c \geqslant -1$ *depends on the type of V but not on* $r =$ rank (V) *or i, and is given in the following table.*

Type of V	$O^+(2n,q)$	$O(2n+1,q)$	$O^-(2n+2,q)$	$U(2n,q^{1/2})$	$U(2n-1,q^{1/2})$
c	-1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$

Proof. For $i = 1$, $|T^\perp - W^\perp| = |\Omega - W^\perp|$ is the number of points not perpendicular to the point W, and is easily computed. For $i \geqslant 2$, $T^\perp/T$ has rank $n - i + 1$ and the same type as V; each of its points outside $W^\perp/T$ corresponds to a coset (containing q^{i-1} points) of T outside $W^\perp$.

Throughout the rest of this section and the next, G will be assumed to act *antiflag transitively* on the geometry and *primitively on the set Ω of points*. Let $d + 1$ denote the rank of G.

LEMMA 9.3. *There is a chain of G_x-invariant subspaces* $0 = W_{-1}(x) \subset x = W_0(x) \subset W_1(x) \subset \cdots \subset W_d(x) = V$ *with the following properties*:

(i) $W_i(x)^\perp = W_{d-i-1}(x)$ *(whence, in particular, $W_i(x)$ is t.i. or t.s. if and only if* $i \leqslant \frac{1}{2}(d-1)$);

(ii) G_x *is transitive on* $W_i(x) - W_{i-1}(x)$ *for each i*;

(iii) $y \in W_{i-1}(x)$ *implies* $W_1(y) \subseteq W_i(x)$ *for* $i \geqslant 1$;

(iv) $W_i(x^g) = W_i(x)^g$ *for all i, x, g*;

(v) $W_1(x) \cap W_1(y)$ *is a hyperplane of $W_1(x)$ if* $y \in W_1(x) - \{x\}$ *and* $d \geqslant 4$.

Proof. Let L be a line on x fixed by some $P \in \mathrm{Syl}_p(G_x)$. For $y \in L - x$, all P_y-orbits on $V - x^\perp$ have length at least $q^{(2r-1-c)-1}$ by (9.2), so P_y is transitive on $y^\perp - L^\perp$ (again by (9.2)). Set $W_1(y) = \langle L^g \mid g \in G_y \rangle$. Then

$$W_1(y)^\perp = \bigcap \{(L^\perp)^g \mid g \in G_y\},$$

and G_y is transitive on $y^\perp - W_1(y)^\perp$. Define $W_1(y^g) = W_1(y)^g$ for all $g \in G$.

If $W_1(x)^\perp = x$, we are finished (and $d = 2$). So suppose $W_1(x)^\perp \neq x$. Then $W_1(x) \cup W_1(x)^\perp \neq x^\perp$, by (9.1). Since G_x is transitive on $x^\perp - W_1(x)^\perp$, it follows that $W_1(x) \subseteq W_1(x)^\perp$, that is, $W_1(x)$ is t.i. or t.s. Also, G_x is transitive on $W_1(x) - x$. (For, $W_1(x)$ is naturally isomorphic to the dual space of $V/W_1(x)^\perp$. Now G_x has two orbits on 1-spaces of $V/W_1(x)^\perp$, namely those in $x^\perp/W_1(x)^\perp$ and those not in $x^\perp/W_1(x)^\perp$; so it has two orbits on the points of $W_1(x)$, namely x and $W_1(x) - x$.) Consequently, $W_1(x) = \bigcup \{L^g \mid g \in G_x\}$.

Now proceed by induction, assuming that $i \leqslant \frac{1}{2}(d-1)$ and that t.i. or t.s. subspaces $W_0(x), W_1(x), \ldots, W_i(x)$ have been defined, subject to (ii)–(iv). Set $W_{d-j-1}(x) = W_j(x)^\perp$ for $0 \leqslant j \leqslant i$, and $m = \dim W_i(x)$. By (9.2), the P-orbits on $V - W_i(x)^\perp$ have length at least q^{2r-m+c}, and hence the P_y-orbits have length at least $q^{2r-m-c-1}$. We may assume that $m \neq r$, since otherwise we are finished. Again by (9.2), $2r - m + c - 1 > m - 1$. Thus, $W_i(y) \subseteq W_i(x)^\perp$ and $\langle W_i(x), W_i(y) \rangle$ is t.i. or t.s., where $W_i(x) \neq W_i(y)$ by primitivity. Since P_y acts on $W_i(y)^\perp - \langle W_i(x), W_i(y) \rangle^\perp$ with orbit lengths at least $q^{2r-(m+1)+c}$, (9.2) implies that $W_i(y)$ is a hyperplane of $\langle W_i(x), W_i(y) \rangle$ and P_y is transitive on $W_i(y)^\perp - \langle W_i(x), W_i(y) \rangle^\perp$.

Set $W_{i+1}(y) = \langle W_i(x)^g \mid g \in G_y \rangle$. Then G_y fixes $W_{i+1}(y)$ and is transitive on $W_i(y)^\perp - W_{i+1}(y)^\perp$. If $W_{i+1}(y) = W_i(y)^\perp$, the proof is ended. Otherwise, $W_{i+1}(y) \cup W_{i+1}(y)^\perp \neq W_i(y)^\perp$, by (9.1). As before, this implies that $W_{i+1}(y) \subseteq W_{i+1}(y)^\perp$ and G_y is transitive on $W_{i+1}(y) - W_i(y)$. This completes the inductive step.

Finally, (v) was proved in our argument (letting $i = 1$), since $m \neq r$ in that case.

Definition. The geometry $\mathscr{G}$ consists of the points of Ω, together with those lines joining x to points of $W_1(x)$ for all $x \in \Omega$. The point graph of $\mathscr{G}$ is Γ.

Lemma 9.4. (i) *Γ is metrically regular.*

(ii) $d \leqslant 4$.

(iii) *If V has type $O(2n+1, q)$, then the conclusions of Theorem IV hold.*

(iv) *If $d = 2$ then the conclusions of Theorem IV hold.*

Proof. (i) This follows from (9.3 iii).

(ii) If $d \geqslant 5$ then $W_2(x)$ is t.i. or t.s., and hence (3.1) yields a contradiction.

(iii) Note that $W_2(x)$ is either $x^\perp$ or t.s., and hence $|W_2(x)| = (q^h - 1)/(q - 1)$ for some h. If $d = 2$ then G has rank 3 on points, and Kantor–Liebler [14, (1.3)] applies, since q is odd. If $d = 3$ then (3.1) and (3.2) show that $\mathscr{G}$ is the generalized hexagon associated with $G_2(q)$, embedded naturally in V of type $O(7, q)$. Then $G \trianglerighteq G_2(q)$ as in Section 7.

(iv) Again, Perin [19] and Kantor–Liebler [14] apply.

Notation. $e_2 = e$ and $f_1 = f$ are defined as in Section 3; $W(x) = W_1(x)$, and $m = \dim W(x)$.

LEMMA 9.5. *$d = 4$ is impossible.*

Proof. If $d = 4$ then the chain in (9.3) is

$$0 \subset x \subset W(x) \subset W(x)^\perp \subset x^\perp \subset V,$$

the differences being orbits of G_x. By (9.3 v), $f = m - 1$. Let N_{r-m} denote the number of points of $W(x)^\perp/W(x)$. Then $|W(x)^\perp - W(x)| = q^m N_{r-m}$, as in the proof of (9.2). As in Section 3, a count of pairs (y, z) with $d(x, y) = d(y, z) = 1$, $d(x, z) = 2$, yields

$$(q^m - q)(q^m - q^{m-1}) = q^m N_{r-m}(q - 1)(q^e - 1).$$

Thus, e divides $m - 1$.

Since $W(x) \neq W(y)$ for $x \neq y$, $W(x)$ is not a clique. Let $y, z \in W(x)$ be non-adjacent points. Then

$$\begin{aligned} e &= \dim W(y) \cap W(z) \\ &\geqslant \dim W(x) \cap W(y) \cap W(z) \\ &\geqslant m - 2, \end{aligned}$$

since $W(x) \cap W(y)$ and $W(x) \cap W(z)$ are hyperplanes of $W(x)$. Now $N_{r-m} \neq 1$ and $e \mid m - 1$ force $m \leqslant 3$. Clearly $m > 2$, since Γ is connected. Thus, $m = 3$, and $N_{r-m} = (q^2 - 1)/(q^e - 1)$. Then $e = 1$ and $|W_2(x)| = |W_1(x)| + q^m N_{r-m} = (q^5 - 1)/(q - 1)$, in contradiction to (3.1).

10. THE CASE $d = 3$

In this section we continue the proof of Theorem IV in the primitive case. By Section 9 we may assume that $d = 3$ and V is not of type $O(2r + 1, q)$. The chain of subspaces in (9.3) is now

$$0 \subset x \subset W(x) \subset x^\perp \subset V,$$

with $W(x)$ maximal t.i. or t.s. Set $k = |x^\perp - x|$ and $v_i = (q^i - 1)/(q - 1)$.

LEMMA 10.1. *V has type $O^+(2r, q)$ with $r = 4$, 5 or 6, while $f = r - 2$ and $e = 2$.*

Proof. As usual, count the pairs (y, z) with $d(x, y) = d(y, z) = 1$, $d(x, z) = 2$, this time obtaining

$$(v_r - 1)(v_r - v_f) = (k - (v_r - 1))v_e .$$

In particular, $k \leqslant (v_r - 1)(v_r - v_f - 1) \leqslant v_r(v_r - 1)$. However, k is easily computed for each type, and the types $O^-(2r + 2, q)$ and $U(2r + 1, q^{1/2})$ fail to satisfy this inequality. Moreover, in the case $U(2r, q^{1/2})$, we have $k = (q^r - 1)(q^{r-1/2} + 1)/(q - 1)$, whence

$$v_r - v_f = q^{r-3/2}v_e .$$

and $f = r - 3/2$, which is absured.

Thus, V has type $O^+(2r, q)$. This time,

$$k - v_r + 1 = q^{r-1}(q^{r-1} - 1)/(q - 1) = q^{r-2}(v_r - 1),$$

so

$$q^{r-2}v_e = v_r - v_f ,$$

whence $f = r - 2$, $e = r - f = 2$. By definition, $f \geqslant 2$, so $r \geqslant 4$.

Let y, z be nonadjacent vertices in $W(x)$. Then

$$\begin{aligned} 2 = e &\geqslant \dim W(x) \cap W(y) \cap W(z) \\ &\geqslant 2(r - 2) - r \qquad (*) \\ &= r - 4, \end{aligned}$$

whence $r \leqslant 6$, as required.

LEMMA 10.2. $r = 4$.

Proof. Suppose $r = 5$ or 6. Call the span of three noncollinear but pairwise adjacent points a *special plane*; note that all lines of a special plane belong to $\mathscr{G}$. If $y \in W(x) - x$, then

$$| W(x) \cap W(y) - \langle x, y\rangle| = (q^f - q^2)/(q - 1),$$

so $\langle x, y\rangle$ lies in exactly $(q^{f-2}-1)/(q-1)$ special planes. If $f = r-2 = 3$, this number is 1, so the number of special planes is

$$v_5(q^4+1)\,v_4\cdot 1/(q^2+q+1)(q+1),$$

which is not an integer. So $r = 6$.

In this case, we will show that the $\mathscr{G}$-lines and special planes which pass through x form a generalized pentagon with parameters q, q, contradicting the Feit–Higman Theorem (2.3).

Any special plane through x contains $q+1$ $\mathscr{G}$-lines through x, and any such $\mathscr{G}$-line lies in $(q^{f-2}-1)/(q-1) = q+1$ special planes. If xy and xz are lines through x not contained in a special plane, tightness in the inequalities $(*)$ shows that $W(x)\cap W(y)\cap W(z)$ is a line through x, the unique such line lying in special planes with both xy and xz. This yields the generalized pentagon and the desired contradiction.

There are several ways to handle the case $r = 4$. One is to show that $\mathscr{G}$ is a dual polar space (of type $O(7, q)$) in the sense of Cameron [3]; another is to quote transitivity results in Kantor–Liebler [14]. The method used here involves triality, a concept which we now briefly discuss. (Triality is used since not just G, but also its embedding, must be determined for Theorem IV.) We refer to [22] for further discussion of triality.

Let $\mathscr{P}$ be the set of points of the geometry of type $O^+(8, q)$, $\mathscr{L}$ the set of lines, and $\mathscr{M}_1$ and $\mathscr{M}_2$ the two families of *solids* (maximal t.s. subspaces); thus, any plane lies in a unique member of each family. More generally, two solids lie in the same family if and only if their intersection has even dimension. The geometry admits a "triality automorphism" τ mapping $\mathscr{L}\to\mathscr{L}$ and $\mathscr{P}\to\mathscr{M}_1\to\mathscr{M}_2\to\mathscr{P}$ and preserving the natural incidence between $\mathscr{P}\cup\mathscr{M}_1\cup\mathscr{M}_2$ and $\mathscr{L}$ (defined by inclusion or reverse inclusion). Also, τ preserves the "incidence" on $\mathscr{P}\cup\mathscr{M}_1\cup\mathscr{M}_2$, in which a solid is incident with a point contained in it, and two solids are incident if they meet in a plane. This automorphism induces an automorphism of $P\Omega^+(8, q)$.

Before continuing with the proof, we outline the way in which the examples of Theorem IV (iii) arise. Let v be a nonsingular "point", so that $v^\perp\cap\mathscr{P}$ carries a geometry of type $O(7, q)$. If $M_i\in\mathscr{M}_i$ $(i = 1, 2)$, then $v^\perp\cap M_i$ is a plane, contained in a unique member M_i^* of $\mathscr{M}_{3-i}$; thus v induces bijections between $\mathscr{M}_1$, $\mathscr{M}_2$, and the set of planes (maximal t.s. subspaces) of $v^\perp\cap\mathscr{P}$. These bijections are invariant under $G = \Omega^+(8, q)_v$, which acts transitively on each set. Now apply triality: G^τ is an irreducible subgroup of $\Omega^+(8, q)$, transitive on $\mathscr{P}$, and preserving a "geometry" on $\mathscr{P}$ isomorphic to the dual polar space of t.s. planes of $v^\perp\cap\mathscr{P}$. (Strictly, in place of G^τ, we use the inverse image in $\Omega^+(8, q)$ of $(G/Z)^\tau$, where $Z = Z(\Omega^+(8, q))$.) G is transitive on disjoint pairs of planes of $v^\perp\cap\mathscr{P}$, and hence on disjoint pairs of elements of $\mathscr{M}_2$; hence G^τ is transitive on nonperpendicular members of $\mathscr{P}$, that is, antiflag transitive. Note that G^τ and

$G^{\tau^{-1}}$ lie in different conjugacy classes in $\Omega^-(8, q)$. Note also that $G^\tau = \Omega(7, q)$ only if q is even; for q odd, G^τ contains the element $-1 \in \Omega^+(8, q)$.

The process can be continued one further time. If w is a nonsingular vector, then $G_x{}^\tau$ acts transitively (and even antiflag transitively) on $w^\tau \cap \mathscr{P}$, preserving a geometry which is the $G_2(q)$ hexagon, naturally embedded.

We return to the proof. There are $(q^4 - 1)(q^3 + 1)/(q - 1) = (q + 1)(q^2 + 1)(q^3 + 1)$ points, and equally many subspaces $W(x)$. Since $f = e = 2$, $\dim W(x) \cap W(y) = 2$ or 0 for $x \neq y$, and so all subspaces $W(x)$ belong to the same family; without loss of generality, $\{W(x) \mid x \in \mathscr{P}\} = \mathscr{M}_1$.

Now take $M \in \mathscr{M}_2$. If L is a $\mathscr{G}$-line in M and x a point of M not on L, then $\langle x, L\rangle$ is contained in a unique member $W(y)$ of $\mathscr{M}_1$, and $M \cap W(y) = \langle x, L\rangle$. Since $\mathscr{G}$ has no triangles, we have $y \in L$, and $\langle x, y\rangle$ is a $\mathscr{G}$-line. Thus, the $\mathscr{G}$-lines in M form a (possibly degenerate) generalized quadrangle. Call M *special* if this quadrangle is nondegenerate. If M is special, then the $\mathscr{G}$-lines in M are the absolute lines of a symplectic polarity $x \leftrightarrow M \cap W(x)$; so the quadrangle is of type $Sp(4, q)$.

Let Λ be the set of special solids, and form Λ^τ. This is a set of points. We claim that, *for any solid W, $W \cap \Lambda^\tau$ is a plane.* This follows from the assertion that, for any $x \in \mathscr{P}$, a special solid contains x if and only if it meets $W(x)$ in a plane. (If $E = M_1 \cap M_2$ is a plane, with $M_1 \in \mathscr{M}_1$, $M_2 \in \mathscr{M}_2$, then $E^{\tau^{-1}}$ is the set of members of $\mathscr{M}_2$ containing the point $M_1^{\tau^{-1}}$ and meeting the solid $M_2^{\tau^{-1}}$ in a plane.)

The following result now identifies Λ^τ (and hence Λ).

THEOREM 10.3. *Let Φ be a subset of Ω, the point set of a geometry of type $O^-(2r, q)$, $r \geqslant 3$. Suppose that, for every t.s. r-space U of Ω, $U \cap \Phi$ is an $(r - 1)$-space. Then* $\dim\langle\Phi\rangle = r - 1$, *and so $\Phi = \Omega \cap v^\perp$ for some nonsingular vector v.*

Proof. We treat first the case $r = 3$. Identify Ω (the Klein quadric) with the set of all lines of $PG(3, q)$. Then a plane of Ω is either the set of lines on a point or the set of lines in a plane; and a line of Ω is the set of lines in a plane E and on a point $x \in E$. Thus, under this identification, Φ is a set of lines of $PG(3, q)$ having the property that the members of Φ on a point x all lie in a plane E, while those in a plane E all contain a point x. Then $x \leftrightarrow E$ is a symplectic polarity, and Φ its set of absolute lines. Now a symplectic polarity of $PG(3, q)$ can be identified with a point v outside the Klein quadric Ω, its absolute lines corresponding to points of $v^\perp \cap \Omega$.

For $r > 3$, use induction on r. Take two nonadjacent points x, y of Φ. Then $\Omega \cap \langle x, y\rangle^\perp = \Omega'$ is of type $O^+(2r - 2, q)$. We claim that $\Phi \cap \langle x, y\rangle^\perp = \Phi'$ satisfies the conditions of the theorem in Ω' (with $r - 1$ replacing r). If U is a t.s. $(r - 1)$-space in Ω', then $\langle x, U\rangle$ is a t.s. r-space, and $\langle x, U\rangle \cap \Phi$ an $(r - 1)$-

space containing x; so $U \cap \Phi = U \cap \Phi'$ is an $(r-2)$-space. By induction, $\dim\langle \Phi \cap \langle x, y\rangle^{\perp}\rangle = 2r - 3$.

Now if $u \in \Phi \cap x^{\perp}$, then the line $\langle x, u\rangle$ contains a unique point of Φ perpendicular to y, so $\Phi \cap x^{\perp} \subseteq \langle x, \Phi \cap \langle x, y\rangle^{\perp}\rangle$, and $\dim\langle \Phi \cap x^{\perp}\rangle = 2r - 2$.

We claim that $\Phi \subseteq \langle x, y, \Phi \cap \langle x, y\rangle^{\perp}\rangle$; from this $\dim\langle\Phi\rangle = 2r - 1$ follows. Choose $z \in \Phi$; we may suppose $z \notin x^{\perp}$, $z \notin y^{\perp}$. Then $(\Phi \cap \langle x, y\rangle^{\perp}) \cap z^{\perp}$ spans a space of dimension $2r - 4$. Choose $w \in \Phi \cap \langle x, y, z\rangle^{\perp}$. Now $\langle \Phi \cap w^{\perp}\rangle$ contains x and y and meets $\langle \Phi \cap \langle x, y\rangle^{\perp}\rangle$ in a subspace of dimension at least $2r - 4$; since $\dim\langle \Phi \cap w^{\perp}\rangle = 2r - 2$, we have

$$z \in \Phi \cap w^{\perp} \subseteq \langle x, y, \Phi \cap \langle x, y\rangle^{\perp}\rangle.$$

Remark. The theorem fails if $r = 2$, $q > 3$: Ω is a ruled quadric (a $(q+1) \times (q+1)$ square lattice), and there are $(q+1)!$ sets Φ satisfying the hypothesis of (10.3), only $(q+1)q(q-1)$ of which are conics.

Completion of the proof of Theorem IV. It remains to identify G. Let H be the group induced by G^{τ} on the $0(7, q)$ geometry Λ^{τ}. Then H is transitive and has rank 4 on the set of planes contained in Λ^{τ}.

If E is a plane, then H_E is transitive on the q^6 planes disjoint from E. Since any point outside E lies on q^3 such planes, every point-orbit outside E of a Sylow p-subgroup P of H_E has length divisible by q^3. Let L be a line of E fixed by P. Since L only lies in q planes $E' \neq E$, it follows that $P_{E'}$ is transitive on $E' - E$. Also, H is transitive on the pairs (E, E') of planes for which $E \cap E'$ is a line. Thus, $H_E{}^E$ is antiflag transitive. By (2.1), $H_E{}^E$ is 2-transitive.

If x is any point of E, then $C_H(x)_E$ is transitive on E/x. Thus, $C_H(x)$ is transitive on $x^{\perp}/x$.

Since H_E is transitive on the q^6 planes disjoint from E, we have $q^6 \mid |H|$. Let Q denote the centralizer of both x and $x^{\perp}/x$ in $\Omega(7, q)$. Then $H \cap Q \neq 1$ since $q^5 \nmid |H_x^{x^{\perp}/x}|$. But Q is elementary abelian of order q^5, and is $C_H(x)$-isomorphic to $x^{\perp}/x$. Then $C_H(x)$ acts irreducibly on Q, and hence $H \cap Q = Q$. If $h \in H$ and $x^h \notin x^{\perp}$, then $H \supseteq \langle Q, Q^h\rangle = \Omega(7, q)$.

This completes the primitive case of Theorem IV.

11. The Imprimitive Case

Throughout this section (which corresponds roughly to Section 7), G satisfies the hypotheses of Theorem IV and is imprimitive on points. We are assuming that V has rank $r \geqslant 3$; however, we will need the case $G \leqslant \Gamma U(4, q)$ in our proof (cf. Remark 3 in Section 14).

Let Δ be a proper block of imprimitivity for G. Then $G_\Delta{}^\Delta$ is transitive, while $G_x = G_{x\Delta}$ is transitive on $V - x^{\perp}$ for $x \in \Delta$. Thus, $\Delta \subseteq \Delta^{\perp}$ (since the relation of non-orthogonality of points is connected), and G_Δ is transitive on $V - \Delta^\perp$. Then $\langle\Delta\rangle$ is t.i. or t.s., and (by the duality between $V/\langle\Delta\rangle^\perp$ and $\langle\Delta\rangle$) G_Δ is transitive on $\langle\Delta\rangle$. Thus, $\Delta = \langle\Delta\rangle$ is a t.i. or t.s. subspace.

From now on, Δ will be a minimal proper block of imprimitivity. Set $\delta = \dim \Delta$, and choose $x \in \Delta$. There are $|\Delta^G|\,(q^\delta - 1)/(q - 1)$ points. Counting the pairs (Δ_1, y_1) with $\Delta_1 \in \Delta^G \cap y_1^\perp$, we also find $|\Delta^G|\,|\Delta^\perp|/|\Delta^G \cap x^\perp|$ to be the number of points. (Here, as before, $\Delta^G \cap x^\perp$ is the set of members of Δ^G contained in $x^\perp$.) Thus, $|\Delta^\perp| = |\Delta^G \cap x^\perp|(q^\delta - 1)/(q - 1)$.

Lemma 11.1. *$\Delta^G \cap x^\perp = \Delta^G \cap \Delta^\perp$ partitions $\Delta^\perp$.*

Proof. Suppose $\Delta' \in \Delta^G$ and $\Delta' \cap \Delta^\perp \neq 0, \Delta'$. Pick $y \in \Delta$ with $\Delta' \not\subseteq y^\perp$. Then $|\Delta' - y^\perp| = q^{\delta-1}$, so

$$|\Delta^\perp - \Delta| \geqslant |\Delta^G - (\Delta^G \cap y^\perp)| = |V - y^\perp|/q^{\delta-1}.$$

However, a check of each classical geometry (computing $|\Delta^\perp - \Delta|$ as in (9.2)) shows that this inequality never holds.

Thus, $\Delta' \in \Delta^G$ and $\Delta' \cap \Delta^\perp \neq 0$ imply that $\Delta' \subseteq \Delta^\perp$. Then $\Delta^\perp$ is partitioned by $\Delta^G \cap \Delta^\perp$. Since $|\Delta^G \cap x^\perp| = |\Delta^\perp|/|\Delta|$, the result follows.

Corollary 11.2. *If W is an intersection of subspaces $(\Delta^g)^\perp$, then W is partitioned by $W \cap \Delta^G$.*

Lemma 11.3. *There is a subspace $\Delta' \in \Delta^G \cap \Delta^\perp$, $\Delta' \neq \Delta$.*

Proof. Suppose not. By (11.1), $\Delta^\perp - \Delta = \emptyset$, so Δ is a maximal t.i. or t.s. subspace. Note that $|\Delta^G| = 1 + q^\epsilon$ where $\epsilon = r - c = \delta + c$, and c is as in (9.2) (so $-1 \leqslant c \leqslant 1$).

Choose any $\Delta' \in \Delta^G - \{\Delta\}$. Let r be a primitive divisor of $q^\delta - 1$ (if one exists; cf. (2.4)), and $R \in \mathrm{Syl}_r(G_{\Delta\Delta'})$. Then R acts contragrediently on Δ and Δ'. Since $\delta > 2$, these actions are not isomorphic; so R can fix no further member of Δ^G. Thus r divides $(q^\epsilon + 1) - 2$, whence $\epsilon = \delta$, and V is of type $O(2\delta + 1, q)$, q odd. If $q^\delta = 2^6$, the same argument applies with $r = 7$.

Since q is odd, Theorems I and II imply that $G^{\Delta}_{\Delta\Delta'}$ contains $Sp(\delta, q)$. Let $g \in G_{\Delta\Delta'}$ be a p-element inducing a transvection on Δ. Then g also induces a transvection on Δ'. Thus, g centralizes a $2(\delta - 1)$-space of $\langle\Delta, \Delta'\rangle$, as well as $\langle\Delta, \Delta'\rangle^\perp$. Since $(2\delta - 1) + \delta > 2\delta + 1$, each member Δ'' of Δ^G meets $C_V(g)$ nontrivially. But Δ'' is a block of imprimitivity, and hence must be fixed by g. Let H denote the subgroup of $G_{\Delta\Delta'}$ generated by all such elements g; thus, H^Δ is $Sp(\delta, q)$ or $SL(\delta, q)$. If $\Delta'' \not\subseteq \langle\Delta, \Delta'\rangle$, then H must act on Δ'', and hence also on the $(\delta - 1)$-space $\Delta'' \cap \langle\Delta, \Delta'\rangle$, which is absurd.

Digression. It is convenient to show at this point that the case $\delta = 2$, $G \leqslant \Gamma U(4, q^{1/2})$ cannot occur. Note that (11.1) and the remarks preceeding it apply, and that $|\Delta^G| = q^{3/2} + 1$. Then $G_{\Delta\Delta'}^{\Delta} \geqslant SL(2, q)$ (if $q \neq 4$), while $q < |E| \leqslant q^{3/2}$. An element g of $G_{\Delta\Delta'}$ whose order is a primitive divisor of $q^2 - 1$ will thus centralize E. (Recall that q is a square here.) But, if X denotes $\Gamma U(4, q^{1/2})$, then $|C_X(\Delta)| = q^2$ and g acts fixed-point-freely on $C_X(\Delta)$; so $E = 1$, a contradiction. A similar argument applies if $q = 4$.

LEMMA 11.4. *If $\Delta' \in \Delta^G$, $\Delta' \not\subseteq \Delta^\perp$, and $W = \langle \Delta, \Delta' \rangle$, then $W \cap \Delta^G$ partitions W.*

Proof. By (11.2), $W^\perp = \Delta^\perp \cap (\Delta')^\perp$ is partitioned by the members of Δ^G it contains. Since the case where $W^\perp$ is anisotropic (or, equivalently, Δ maximal t.i. or t.s.) has been excluded by (11.3), $W^\perp$ is spanned by these members of Δ^G, and (11.2) applies to $W^{\perp\perp}$. Since the bilinear form defining the geometry is nondegenerate (cf. remark at the opening of Part II), $W^{\perp\perp} = W$, concluding the proof. (Remark. It is necessary to exclude the case $G \leqslant \Gamma O(2m + 1, q)$, q even, here. The Lemma fails for $\Omega(m + 1, 4) \trianglelefteq G \leqslant \Gamma O(2m + 1^6 2)$, m even.)

LEMMA 11.5. *V is orthogonal and $\delta = 2$.*

Proof. Choose $\Delta' \not\subseteq \Delta^\perp$, $\Delta' \in \Delta^G$. Then $\Delta' \cap \Delta^\perp = 0$, by (11.1), so $W = \langle \Delta, \Delta' \rangle$ is nonsingular. If $y_1, y_2 \in W - x^\perp$, and $y_i \in \Delta_i \in \Delta^G$ $(i = 1, 2)$, then an element of G_x mapping y_1 to y_2 also maps Δ_1 to Δ_2 and so fixes W, since $W = \langle \Delta, \Delta_i \rangle$ $(i = 1, 2)$ by (11.4). So $G_W{}^W$ is antiflag transitive and imprimitive; and so (11.3) and the subsequent degression give the result.

Remark. The $O^+(4, q)$ geometry is a ruled quadric, and $\Omega^+(4, q)$ has two natural systems of imprimitivity.

DEFINITION. Let $\mathscr{L}$ be the lattice of all t.s. subspaces which are intersections of members of $(\Delta^\perp)^G$.

LEMMA 11.6. *$\mathscr{L}$ is the lattice of all* t.i. *or* t.s. *subspaces of a classical geometry of type $U(n/2, q)$ (over $GF(q^2)$).*

Proof. By (11.2), each member of $\mathscr{L}$ is partitioned by the members of Δ^G it contains. If M is a maximal member of $\mathscr{L}$, then M is a maximal t.s. subspace. (For, if $x \in M^\perp - M$, then the member of Δ^G containing x would be in $M^\perp$, by (11.1).) If $r = \dim M > 4$, then M is a projective space with $q^2 + 1$ points per line, exactly as in (7.2). If $\Delta \not\subseteq M$ then $\Delta^\perp \cap M = \langle \Delta, M^\perp \rangle^\perp$ has dimension $n - (2 + n - r) = r - 2$, so $\Delta^\perp \cap M$ is a hyperplane of our new projective space M.

Note that an $(r - 2)$-space N in $\mathscr{L}$ lies in at least two maximal members of $\mathscr{L}$, since $|N^\perp| > (q^r - 1)/(q - 1)$. Now, if M and M' are r-spaces in $\mathscr{L}$ with nonzero intersection, let N be an $(r-2)$-space in $\mathscr{L}$ with $N \subseteq M'$, $N \cap M \subset M' \cap M$. There is a subspace $U \in \mathscr{L}$ with $N = (M \cap N) \oplus U$, $M' = (M \cap M') \oplus U$; and so $U^\perp \cap M = M' \cap M$. Now, if M'' is another r-space in $\mathscr{L}$ containing N, then $N \cap M \subseteq M'' \cap M \subseteq U^\perp \cap M = M' \cap M$, so $M'' \cap M = N \cap M \subset M' \cap M$. Continuing, we find that there exist disjoint r-spaces in $\mathscr{L}$.

It follows from Tits [23] that $\mathscr{L}$ is a classical polar space.

Now if M and M' are disjoint maximal subspaces of $\mathscr{L}$ and $\langle M, M'\rangle \neq V$, then there is a member of Δ^G disjoint from $\langle M, M'\rangle$. So $n = \dim V = 2r$ or $2r+2$. If $n = 2r-2$ then V has type $O^-(2r+2, q)$; then $|\Delta^G| = (q^r - 1)(q^{r-1}+1)/(q^2-1)$, and $\mathscr{L}$ is of type $U(r+1, q)$. Similarly, if $n = 2r$, then V has type $O^-(2r, q)$, and the same argument shows $\mathscr{L}$ has type $U(r, q)$.

Next suppose that $r = 4$. Then $\mathscr{L}$ is the lattice of points and lines of a geometry $\mathscr{G}$. Arguing as above, we find that $\mathscr{G}$ is a generalized quadrangle with parameters $s = q^2$, $t = q$ or q^3 according as V has type $O^+(8, q)$ or $O^-(10, q)$.

If $\Delta \not\subseteq \Delta'^\perp$, then $|\langle \Delta, \Delta'\rangle \cap \Delta^G| = q+1$ by (11.4), and for any $\Delta'' \in \langle \Delta, \Delta'\rangle \cap \Delta^G$, $(\Delta'')^\perp \supseteq \Delta^\perp \cap (\Delta')^\perp$. Thus a theorem of Thas [21] identifies the quadrangle with $t = q^3$ as that of type $U(5, q)$.

In the case $t = q$, the points and lines of the quadrangle are certain lines and solids of the $O^+(8, q)$ geometry. Any two of the solids are disjoint or meet in a line, and so they all belong to the same class. Applying the triality map (Section 10), the dual quadrangle is embedded as a set of points and lines in $O^-(8, q)$, satisfying the hypotheses of Buekenhout–Lefèvre [1]. Thus the dual of $\mathscr{L}$ is of type $O^-(6, q)$, and $\mathscr{L}$ itself of type $U(4, q)$.

This proves (11.6).

We can now complete the proof of Theorem IV as follows. By (11.6), $\mathscr{L}$ is uniquely embeddable in a projective space derived from a vector space $V(n/2, q^2)$. Proceeding as in Section 7, we obtain the original space V by restricting the scalars, and repeat the arguments of that section to show that $q = 2$ and that G is primitive and antiflag transitive on the $U(n/2, q)$ geometry. Now by Section 10, $G \geq SU(n/2, q)$, as required.

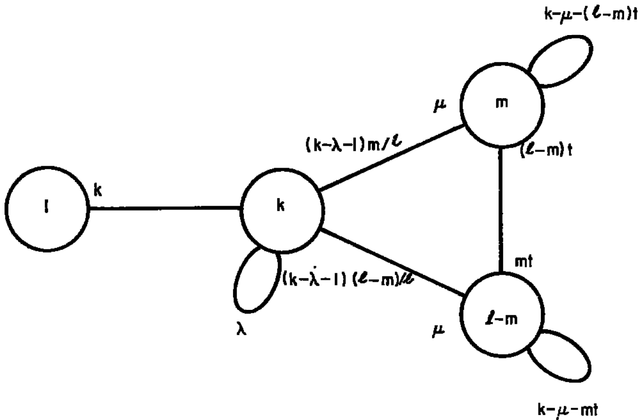
12. Rank 4 Subgroups of Rank 3 Groups

In this section, G will denote a primitive rank 3 permutation group on a set X, and H a subgroup of G having rank 4 on X.

Let k, l, λ, μ be the usual parameters of G, as defined in Higman [9], and let I, A, B be the adjacency matrices corresponding to the orbits $\{x\}$, $\Delta(x)$ and $\Gamma(x)$ of G_x, $x \in X$. If k, r, s are the eigenvalues of A, then $\lambda = k + r + s + rs$, $\mu = k + rs$, $k(k - \lambda - 1) = l\mu$.

We assume that H_x splits $\Gamma(x)$ into two orbits $\Gamma_1(x)$ and $\Gamma_2(x)$, of lengths

m, $l - m$ and with adjacency matrices C, $B - C$ respectively. Set $mt = |\Gamma_1(x) \cap \Delta(y)|$ for $y \in \Gamma_2(x)$. Then, with respect to the Δ-graph, the intersection numbers for H are as in the following diagram.



Then $AC = (k - \lambda - 1)(m/l)A + (k - \mu - (l - m)t)C + mt(B - C)$. applying this to an eigenvector of A and C with eigenvalues r, θ, respectively, yields

$$r\theta = (k - \lambda - 1)(m/l)r + (k - \mu - (l - m)t)\theta + mt(-r - 1 - \theta).$$

(Since $A + B + I$ is the all $- 1$ matrix, $-r - 1$ is an eigenvalue of B.) simplifying,

$$(r(s + 1) + lt)\theta = -(m/l)(r + 1)(r(s + 1) + lt).$$

Similarly, if φ is an eigenvalue of C corresponding to the eigenvalue s of A,

$$(s(r - 1) + lt)\varphi = -(m/l)(s + 1)(s(r + 1) + lt).$$

But the centralizer algebra of H has dimension 4, so exactly one of the eigenspaces of A must split into two eigenspaces for C. If this corresponds to r, then θ is not unique, so

$$r(s + 1) + lt = 0. \qquad (12.1)$$

Since $r \neq s$, it follows that

$$\varphi = -m(s + 1)/l \qquad (12.2)$$

But φ must be an integer, so

$$l/(l, s + 1) \text{ divides } m. \qquad (12.3)$$

Also, for $y \in \Gamma_1(x)$, $|\Delta(x) \cap \Gamma_2(y)| = m(l-m)t/k$, so

$$kl \text{ divides } m(l-m)r(s+1). \tag{12.4}$$

Remarks. (1) The results of this section can be used to give an alternative proof of (10.2).

(2) Of course, the same results hold in a more general situation (involving association schemes).

13. Theorem V

The proof of Theorem V follows (and was inspired by) the pattern of Perin's Theorem [20] discussed in Section 8. Suppose what G satisfies the hypotheses of Theorem V. If G_x is transitive on the points outside $x^\perp$, then G is antiflag transitive, and Theorem IV applies. So we may assume that G_x is transitive on $x^\perp - x$, and splits $V - x^\perp$ into two orbits. We use the notation of the last section.

Suppose first that $G \leqslant \Gamma Sp(2m, q)$. If G has two orbits on the nonsingular 2-spaces containing x, then the stabilizer of any projective line (singular or not) acts 2-transitively on it. By (4.1), G is antiflag transitive, contrary to assumption. So G is transitive on the nonsingular 2-spaces containing x; and if W is one such, then $G_W{}^W$ has rank 3, with subdegrees $1, h, q-h$. As in (8.5), $(q, h) = (2, 1)$, $(3, 1)$, $(4, 2)$ or $(9, 3)$.

We have $k = q(q^{2m-2}-1)/(q-1)$, $l = q^{2m-1}$, $m = q^{2m-2}h$. Also $r, s = \pm q^{m-1} - 1$. By (12.4),

$$q^{2m}(q^{2m-2}-1)/(q-1) \text{ divides } q^{4m-4}h(q-h)\, q^{m-1}(q^{m-1} \mp 1),$$

whence

$$q^{m-1} \mp 1 \text{ divides } (q-1)h(q-h).$$

This is impossible if $m \geqslant 4$; and none of the specific values of q and h satisfy it when $m = 3$. So this case cannot occur.

The case V unitary is ruled out by Kantor and Liebler [14, (6.2)].

Suppose $G \leqslant \Gamma O(2m+1, q)$, $m \geqslant 3$, q odd. Let r be a primitive division of $q^{2m-2} - 1$ (see (2.4)), and $R \in \mathrm{Syl}_r G_x$. Then $W = C_V(R)$ is a nonsingular 3-space, and $N_G(R)^W$ has rank 2 or 3. If $q > 3$ then $N_G(R)^W$ contains $\Omega(3, q)$ or (if $q = 9$) A_5. Proceeding as in [12], (compare the proof of (6.2)) we obtain the contradiction $G \geqslant \Omega(2m+1, q)$.

The case $q = 3$ is somewhat harder. Here, choose $r \mid 3^{2m-2} - 1$ or $r \mid 3^{m-1} - 1$ according as m is odd or even. (The case $m = 2$ is not difficult and is omitted.) Then $N_G(R)^W$ contains $\Omega(3, 3)$ or D_8, while $N_G(R)^{W^\perp}$ contains no D_8. (In fact, $N_G(R)^{W^\perp} \leqslant \Gamma O^{\pm}(2, 3^{m-1})$ by Sylow's theorem.) Thus, there is an involution $t \in C_G(W^\perp)$ with $t^W = \mathrm{diag}(-1, -1, 1)$. Now note that G has 2 orbits of pairs (x, b) with x a singular point and $b \in x^\perp$ a 1-space of length 1 (or, alternatively, length -1). If t induces -1 on b, then $t^{b^\perp}$ is a reflection while $G_b^{b^\perp}$ has at most 2 point-orbits. We may then assume that any two G_b-conjugates of t commute. (For otherwise, the product of two such non-commuting conjugates has a product of order 3 centralizing $y^\perp/y$ for some point $y \in b^\perp$. The argument in [14] or (6.2) now yields the contradiction $G \geq \Omega(2m+1, 3)$.) Since $G_b^{b^\perp}$ has at most 2 point-orbits, it cannot be an irreducible monomial group. It follows that G_b fixes an anisotropic 1-space or 2-space $T \subset b^\perp$, and is transitive on the points of $b^\perp \cap T^\perp$. Now one of the G_x-orbits of nonsingular 1-spaces in $x^\perp$ has length $\frac{1}{2}3^m(3^m \pm 1) \cdot \frac{1}{2}(3^{2m-2} - 1)/\frac{1}{2}(3^{2m} - 1)$ or $\frac{1}{2}3^m(3^m \pm 1) \cdot \frac{1}{2}(3^{m-1} - \epsilon)(3^{m-2} + \epsilon)/\frac{1}{2}(3^{2m} - 1)$ with $\epsilon = 1$ or -1. Since this is not an integer, we again obtain a contradiction.

Finally, consider the case $G \leqslant \Gamma O^{\pm}(2m, q)$, $m \geq 3$, $q > 2$, in which $x^\perp/x$ has $(q^{m-1} \mp 1)(q^{m-2} \pm 1)/(q-1)$ points. If $m = 3$, use [14, Section 5]. We therefore assume that $m \geqslant 4$, and use $r \mid q^{m-2} \pm 1$ and $R \in \mathrm{Syl}_r G_x$ as before, temporarily excluding the case $O^-(8, q)$ with q a Mersenne prime. This time $W = C_V(R)$ is a nonsingular 4-space, with $N_G(R)^W$ of rank 2 or 3. Moreover, since R fixes no points of $x^\perp/x$, necessarily W has type $O^-(4, q)$. Thus, $N_G(R)^W$ contains $\Omega^-(4, q)$ or (if $q = 3$) A_5. As in [14, Section 12], we obtain the contradiction $G \geqslant \Omega^{\pm}(2m, q)$.

This leaves the possibility $G \leqslant \Gamma O^-(8, q)$ with q Mersenne. We may assume that $-1 \in G$; note that $-1 \not\subset \Omega^-(8, q)$. Let L be a line, and let $R \in \mathrm{Syl}_2 C_G(L)$. Since $G_L{}^L \geqslant SL(2, q)$, it follows that $R \neq 1$. Set $W = C_V(R)$. Then $\dim W = 4$ or 6, while $N_G(R)^W$ is line-transitive. Also, $N_G(R)_L^L \geqslant SL(2, q)$ by the Frattini argument. If $\dim W = 6$ then $N_G(R)^W \geqslant \Omega^{\pm}(6, q)$ by [14, Section 5], and we can proceed as before. If $\dim W = 4$ then $N_G(R)^W \geqslant O^+(4, q)$ while $N_G(R)^{W^\perp}$ normalizes the fixed-point-free 2-group $R^{W^\perp}$. Then $C_G(W^\perp)^W$ contains $\Omega^+(4, q)$ if $q \neq 3$, and $G \geqslant \Omega^-(8, q)$. Suppose that $q = 3$. Then a Sylow 3-subgroup of $N_G(R)^{W^\perp}$ has order 1 or 3, so $C_G(W^\perp)$ has an element g or order 3. There is a point $x \in W$ fixed by g, and $g \in C_G(x^\perp/x)$. As usual, this implies that $G \geqslant \Omega^-(8, q)$.

This contradiction completes the proof of Theorem V.

Remark. If $G < O^{\pm}(2m, 2)$, the argument breaks down when $r \mid 2^{m-2} \pm 1$ and $\dim W = 4$, but $|N_G(R)^W| = 10$ or 20.

14. Concluding Remarks

1. The method used in our proofs for employing p-groups also works for suitable permutation representations of the exceptional Chevalley groups.

2. After classifying antiflag transitive groups, it is natural to ask about transitivity on incident point-hyperplane pairs (where the hyperplane is not the polar of the point in the case of a classical geometry). If a group G is transitive on all such pairs in $PG(n-1, q)$, then it is transitive on incident point-line pairs, and hence 2-transitive on points (Kantor [12]); so Theorem I applies. However, for classical geometries, results are known only in the unitary case Kantor–Liebler [14]).

3. In the proofs of (8.4)–(8.6) and Theorem V, as well as that of Perin's theorem outlined at the beginning of Section 6, results were employed which had been proven using group theoretic classification theorems. However, the proofs of Theorems I–IV are not dependent on such classification theorems. Even when [14] was invoked, the required results used nothing more than (2.4) and elementary generational properties of classical groups. Of course, proofs have occasionally been lengthened by the requirement that group theoretic classification theorems not be used even implicitly.

4. However, it should be noted that [14] produces a proof of the rank 2 analogue of Theorem IV, as follows. We assume that V does not have type $O^{+}(4, q)$. The primitive case proceeds as in Sections 9, 10. In the imprimitive case, the block Δ of Section 11 is a t.i. or t.s. line. If $x \in \Delta$ then G_x is transitive on $\Delta^G - \{\Delta\}$ and hence on $x^\perp - \Delta$. Thus, G has one orbit of points and two orbits each of lines and incident point-line pairs. Now [14, Sect. 5] applies.

APPENDIX: *The $G_2(q)$ Hexagon*

This appendix contains new and elementary proofs of the existence and uniqueness statements in Section 3, as well as further properties of the hexagons (including antiflag transitivity).

Assume that $\mathscr{G}$ is an in (3.2), and set $W(x) = W_1(x)$. We will prove several properties of $\mathscr{G}$, from which an explicit construction will easily follow.

LEMMA A.1. (i) *For any points x, y of $\mathscr{G}$,*

$$\langle x, y\rangle = \bigcap \{W_2(u) \mid u \in W_2(x) \cap W_2(y)\};$$

if $d(x, y) = 1$ or 2, then all points of $\langle x, y\rangle$ are points of $\mathscr{G}$.

(ii) *If $z \notin \langle x, y\rangle$ and $d(x, y) = 1$ or 2, then $W_2(z) \cap \langle x, y\rangle$ is either $\langle x, y\rangle$ or a point.*

(iii) *Either* dim $V = 6$ *and V is symplectic, or* dim $V = 7$ *and V is orthogonal*; *in either case, the points and lines of $\mathscr{G}$ consist of all points and certain t.i. or t.s. lines of V, and $W_2(x)$ consists of all points of $x^\perp$.*

Proof. (i) The first statement follows from axiom (f) of Section 3; the second from the fact that $\langle x, y\rangle \subseteq W(u)$ if $u \in W(x) \cap W(y)$.

(ii) If $d(x, y) = 1$, this follows from the axioms for a generalized hexagon. Suppose $d(x, y) = 2$, and set $u = W(x) \cap W(y)$. Then $W(u) \cap W_2(z)$ is a subspace meeting each line of $W(u)$ on u, and our assertion follows.

(iii) This follows easily from (ii) and the fact that $\mathscr{G}$ has exactly $(q^6 - 1)/(q - 1)$ points (cf. Yanushka [27, Sect. 3].)

LEMMA A.2. *Let a and b be opposite points, and set $H = \langle W(a), W(b) \rangle$.*

(i) *$H = E \oplus F$, where E and F are t.i. or t.s. planes such that, for $e \in E$, $f \in F$, $\langle e, f \rangle$ is a $\mathscr{G}$-line if and only if it is a (t.i. or t.s.) line (Call these $E \mid F$-lines.)*

(ii) *If $e \in E$, then $W(e) = \langle e, e^\perp \cap F \rangle$.*

(iii) *If x is a point on no $E \mid F$-line, then $W(x)$ meets exactly $q + 1$ $E \mid F$-lines, and the points of intersection are collinear.*

(iv) *If V has type $Sp(6, q)$, then q is even.*

Proof. (i) Since $W(a) \cap W(b) = 0$, $\dim H = 6$. Let $a = x_1, x_2, x_3$, $b = x_4, x_5, x_6$ be the vertices of an ordinary hexagon in $\mathscr{G}$. Then $x_2, x_6 \in W(a)$ and $x_3, x_5 \in W(b)$. Set $E = \langle x_2, x_4, x_6 \rangle$ and $F = \langle x_1, x_3, x_5 \rangle$. Then E and F are t.i. or t.s. and $H = E \oplus F$. Also $W(x_{2i}) = \langle x_{2i}, x_{2i}^\perp \cap F \rangle$. We can thus vary $x_2, x_6 \in W(a) \cap E$, and also move around the ordinary hexagon, in order to show that each t.i. or t.s. line $\langle e, f \rangle$ is a $\mathscr{G}$-line.

(ii) This is clear from the above proof. (In fact, the points of $E \cup F$ and the $E \mid F$-lines form a degenerate subhexagon with $s = 1$, $t = q$.)

(iii) If u lies in $W(x)$ and also on an $E \mid F$-line $\langle e, f \rangle$, then $e \in E \cap x^\perp$ and $f \in F \cap x^\perp$. Here, $E \cap x^\perp$ and $F \cap x^\perp$ are lines spanning a nonsingular 4-space U. If $e_1 \in E \cap x^\perp$ then U contains a $\mathscr{G}$-line $\langle e_1, f_1 \rangle$ met by $W(x)$. Thus, $W(x) \cap U$ is the desired set of points, and is clearly a line.

(iv) If V has type $Sp(6, q)$, then U has type $Sp(4, q)$. But the $Sp(4, q)$ quadrangle contains six lines forming a 3×3 grid (such as $E \cap x^\perp, F \cap x^\perp$, $W(x) \cap U$, and any three $E \mid F$-lines in U) if and only if q is even.

Remark. Because of (A.2iv), and the isomorphism between the $Sp(6, q)$ and $O(7, q)$ geometries when q is even, we will assume from now on that V has type $O(7, q)$. Now H has type $O^+(6, q)$, and the line mentioned in (iii) is $W(x) \cap H$. Also, $O(7, q) = SO(7, q) \times \{\pm 1\}$, so we may where necessary assume that linear automorphisms of $\mathscr{G}$ have determinant 1.

The next lemma is more technical, and concerns generating $\mathscr{G}$.

LEMMA A.3. *Let S be a set of points, containing at least one pair a, b of opposite points, and such that $W(a) \cap b^\perp \subseteq S$ for any such pair. Then either $S = E \cup F$ for E, F as in* (A.2i), *or S consists of all points of $\mathscr{G}$.*

Proof. Certainly $S \subseteq E \cup F$. Let $\mathscr{G}_0$ consist of S together with the set of lines meeting it at least twice. We will show that $\mathscr{G}_0$ is a (possibly degenerate) subhexagon.

Let L be a line of $\mathscr{G}_0$ and $x \in S - L$; we must show that the unique point u of L nearest x lies in S. Let $y \in L - u$. Since x is opposite some point of E, our hypothesis implies that each line on x meets $S - \{x\}$. If $d(x, u) = 1$, pick $z \in S \cap W(x)$ with $d(y, z) = 3$, and note that $u \in W(y) \cap z^-$, so $u \in S$. If $d(x, u) = 2$ then $d(x, y) = 3$ and $u \in W(y) \cap x^-$ is in S.

Thus, $\mathscr{G}_0$ is a subhexagon. Choose $a \in S$. Then $S \cap W(a)$ has the following properties: it meets every line on a at least twice; if $x, y \in S \cap W(a)$ and $W(a) = \langle a, x, y \rangle$, then $\langle x, y \rangle \subseteq S$. (For if $b \in x^\perp \cap y^- \cap S$ and b is opposite a, then $\langle x, y \rangle = W(a) \cap b^\perp$.) Thus, $S \cap W(a)$ is a (possibly degenerate) subplane of $W(a)$.

If each line of $\mathscr{G}_0$ has size 2, then $S = E \cup F$. So suppose that some line of $\mathscr{G}_0$ on a has at least three points. Then $S \cap W(a)$ is nondegenerate, and hence is all of $W(a)$. Thus $\mathscr{G} = \mathscr{G}_0$.

LEMMA A.4. *Suppose $\mathscr{G}$ and $\mathscr{G}'$ are both embedded in V as in Section 3. Let $x_1, \ldots, x_6$ and $y_1, \ldots, y_6$ be the vertices of ordinary hexagons in $\mathscr{G}$ resp. $\mathscr{G}'$. Then there is an element of $GL(V)$ mapping x_i to y_i $(i = 1, \ldots, 6)$ and inducing an isomorphism of $\mathscr{G}$ onto $\mathscr{G}'$.*

Proof. The orthogonal geometries determined by $\mathscr{G}$ and $\mathscr{G}'$ as in (A.1iii) are equivalent under $GL(V)$; so we may suppose that they are equal. There is an orthogonal transformation taking x_i to y_i $(i = 1, \ldots, 6)$, so we may assume that $x_i = y_i$ for each i. Set $E = \langle x_2, x_4, x_6 \rangle$, $F = \langle x_1, x_3, x_5 \rangle$. By (A.2ii), $W(e)$ and $W(f)$ are the same whether computed in $\mathscr{G}$ or $\mathscr{G}'$ (where $e \in E, f \in F$).

Pick a point x on no $E \mid F$-line, and call $W_H(x)$ the line of points in (A.2ii). Then $W_H(x)$ is one of the $q - 1$ lines in $U = \langle E \cap x^\perp, F \cap x^\perp \rangle$ meeting each $E \mid F$-line of U, other than $E \cap x^\perp$ and $F \cap x^\perp$. But $O(7, q)_{EFx}$ is transitive on these lines, so we may assume that $W(x)$ is the same in $\mathscr{G}$ and $\mathscr{G}'$ for some such x.

Now if S is the set of points u of V such that $W(u)$ is the same in both $\mathscr{G}$ and $\mathscr{G}'$, then (A.2ii) shows that (A.3) applies, and we conclude that $\mathscr{G} = \mathscr{G}'$.

COROLLARY A.5. *The group $\mathrm{Aut}_V(\mathscr{G})$ of automorphisms of $\mathscr{G}$ induced by elements of $SL(V)$ is transitive on the set of ordered ordinary hexagons of $\mathscr{G}$. In particular, $\mathrm{Aut}_V(\mathscr{G})$ is antiflag transitive.*

(Recall the remark following (A.2).)

COROLLARY A.6. (i) *There is a subgroup $K \cong SL(3, q)$ of $\mathrm{Aut}_V(\mathscr{G})$ fixing E and F and centralizing $H^\perp$.*

(ii) *The stabilizer of E in $\mathrm{Aut}_V(\mathscr{G})$ induces $SL(3, q)$ on it.*

(iii) $|\mathrm{Aut}_V(\mathscr{G})| = (q^6 + 1) q^6(q^2 - 1)$ *and* $Z(\mathrm{Aut}_V(\mathscr{G})) = 1$.

Proof. (i) is clear from (A.4).

(ii) We show first that $\mathrm{Aut}_V(\mathscr{G})_{EF}^E = SL(3, q)$. Set $G' = \mathrm{Aut}_V(\mathscr{G})_{EF}$. If x lies on no $E \mid F$-line, then $|G': G'_x|$ is equal to the index in $GL(3, q)$ of the stabilizer of a non-incident point-line pair (p, L); and no non-identity (p, L)-homology fixes x, since the group of (p, L) homologies permutes regularly the $q - 1$ lines called $W_H(x)$ in the proof of (A.4). So $(G'_x)^E$ has index at least $q - 1$ in $GL(3, q)_{pL}$; and hence $(G')^E = SL(3, q)$.

Now, given E, there are q^2 choices for F, permuted transitively by $\mathrm{Aut}_V(\mathscr{G})_E$. So $|\mathrm{Aut}_V(\mathscr{G})_E^E: \mathrm{Aut}_V(\mathscr{G})_{EF}^E|$ divides $(q^2, q - 1)$, whence $\mathrm{Aut}_V(\mathscr{G})_E^E = SL(3, q)$.

(iii) Clear.

THEOREM A.7. *Each $O(7, q)$ space has one and only one class of generalized hexagons embedded as in Section 3. An $Sp(6, q)$ space has such a hexagon if and only if q is even.*

Proof. Uniqueness follows from (A.4), and the assertion about $Sp(6, q)$ from (A.2iv). The preceding lemmas (especially (A.1), (A.2) and (A.6)) tell us exactly how $\mathscr{G}$ must look, and hence how to construct $\mathscr{G}$.

Construction. Let V be a vector space carrying a geometry of type $O(7, q)$, and E and F t.s. planes such that $H = \langle E, F\rangle$ is nonsingular of dimension 6. Let $K < O(7, q)$ fix E and F, centralize $H^\perp$, and induce $SL(3, q)$ on both E and F. If $\{e_1, e_2, e_3\}$ is a basis for E and $\{f_1, f_2, f_3\}$ the dual basis for F, then the matrices of g^E and g^F with respect to these bases are inverse transposes of one another for all $g \in K$. Let $H^\perp = \langle d\rangle$.

We must use the $E \mid F$-lines $\langle e, f\rangle$, with $e \in E$, $f \in e^\perp \cap F$, as $\mathscr{G}$-lines; set $W(e) = \langle e, e^\perp \cap F\rangle$, $W(f) = \langle f, f^\perp \cap E\rangle$. Note that K is transitive on the $(q^2 + q + 1)(q + 1)(q - 1)$ points on the union of the $E \mid F$ lines but not in $E \cup F$, on the $(q^2 + q + 1)(q^3 - q^2)$ points on no $E \mid F$-line, and on the $(q^2 + q + 1)(q^3 - q^2)$ lines contained in the union of the $E \mid F$-lines but not meeting E or F.

Pick an $E \mid F$-line $\langle e, f\rangle$, a point $u \in \langle e, f\rangle - \{e, f\}$, and a plane $W(u) \supset \langle e, f\rangle$ with $W(u) \neq W(e), W(f)$. Write $W(u^g) = W(u)^g$ for all $g \in K$. The new points must be the t.s. points of $V - H$, and the new $\mathscr{G}$-lines must be the lines of $W(u^g)$ through u^g, for all $g \in K$. We must show that this is well-defined and yields a generalized hexagon. This will be done in several steps.

(1) If $u^g = u$ then $W(u^g) = W(u)$; so $W(u^g)$ is well-defined. For, $|K_u| = q^3(q - 1)$, and K_u fixes $W(e)/\langle e, f\rangle$ and $W(f)/\langle e, f\rangle$. Thus, each p-element of K_u fixes every plane containing $\langle e, f\rangle$. Suppose the order of g divides $q - 1$. We may assume that $e = \langle e_1\rangle, f = \langle f_2\rangle, u = \langle e_1 + f_2\rangle$, in the above notation. Then we find that $e_1^g = \alpha e_1$, $f_2^g = \alpha f_2$, $e_2^g = \alpha^{-1} e_2$, $f_1^g = \alpha^{-1} f_1$, whence

$e_3^g = e_3$, $f_3^g = f_3$. Since $W(u)$ contains a unique point of $\langle e_3, f_3, d\rangle$, it is fixed by g.

(2) If $W(u^g) = W(u)$, then $u^g = u$. For suppose $W(u^g) = W(u)$. Then g fixes $W(u) \cap E = e$ and $W(u) \cap F = f$. Here, K_{ef} is the stabilizer of a flag of $PG(2, q)$, of order $q^3(q-1)^2$. If $u^g \neq u$, then the order of g divides $q-1$ and g is diagonalizable. We may assume $u = \langle e_1 - f_2\rangle$, $W(u) = \langle e_1, f_2, e_3 - f_3 + d\rangle$. If $g^E = \mathrm{diag}(\alpha, \beta, \gamma)$, then $(e_3 + f_3 + d)^g = \gamma e_3 - \gamma^{-1} f_3 + d$, so $\gamma = 1$, whence $\alpha\beta = 1$, and $u^g = u$.

(3) If L is a $\mathscr{G}$-line on u then $L \subset W(u)$. (For, we may assume $L \subseteq H$ and $L^g \subset W(u)$, so $u = L \cap H = u^g$ and $L \subset W(u)$.) The total number of $\mathscr{G}$-lines is then

$$(q^2 - q - 1)(q + 1) + (q^2 + q - 1)(q + 1)(q - 1)q = (q^6 - 1)/(q - 1).$$

Thus, each point $x \notin H$ lies on

$$(q^2 - q - 1)(q - 1)(q - 1)q \cdot q/(q^2 + q + 1)(q^3 - q^2) = q - 1$$

$\mathscr{G}$-lines.

(4) Fix $x \in V - H$. Then $K_x = SL(2, q)$ fixes the non-perpendicular points $E_x^\perp \cap F$ and $F_x^\perp \cap E$, where $E_x = E \cap x^\perp$ and $F_x = F \cap x^\perp$ are lines. Here, K_x acts on the nonsingular 4-space $\langle E_x, F_x\rangle$, fixing $q + 1$ lines. Each $\mathscr{G}$-line L on x meets one of these lines (since $W(L \cap H)$ is t.s. and contains x and points of E and F); but none meets E_x or F_x. The action of K_x then implies that they all meet the same line M, and they lie in a plane $W(x) = \langle x, M\rangle$.

By transitivity, each line of H missing $E \cup F$ occurs as $W(x) \cap H$ for some $x \in V - H$. Since the numbers of such x and such lines are the same, distinct points x yield distinct $W(x)$. It follows that, for any two distinct points a, b of V, we have $W(a) \neq W(b)$.

(5) Points a, b are perpendicular if and only if $d(a, b) \leqslant 2$. For, if $d(a, b) \leqslant 2$ then $a, b \in W(c)$ for some c, and $W(c)$ is t.s. But the number of such pairs is the same as the number of pairs of perpendicular points.

(6) $\mathscr{G}$ has no k-gons for $k \leqslant 5$. For, let $a_1, \ldots, a_k$ be the vertices of a k-gon. Then $d(a_i, a_j) \leqslant 2$ for all i, j so $\langle a_1, \ldots, a_k\rangle$ is a t.s. plane, which must be both $W(a_1)$ and $W(a_2)$, contradicting (4).

(7) $\mathscr{G}$ is a generalized hexagon. This follows from the same counting argument as in Section 3.

This completes the proof of (A.7).

Remarks. Further properties of the group $G_2(q) = \mathrm{Aut}_V(\mathscr{G})$ are found in (5.4). Additional information, such as simplicity when $q \neq 2$ and identification with $PSU(3, 3) \rtimes Z_2$ if $q = 2$, is left to the reader, and can be found in Tits [22].

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