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Some Results on 2-Transitive Groups

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§ 1. Introduction

A fundamental result of Tits [19] classifies all finite groups having an irreducible BN-pair of rank $\geqq 3$ and which is faithful in the sense that $\bigcap_{g \in G} B^g = 1$ (see [3] or [14] for the definition of a BN-pair). A group G is said to have a faithful split BN-pair of rank n if it has a faithful BN-pair of rank n, $H = \bigcap_{n \in N} B^n$, and $B = XH$ with $X \trianglelefteq B$ and $X \cap H = 1$. If in addition G is finite, X is a p-group and H is an abelian p'-group, following Richen [14] we say that G has a faithful split BN-pair at characteristic p. The main purpose of this paper is to prove two results on finite 2-transitive groups which can be applied to groups having a faithful split BN-pair at characteristic p of rank 1 or 2.

Recent results of Shult [16] and Hering, Kantor and Seitz [9] classify all finite groups having a faithful split BN-pair of rank 1. In Theorem A we handle the special case of this classification in which H is abelian. The proof of this theorem is much more elementary than those of [16] and [9], and consists of showing that H is cyclic, so that a result of Kantor, O'Nan and Seitz [11] applies. Our methods are a combination of ideas found in [9] and [11].

In Theorem C we classify all 2-transitive groups in which the stabilizer of a point has a normal nilpotent subgroup transitive on the remaining points. For groups of odd degree this amounts to a straightforward application of a result of Shult [15]. In the even degree case we reduce to the result of Hering, Kantor and Seitz [9]. Once again our approach is based in part on ideas in [9] and [11].

Theorem C has the following application to a group G having a faithful split BN-pair at characteristic p of rank 2. Let s be a fundamental reflection in the Weyl group N/H, and let $P = \langle B, B^s \rangle$ be the corresponding maximal parabolic subgroup. Then, in its natural 2-transitive representation on the cosets of B, P either has a split BN-pair at characteristic p of rank 1 or is the solvable 2-transitive group of degree 9 and order $9 \cdot 8 \cdot 2$. Thus, excluding this exceptional case, we must have $X \cap X^s \trianglelefteq X$.

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Moreover, a great deal is now known about the structure of these maximal parabolic subgroups. It is hoped that this information will eventually be useful in the complete classification of groups having a faithful split BN-pair at characteristic p of rank 2.

As a further application of Theorem C, we classify all finite 2-transitive groups G having an involution fixing just 2 points α and β and weakly closed in a Sylow 2-subgroup of $G_{\alpha\beta}$ (with respect to $G_{\alpha\beta}$).

Our notation is that of [9, 11] and, for the most part, [21]. Let G be a permutation group on a set Ω. Let X be a subset of G. Then $\Delta(X)$ is the set of fixed points of X and W_X is the pointwise stabilizer of $\Delta(X)$ in $N(X)$. Also, if $Y \subseteq N(X)$ then $Y^{\Delta(X)}$ is the set of permutations induced by Y on $\Delta(X)$.

Throughout our proofs we will use many known facts concerning the groups being characterized. These can be found in [11], § 2 and [9], § 3.

§ 2. Split BN-Pairs of Rank 1 at Characteristic p

The following result is a special case of [9] and [16].

Theorem A. *Let G be a permutation group 2-transitive on a finite set Ω. Let $\alpha, \beta \in \Omega$, $\alpha \neq \beta$. Suppose that G_α has a normal subgroup Q regular on $\Omega - \alpha$ and that $G_{\alpha\beta}$ is abelian. Then either G has a normal subgroup which is sharply 2-transitive on Ω or G is one of the following groups in its usual 2-transitive representation: $PSL(2, q)$, $PGL(2, q)$, $Sz(q)$, $PSU(3, q)$, $PGU(3, q)$, or a group of Ree type.*

Proof. We shall use induction on $|G|$ to show that $G_{\alpha\beta}$ is cyclic. Then by a result of Kantor, O'Nan and Seitz [11], either G has a regular normal subgroup N and QN is sharply 2-transitive, or G is one of the groups listed above.

Assume that $G_{\alpha\beta}$ is not cyclic. We may suppose that $G_{\alpha\beta}$ has even order (Bender [1], Suzuki [17]). Let T be the subgroup of $G_{\alpha\beta}$ generated by the involutions in $G_{\alpha\beta}$. If $|T|=2$ then we may assume that T fixes more than 2 points (Hering [8]).

Lemma 1. *Let X be a non-empty subset of $G_{\alpha\beta}$ fixing at least 3 points.*

(i) *$C_0(X) = \langle C_{Q^g}(X) \mid \alpha^g \in \Delta(X) \rangle$ is 2-transitive on $\Delta(X)$, and $|C_Q(X)| = |\Delta(X)| - 1$.*

(ii) *X is weakly closed in $G_{\alpha\beta}$.*

Proof. (i) If $\beta, \gamma \in \Delta(X) - \alpha$ let $\gamma = \beta^g$, $g \in Q$. For any $x \in X$, $\beta^{gx} = \beta^g = \beta^{xg}$, so that $[x, g] \in Q \cap G_{\alpha\beta} = 1$. Thus $C_Q(X)$ is transitive on $\Delta(X) - \alpha$, and (i) follows.

(ii) If $g \in G$ and $X^g \leq G_{\alpha\beta}$ then $\alpha, \beta, \alpha^g, \beta^g \in \Delta(X^g)$. By (i), $\alpha^{gh} = \alpha$ and $\beta^{gh} = \beta$ with $h \in C(X)$. Thus, $X^g = X^{gh} = X$ as $G_{\alpha\beta}$ is abelian.

Lemma 2. *$n=|\Omega|$ is even.*

Proof. Suppose that n is odd. Let S be a Sylow 2-subgroup of G_α such that S_β is a Sylow 2-subgroup of $G_{\alpha\beta}$. Then $S=(S\cap Q)\,S_\beta$. Also, $S'\leqq Q$ and $S_\beta^g\cap(S\cap Q)=1$ for all $g\in G$. Thus, if K is the kernel of the transfer of G into $S/S\cap Q\approx S_\beta$ then $K\cap S_\beta=1$. Also, $K\geqq S\cap Q$, so that $K\geqq Q$. By a result of Suzuki [17], K is one of the groups being characterized. Since $S_\beta\leqq \operatorname{Aut} K$, it is easy to see that $S_\beta\cdot K_{\alpha\beta}=G_{\alpha\beta}$ is non-abelian, a contradiction.

Lemma 3. *Let $1\neq t\in T$.*

(i) *If $|\Delta(t)|>2$, then $C(t)=C(W_t)$; and*

(ii) *If $|\Delta(t)|=2$ then t inverts Q.*

Proof. By Lemma 1 (ii) we have (i), and (ii) follows from the fact that $C_Q(t)=1$.

Lemma 4. *Let $T^{\#}=\{t_i\,|\,i=1,\ldots,m\}$. Set $k_i=|\Delta(t_i)|$. Let c be the number of involutions $(\alpha\,\beta)\ldots$ and d the number of regular involutions $(\alpha\,\beta)\ldots$.*

(i) *$(n-k_i)/k_i(k_i-1)$ is an integer >1.*

(ii) *$\sum_{i=1}^{m}(n-k_i)/k_i(k_i-1)=c-d$.*

Proof ([11], Lemma 4.3). (i) By Lemmas 1 (ii) and 3 (ii), t_i is weakly closed in $G_{\alpha\beta}$. Following Witt [22], we call a subset of Ω a line if it has the form $\Delta(t_i^g)$, $g\in G$. Then there is a unique line through two distinct points of Ω, and $(n-1)/(k_i-1)$ lines through α. If $\gamma\notin\Delta(t_i)$ then t_i fixes the line through γ and γ^{t_i}, and, since k_i is even, no two fixed lines of t_i meet. Thus, $k_i\,|\,n$, so that $(n-k_i)/k_i(k_i-1)$ is an integer. If this number is 1 then $n=k_i^2$. The points and lines then form an affine plane with a 2-transitive collineation group G. By a result of Ostrom and Wagner ([13], Theorem 1) G has a regular normal subgroup, which is not the case.

(ii) Since there are $c-d$ non-regular involutions $(\alpha\,\beta)\ldots$, α is moved by $(c-d)(n-1)$ non-regular involutions. On the other hand, there are $n(n-1)/k_i(k_i-1)$ conjugates of t_i, of which $(n-1)/(k_i-1)$ fix α and $(n-k_i)(n-1)/k_i(k_i-1)$ move α. Thus,

$$(c-d)(n-1)=\sum_{i=1}^{n}(n-k_i)(n-1)/k_i(k_i-1),$$

as required.

Lemma 5. *$|T|\geqq 4$.*

Proof. Otherwise, $T=\langle t\rangle$ has order 2. Set $\Delta=\Delta(t)$ and $W=W_t$. Set $k=|\Delta|>2$. By Lemma 4, $n-k=(c-d)\,k(k-1)$.

By Lemma 1 each involution $x \in C(t) - \{t\}$ is regular on Δ. If x and x_1 are involutions in $C(t) - \{t\}$ and $x^\Delta = x_1^\Delta$ then $x^{-1} x_1 \in \langle t \rangle$ by Lemma 3 (i). There are thus two possibilities:

(a) $C(t)^\Delta$ has a regular normal subgroup, $k > 4$ and $c = 2$ or

(b) $C_0(t)^\Delta = PSL(2, q)$ and $c = 2 \cdot (q-1)/2$.

Moreover, in (a) we have $d = 0$ by Lemma 4, and $n - k = 2k(k-1)$. In (b), $d = 0$ or $(q-1)/2$, and $n \leqq q^3 + 1$.

We note that W is semiregular on $\Omega - \Delta$. For, let $1 \neq U < W$ and $\Delta(U) \supset \Delta(t)$. Then t fixes k points of $\Delta(U)$, and it is easy to see that $|\Delta(U)| = k^2$ in case (a) and $|\Delta(U)| = q^3 + 1$ in case (b). Thus, (a) holds. However, U is weakly closed in $G_{\alpha\beta}$ by Lemma 1. Thus, G_α contains

$$(n-1)/(|\Delta(U)| - 1) = (2k+1)(k-1)/(k^2 - 1)$$

conjugates of U, a contradiction.

Let $t' = (\alpha\beta) \ldots$ be a conjugate of t. By Lemma 3(i), t' centralizes W. Then W is semiregular and faithful on $\Delta(t')$, so that W is cyclic of order dividing k. If (a) holds, W is a 2-group and $G_{\alpha\beta}/W$ is cyclic, whereas $G_{\alpha\beta}$ is non-cyclic. Thus, (b) holds and $G_{\alpha\beta}/W$ is cyclic of order dividing $q-1$. Once again, $G_{\alpha\beta}$ is cyclic, which is not the case.

We now use the notation of Lemma 4, where t_1 is chosen so that $|\Delta(t_1)|$ is maximal.

Lemma 6. (i) $T \cap W_{t_1} = \langle t_1 \rangle$.

(ii) $|T| = 4$.

Proof. (i) Suppose that $t_2 \in T \cap W_{t_1}$. By the maximality of $|\Delta(t_1)|$ we have $C_Q(t_1) = C_Q(t_2) = C_Q(t_1 t_2)$. By the Brauer-Wielandt Theorem [20], $(k_1 - 1)^3 = (n-1)(k_1 - 1)^2$, a contradiction.

(ii) Since $|T^{\Delta_1}| \leqq 2$ this follows from (i).

We can now complete the proof of Theorem A. By Lemma 6, $T^{\Delta(t_1)} \neq 1$. Also, a conjugate $(\alpha\beta) \ldots$ of t_1 is regular on $\Delta(t_1)$. If $C(t_1)^{\Delta(t_1)}$ has a regular normal subgroup then $C(t_1)^{\Delta(t_1)}$ has just 2 involutions $(\alpha\beta) \ldots$. Since $C(t_1) = N(W_{t_1}) = C(W_{t_1})$ (Lemmas 1 and 3), $c = 4$, whereas $c \geqq 2(|T| - 1) = 6$ by Lemma 4. It follows that $C(t_1)^{\Delta(t_1)} = PGL(2, q_1)$. In particular, $|\Delta(T)| = 2$.

Let $|\Delta(t_i)| > 2$. Then $T^{\Delta(t_i)}$ fixes just 2 points. Also, $C(t_i)^{\Delta(t_i)}$ contains a regular involution. Thus, $C(t_i)^{\Delta(t_i)} = PGL(2, q_i)$. Since $C(t_i) = C(W_{t_i})$ and $T \cap W_{t_i} = \langle t_i \rangle$, $c = 2(q_i - 1)$. In particular, $q_1 = q_i$.

By the Brauer-Wielandt Theorem [20],

$$n - 1 = |Q| = (k_1 - 1)(k_2 - 1)(k_3 - 1).$$

If $|\Delta(t_3)| = 2$ then $|\Delta(t_2)| > 2$, so that $k_1 = k_2 = q_1 + 1$ and $n - 1 = q_1^2$. However, by Lemma 4 (i) $(n - k_1)/k_1(k_1 - 1) = (q_1^2 - q_1)/(q_1 + 1)\, q_1$ is an integer, which is impossible.

Thus, $k_1=k_2=k_3=q_1+1$ and $n-1=q_1^3$. By Lemma 4 (ii),

$$2(q_1-1)=c\geqq c-d=\sum_i (n-k_i)/k_i(k_i-1)=3(q_1-1),$$

a final contradiction.

The definitions needed for the following corollary have been given in § 1.

Corollary. *Let G have a faithful split BN-pair of rank 1 at characteristic p. If G has no regular normal subgroup then G is one of the following groups in its usual 2-transitive representation: $PSL(2,q)$, $PGL(2,q)$, $PSU(3,q)$, $PGU(3,q)$, $Sz(q)$, or a group of Ree type. If G has a regular normal subgroup N, then $|N|=2^a$ and $p=2^a-1$ is a Mersenne prime, or $p=2$ and $|N|=q$ for $q=9$ or q a Fermat prime.*

Proof. We apply Theorem A in the case where Q (which is X in the notation of [14]) is a p-group for some prime p. We need only check the case where G has a regular normal subgroup N of order q^a for q a prime $q\neq p$. Then $p^b=|Q|=|\Omega|-1=q^a-1$ for some integer b, and the result follows.

§ 3. Nilpotent Q

As in [9], § 6, we call an involution in a permutation group a 2-*involution* provided that it fixes just 2 points. Clearly, these can exist only for permutation groups of even degree. The only known 2-transitive groups containing 2-involutions are S_n with $n\geqq 4$ even, A_n with $n>4$ even, and suitable subgroups of $P\Gamma L(2,q)$ containing $PSL(2,q)$ with q an odd prime power.

Theorem B. *Let G be a finite group 2-transitive on a set Ω, and let $\alpha,\beta\in\Omega$, $\alpha\neq\beta$. Suppose that $G_{\alpha\beta}$ has a non-trivial normal 2-subgroup semiregular on $\Omega-\{\alpha,\beta\}$. Then G acts on Ω as A_6, S_6 or a subgroup of $P\Gamma L(2,q)$ in its usual 2-transitive representation.*

Proof. The hypotheses state, in effect, that G is a transitive extension of the type of transitive group considered by Shult [15]. If the given 2-subgroup of $G_{\alpha\beta}$ contains a Klein group, it is easy to use Shult's result and a result of Suzuki [18] in order to show that G is A_6 or S_6.

We may thus assume that $G_{\alpha\beta}$ has a central 2-involution z. If $g\in G_\alpha$ and $z z^g=z^g z$ then z^g fixes β, so that $g\in G_{\alpha\beta}$ and $z^g=z$. By Glauberman's Z^*-Theorem [5], $G_\alpha=O(G_\alpha)\,C(z)_\alpha$. As $C(z)_\alpha=G_{\alpha\beta}$, $O(G_\alpha)$ is transitive on $\Omega-\alpha$. By the Feit-Thompson Theorem [4], $Q=O(G_\alpha)$ is solvable.

Let X be any subset of G fixing at least 3 points. Then $\langle z^g|\alpha^g=\alpha$ and $\beta^g\in\Delta(X)-\alpha\rangle$ is transitive on $\Delta(X)-\alpha$. Thus, $C(X)^{\Delta(X)}$ is 2-transitive and satisfies our hypotheses.

We may assume that there is an involution $t \neq z$ in $G_{\alpha\beta}$ (Hering [8]). Since $z \in Z(G_{\alpha\beta})$, $C_Q(t)_\beta = C_Q(tz)_\beta$. By the Brauer-Wielandt Theorem [20],

$$\begin{aligned}|Q|\,|C_Q(t)_\beta|^2 &= |Q_\beta|\,|C_Q(t)|\,|C_Q(tz)|\\ &= |Q_\beta|\,(|\Delta(t)|-1)\,|C_Q(t)_\beta|\,(|\Delta(tz)|-1)\,|C_Q(tz)_\beta|\\ &= |Q_\beta|\,|C_Q(t)_\beta|^2\,(|\Delta(t)|-1)(|\Delta(tz)|-1).\end{aligned}$$

Thus, $n-1=|Q:Q_\beta|=(|\Delta(t)|-1)(|\Delta(tz)|-1)$. Then t and tz each fixes at least 3 points. It follows by induction that $C(t)^{\Delta(t)} \trianglerighteq PSL(2,q)$ and $C(tz)^{\Delta(tz)} \trianglerighteq PSL(2,q')$ for some q and q'. Then $n-1=qq'$.

Let $\gamma \notin \Delta(t)$. Then t centralizes the conjugate z' of z lying in $G_{\gamma\gamma^t}$. Clearly, $z'^{\Delta(t)}$ fixes no points. Here $C(t)^{\Delta(t)}$ contains precisely $(q^2-q)/2$ regular involutions, all of which are conjugate. Thus, each such regular involution has the form $z'^{\Delta(t)}$ for the same number m of conjugates z' of z. Since $\langle t\rangle$ has precisely $(n-q-1)/2$ non-trivial orbits, it follows that $(n-q-1)/2 = m(q^2-q)/2$. Then $n-1=q(1+m(q-1))$ and $q'-1=m(q-1)$. Interchanging the roles of q and q' we find that $q=q'$ and $n=q^2+1$. Note also that, since $C(t)^{\Delta(t)}$ and $C(tz)^{\Delta(tz)}$ each contain involutions fixing 0 points and involutions fixing 2 points, both groups have subgroups acting as $PGL(2,q)$.

Let A be a minimal normal subgroup of G_α contained in $O(G_\alpha)$. Then $A=C_A(z)\,C_A(t)\,C_A(tz)=A_\beta\,C_A(t)\,C_A(tz)$ and A fixes no points of $\Omega-\alpha$. Clearly, $C(t)_{\alpha\beta}$ acts on $C_A(t)$. Since we may assume that $C_A(t) \nleqq G_{\alpha\beta}$, it follows that $C_A(t)$ is transitive on $\Delta(t)-\alpha$. If also $C_A(tz) \nleqq G_{\alpha\beta}$ then $C_A(tz)$ is transitive on $\Delta(tz)-\alpha$ and, by the Brauer-Wielandt Theorem [20], $|A|\,|C_A(t)_\beta|^2 = |A_\beta|\,|C_A(t)|\,|C_A(tz)| = |A_\beta|\,q\,|C_A(t)_\beta|\,q\,|C_A(tz)_\beta|$, so that $|A:A_\beta|=q^2=n-1$. Then A is transitive on $\Omega-\alpha$ and is abelian, so that A is regular on $\Omega-\alpha$. The theorem now follows from [9].

We may thus assume that $C_A(tz) \leqq G_{\alpha\beta}$. Let B/A be a minimal normal subgroup of G_α/A such that $A<B\leqq O(G_\alpha)$. If $C_B(tz) \nleqq G_{\alpha\beta}$ then, as in the preceding paragraph, we find that $|B:B_\beta|=q^2=n-1$, and by [9] we may thus assume that $B_\beta \neq 1$. If $C_B(tz) \leqq G_{\alpha\beta}$ then $B_\beta \neq 1$, as otherwise $C_B(tz)=1=C_B(z)$ and hence $A<B\leqq C(t)$, whereas A is regular on $\Delta(t)-\alpha$.

If $A_\beta \neq 1$ set $P=A_\beta$. If $A_\beta=1$ set $P=B_\beta$. In either case, $P \trianglelefteq G_{\alpha\beta}$ and $|P|$ is a prime power. We note that $|\Delta(P)| \geqq 3$. For otherwise, $P=B_\beta$ is a Sylow subgroup of B and $G_\alpha = BN(P)_\alpha = AN(P)_{\alpha\beta}$, whereas A is not transitive on $\Omega-\alpha$.

Since $P \trianglelefteq G_{\alpha\beta}$ and $C(P)^{\Delta(P)}$ is 2-transitive, P is weakly closed in $G_{\alpha\beta}$ (compare § 2, Lemma 1). Set $|\Delta(P)|=s+1$. Then P has $n(n-1)/(s+1)s$ conjugates in G, $(n-1)/s=q^2/s$ of which are in G_α. Also, $\langle t,z\rangle$ acts on $\Delta(P)$, so that $t^*=t$ or tz fixes at least 3 points of $\Delta(P)$. Recall that $C(t^*)^{\Delta(t^*)}$ has a subgroup acting as $PGL(2,q)$, so that $C(t^*)_{\alpha\beta}$ is transitive on $\Delta(t^*)-\{\alpha,\beta\}$. Since $C(t^*)_{\alpha\beta}$ normalizes P it acts on $\Delta(P)$, and it follows that

$\Delta(t^*) \subseteq \Delta(P)$. Thus, $q \leqq s$. However, q and s are powers of the same prime and $(s+1) \mid (q^2+1)$. This contradiction completes the proof of Theorem B.

Theorem C. *Let G be a 2-transitive permutation group on a finite set Ω. Let $\alpha \in \Omega$, and suppose that G_α has a normal nilpotent subgroup Q transitive on $\Omega - \alpha$. Then there is a normal subgroup Q^* of G_α such that $Q^* \leqq Q$ and Q^* is regular on $\Omega - \alpha$.*

Theorem C easily follows from the following result, which gives the precise structure of those groups satisfying the hypotheses of Theorem C.

Theorem C′. *Let G be a 2-transitive group on a finite set Ω such that G_α has a normal nilpotent subgroup Q transitive on $\Omega - \alpha$. Then either Q is regular on $\Omega - \alpha$ or G has a regular normal subgroup of order p^2, where p is a Mersenne prime. Moreover, G has a normal subgroup M such that $G \leqq \mathrm{Aut}\, M$ and M is one of the following groups in its usual permutation representation: a sharply 2-transitive group, $PSL(2, q)$, $PSU(3, q)$, $Sz(q)$, or a group of Ree type.*

We remark that, if $|\Omega|$ is even, Theorem C′ implies that the group Q is always regular on $\Omega - \alpha$. This fact will be used throughout the inductive proof of Theorem C′.

Proof. Let G be a counterexample to Theorem C′ of minimal order. Then by results of Shult [16] and Hering, Kantor, and Seitz [9], it suffices to show that $Q \cap G_{\alpha\beta} = 1$ for $\alpha \neq \beta$ in Ω. Thus, we suppose that $1 \neq P = Q \cap G_{\alpha\beta} \trianglelefteq G_{\alpha\beta}$. Set $\Delta = \Delta(P)$ and $W = W_p$.

Lemma 1. *G does not contain a regular normal subgroup.*

Proof. Suppose that G has a regular normal subgroup N of order p^a. As G does not satisfy the conclusion of Theorem C′, $p^a \neq p^2$ for p a Mersenne prime. If $p^a = 64$, then, by the nilpotence of Q and a theorem of Huppert [10], $|Q| = 63$ and Q is regular on $\Omega - \alpha$. Thus, neither of the above cases occurs and, by a result of Birkhoff and Vandiver [2], there is a prime r such that $r \mid (p^a - 1)$ and $r \nmid (p^b - 1)$ for $1 \leqq b < a$. Let $Q = R \times L$, where R is a Sylow r-subgroup of Q. Since $(p^a - 1) \mid |Q|$, $R \neq 1$, and, by the conditions on r, R is fixed-point-free and irreducible on N. If $1 \neq x \in L$ then R normalizes $C_N(x)$, so that $C_N(x) = 1$. It follows that Q is fixed-point-free on N and Q is regular on $\Omega - \alpha$, which is not the case.

Lemma 2. *$Z(Q)$ is semiregular on $\Omega - \alpha$.*

Proof. If $g \in Z(Q)_\beta$ then g fixes β^Q pointwise, so that $g = 1$.

Lemma 3. (i) *n is even.*

(ii) *Q has odd order.*

(iii) *G has no normal subgroup of index 2.*

Proof. (i) Otherwise, the Sylow 2-subgroup of $Z(Q)$ is nontrivial and semiregular on $\Omega-\alpha$. The result now follows from Lemma 1 and a theorem of Shult [15].

(ii) This follows from (i) and Lemma 2.

(iii) If $|G:N|=2$ then N is transitive and $Q\leqq N$. Thus, N is 2-transitive. As $|N|<|G|$, N has the required structure, and consequently so does G. This is a contradiction.

Lemma 4. *Set* $k=|\Delta|$.

(i) $N_Q(P)$ *is transitive on* $\Delta-\alpha$ *and* k *is even.*

(ii) $N(P)^{\Delta}$ *is 2-transitive and* $k>2$.

(iii) P *is weakly closed in* $G_{\alpha\beta}$.

(iv) *Call a subset of* Ω *a line if it has the form* Δ^g, $g\in G$. *Then there is a unique line through two distinct points of* Ω *and there are* $(n-1)/(k-1)$ *lines through* α.

(v) $k|n$ *if* Q *is a p-group.*

Proof. Let $\beta'\in\Delta-\alpha$. Then $\beta'=\beta^x$ with $x\in Q$, so that $P^{x^{-1}}\leqq G_{\alpha\beta}\cap Q=P$ and $x\in N_Q(P)$. This proves (i). Similarly, if $P_1,\ldots,P_r$ are the Sylow subgroups of P then $N_Q(P_i)$ is transitive on $\Delta(P_i)-\alpha$ for each i. Let $\alpha^g=\beta$, $g\in G$. Then P normalizes Q^g and $C(P_i)\cap Z(Q^g)\neq 1$. By Lemma 2, $N(P_i)$ is 2-transitive on $\Delta(P_i)$. Since $P^{\Delta(P_i)}\trianglelefteq N(P_i)_{\alpha\beta}^{\Delta(P_i)}$, the minimality of G implies that $P\leqq W_{P_i}$ and hence that $[P, C(P_i)\cap Z(Q^g)]\leqq G_{\alpha\beta}\cap Z(Q^g)=1$ for each i. Therefore, $C(P)\cap Z(Q^g)\neq 1$ and (ii) holds.

Let $P^y\leqq G_{\alpha\beta}$, $y\in G$. Then P fixes $\alpha,\beta,\alpha^{y^{-1}}$ and $\beta^{y^{-1}}$, so that by (ii) $\alpha^{y^{-1}}=\alpha^z$ and $\beta^{y^{-1}}=\beta^z$ with $z\in N(P)$. Thus, $zy\in G_{\alpha\beta}$ and $P=P^{zy}=P^y$, proving (iii).

An elementary result of Witt [22] yields (iv). Moreover, there are $n(n-1)/k(k-1)$ lines. If Q is a p-group then $n-1$ and $k-1$ are powers of p by (i). There are $n(n-1)/k(k-1)$ conjugates of P, so that $k|n$ and (v) holds.

Lemma 5. *Let* X *be a subgroup of* $G_{\alpha\beta}$ *fixing at least 3 points and such that* $(|Q|,|X|)=1$.

(i) $C_Q(X)$ *is transitive on* $\Delta(X)-\alpha$ *and* $|\Delta(X)|$ *is even.*

(ii) $\langle C_{Q^g}(X)|X$ *fixes* $\alpha^g\rangle$ *is 2-transitive on* $\Delta(X)$.

(iii) *If* $X^g\leqq G_{\alpha\beta}$, $g\in G$, *then* X *and* X^g *are conjugate in* $G_{\alpha\beta}$.

Proof. Let $\beta'\in\Delta(X)-\alpha$. Then $\beta'=\beta^y$ with $y\in Q$, so that $X^{y^{-1}}\leqq QX\cap G_{\alpha\beta}=PX$. By the Schur-Zassenhaus Theorem, $X^{y^{-1}}=X^z$ with $z\in P$. Thus, $\beta'=\beta^y=\beta^{zy}$ and $zy\in N_Q(X)=C_Q(X)$. This proves (i) and (ii).

Suppose that $X^g\leqq G_{\alpha\beta}$. Then $\alpha,\beta,\alpha^g,\beta^g\in\Delta(X^g)$. By (ii), $\alpha^{gl}=\alpha$ and $\beta^{gl}=\beta$ with $l\in C(X^g)$. Thus, $X^g=X^{gl}$ and $gl\in G_{\alpha\beta}$, proving (iii).

Lemma 6. *Let $\langle t, u\rangle$ be a Klein group in $G_{\alpha\beta}$.*

(i) $(|\Delta(t)|-1)(|\Delta(u)|-1)(|\Delta(tu)|-1)=(n-1)(|\Delta(\langle t, u\rangle)|-1)^2$.

(ii) $\Delta(t) \neq \Delta(u)$.

Proof. (i) By the Brauer-Wielandt Theorem [20],

$$|C_Q(t)|\,|C_Q(u)|\,|C_Q(tu)|=|Q|\,|C_Q(\langle t, u\rangle)|^2$$

and

$$|C_P(t)|\,|C_P(u)|\,|C_P(tu)|=|P|\,|C_P(\langle t, u\rangle)|^2.$$

Clearly, $|Q|/|P|=n-1$. By Lemma 5(i), $|C_Q(t)|/|C_P(t)|=|\Delta(t)|-1$, with similar equations for u, tu and $\langle t, u\rangle$. This proves (i).

(ii) If $\Delta(t)=\Delta(u)$ then $\Delta(t)=\Delta(\langle t, u\rangle)\subseteq\Delta(tu)$. By (i),

$$(|\Delta(t)|-1)^2(|\Delta(tu)|-1)=(n-1)(|\Delta(t)|-1)^2,$$

which is impossible.

Lemma 7. *Q is a p-group for some prime p.*

Proof. Suppose that Q is not a p-group. Then $Z(Q)\leqq N(P)$ and $Z(Q)$ is not a p-group. By Lemmas 2 and 4 and the minimality of n, $N(P)^\Delta$ has a regular normal elementary abelian 2-subgroup T_0^Δ. We may assume that $T_0\leqq\langle Z(Q^g)\,|\,\alpha^g\in\Delta\rangle=C_0(P)$. As $W\lhd N(P)$, $[C_0(P), W]=1$. Also, $|\Delta|-1\geqq|Z(Q)|>3$. By [9], Lemma 2.7, $C_0(P)$ has a normal Sylow 2-subgroup T of order $k=|\Delta|$, and $C_0(P)=TZ(Q)$.

We now proceed by a series of steps.

(i) $T^\#$ consists of regular involutions. For otherwise, if $x\in T^\#$ then P acts on $\Delta(x)$ without fixed points. In particular, $C(x)^{\Delta(x)}$ has no regular normal 2-subgroup. Let x' be a conjugate of x lying in $G_{\alpha\beta}$. Since $C(x')^{\Delta(x')}$ is nonsolvable and x' acts on Δ, $|\Delta\cap\Delta(x')|=4$. Then $|\Delta|=16$ and $|Z(Q)|=15$ since $|Z(Q)|$ is not a prime power. x' centralizes the subgroup R of $Z(Q)$ of order 3. Since $R^{\Delta(x')}\trianglelefteq C(x')_\alpha^{\Delta(x')}$ we must have $|\Delta(x')|=28$ (see [9], § 3). Also, Q is a $\{3, 5\}$-group. Since $5\nmid|C(x')^{\Delta(x')}|$, $P^{\Delta(x)}$ is a 3-group and hence P fixes a point of $\Delta(x)$, which is not the case.

(ii) $G_{\alpha\beta}$ has even order (Bender [1]).

Let t be an involution in $G_{\alpha\beta}$. Then t normalizes P and T, and either $t^\Delta=1$ or t^Δ fixes $\sqrt{k}$ points. Thus, $|C_T(t)|=k$ or $\sqrt{k}$. Moreover, $|\Delta(t)|>2$ and Lemma 5 applies to t.

We claim that $C(t)^{\Delta(t)}$ has a regular normal subgroup. For otherwise, since $C_T(t)^{\Delta(t)}$ is semiregular, $C(t)^{\Delta(t)}$ has a normal subgroup $PSL(2, q)$ for some $q>3$ (see [9], § 3). Moreover, $|C_T(t)|\leqq 4$. Since $k>4$ we must have $k=16$ and $|\Delta\cap\Delta(t)|=|C_T(t)|=4$. As in (i), $|Z(Q)|=15$, t centralizes the subgroup R of $Z(Q)$ of order 3, and $R^{\Delta(t)}\trianglelefteq C(t)_\alpha^{\Delta(t)}$, which is impossible.

(iii) There is a normal subgroup U of $C(t)$ of order $|\Delta(t)|$ containing $C_T(t)$ and regular on $\Delta(t)$. For let U be the normal closure of $C_T(t)$ in $C(t)$. Since $[C_T(t), W_t] \leqq T \cap W_t = 1$, $U \cap W_t \leqq Z(U)$. Thus, U is a 2-group. Let $B^{\Delta(t)} = O(C(t)_\alpha^{\Delta(t)})$ with $B \geqq W_t$. Then $B/C_B(U)$ has odd order and is transitive on $(U/U \cap W_t)^{\#}$. If $|\Delta(t)| > 4$ our assertion now follows from [9], Lemma 2.7. If $|\Delta(t)| = 4$ then $|\Delta| > 4$ implies that $\Delta(t) \subseteq \Delta$ and hence that $U = C_T(t)$.

(iv) The remainder of the proof closely follows that of [9], § 5. We first show that $G_{\alpha\beta}$ contains no Klein group (cf. [9], Lemma 5.5). For let $\langle t, u\rangle$ be a Klein group in $G_{\alpha\beta}$ with t central in a Sylow 2-subgroup of $G_{\alpha\beta}$. Set $l = |\Delta(\langle t, u\rangle)|$. Since $|\langle t, u\rangle^\Delta| = 2$, $l > 2$. By (ii), t, u and tu fix l or l^2 points. By Lemma 6, at most one of these fixes l points and $n-1 = (l+1)^2(l^i - 1)$, $i = 1$ or 2. Computing $|G|_2$ we find that $|\Delta(t)| = l$ and $C(t)$ contains a Sylow 2-subgroup of G. Let $V \lhd C(u)$ with $|V| = |\Delta(u)|$ (see (iii)). Then $C(t)$ contains a conjugate $\tilde{V}$ of $V\langle u\rangle$, $|\tilde{V}^{\Delta(t)}| \leqq l$, and hence $|\tilde{V} \cap W_t| \geqq 2l^2/l \geqq 8$. If $v \in \tilde{V}^{\#}$ with $|\Delta(v)|$ maximal then $|(\tilde{V} \cap W_t)^{\Delta(v)}| \geqq 4$, which is impossible by Lemma 6.

(v) Next, $n = |\Delta(t)|^2$ and Ut is fused. For $U\langle t\rangle$ contains all involutions in $C(t)$. If $\gamma^t \neq \gamma \in \Omega - \Delta(t)$ then t centralizes an involution in $G_{\gamma\gamma^t}$. No two involutions in Ut fix common points. Thus $n = |\Delta(t)|^2$.

In particular, $|\Delta(t)| - 1$ is not a prime. For otherwise, by considering t^Δ, we find that $C(t) \cap C_Q(P)$ is transitive on $\Delta(t) - \alpha$ and hence $\Delta(t) \subseteq \Delta$. Since $|\Delta(t)| = k$ or $\sqrt{k}$ it follows that $\Delta(t) = \Delta$. Now the points and lines (Lemma 4 (iv)) form an affine plane, and G has a regular normal subgroup (Ostrom and Wagner [13], Theorem 1), which is not the case.

Moreover, it is easy to see that $N(U\langle t\rangle)$ is transitive on the set $\mathscr{I} = Ut$. Since $C(\mathscr{I}) \leqq UW_t$ it follows that $W_t = \langle t\rangle$. Consequently, $C(t)_{\alpha\beta}$ is cyclic.

(vi) $N(U\langle t\rangle)$ is transitive on $\mathscr{I}$ and $C_Q(t)$ is transitive on $\mathscr{I} - \{t\}$ (Lemma 5(i)). We can apply induction to $N(U\langle t\rangle)$. Here $|\mathscr{I}| = |\Delta(t)| = l > 2$ is a power of 2. If $N(U\langle t\rangle)^{\mathscr{I}}$ does not have a regular normal subgroup then $l-1$ is a prime power and hence a prime, and this contradicts (v). Thus, $N(U\langle t\rangle)$ has a normal subgroup R containing $C(\mathscr{I}) = U\langle t\rangle$ such that $R^{\mathscr{I}}$ is regular. Here $|R| = 2l^2$ where $l = |\Delta(t)|$.

Set $A = C_Q(t)$. Since U consists of regular involutions, $U \lhd RA$. Then $U \leqq Z(R)$ as A is transitive on $(R/U)^{\#}$.

(vii) Since $l > 4$ by (v), using [9], Lemma 2.7, we find that $R/U = U_1/U \times U\langle t\rangle/U$, where $C(t)_\alpha$ normalizes $U_1 = [R, C(t)_\alpha]$. Then $U \leqq Z(U_1)$ and A is transitive on $(U_1/U)^{\#}$.

Since $n = l^2$, a Sylow 2-subgroup S of $N(U\langle t\rangle)$ containing a Sylow 2-subgroup of $C(t)_{\alpha\beta}$ is Sylow in G. Clearly $S = U_1 S_{\alpha\beta} \rhd U_1$, $t \in S_{\alpha\beta}$, $S_{\alpha\beta}$ is cyclic and $U_1 \cap S_{\alpha\beta} = 1$.

By Thompson's transfer lemma ([9], Lemma 2.3), t is conjugate to an element u_1 of U_1. Then $u_1 \notin U$. Since A is transitive on $(U_1/U)^{\#}$ and $U \leqq Z(U_1)$, U_1 is elementary abelian. However, $U_1 \leqq C(u_1)$ and $C(u_1)$ has no elementary abelian subgroup of order $l^2 > 2l$. This contradiction proves Lemma 7.

Lemma 8. *$W \cap C(P)$ is a p-group.*

Proof. Let $Q_1 = N_Q(P)$. Suppose that L is a p'-group and $1 \neq L \leqq W \cap C(P)$. Then $[Q_1, L] \leqq Q \cap W \leqq Q \cap G_{\alpha\beta} = P$. Thus, L centralizes Q_1/P, and since L centralizes P, $L \leqq C(Q_1)$.

Suppose that $\Delta \subset \Delta(L)$. By Lemma 5 (ii), $N(L)^{\Delta(L)}$ is 2-transitive. Clearly, $P^{\Delta(L)} \neq 1$. Since $C_Q(L)$ is transitive on $\Delta(L) - \alpha$ (Lemma 5 (i)), $|\Delta(L)| - 1$ is a power of p. As $1 \neq P^{\Delta(L)} \trianglelefteq C(L)^{\Delta(L)}_{\alpha\beta}$, this contradicts the minimality of G. Thus, $\Delta = \Delta(L)$. Also, $P < Q_1 \leqq N_Q(L)$ and $N_Q(L)^{\Delta(L) - \alpha}$ is regular. Consequently, $N_Q(L) = Q_1$.

Let $Q_2 \leqq N_Q(Q_1)$ be such that Q_2/Q_1 is a minimal normal subgroup of $N_Q(Q_1) \cdot L/Q_1$. Then $[L, P, Q_2] = 1$ and $[P, Q_2, L] \leqq [Q_1, L] = 1$. Thus, $[Q_2, L, P] = 1$ and P centralizes $[Q_2, L]$. Then $[Q_2, L] \leqq N(P) \cap Q = N_Q(P) = Q_1$, and L centralizes Q_2/Q_1. Since L centralizes Q_1, L centralizes Q_2, a contradiction.

Lemma 9. *$N(P)$ contains a Sylow 2-subgroup of G.*

Proof. By Lemma 4 (v), $k \mid n$. Since Q is a p-group, $n = p^b + 1$ for some integer b and $k = p^a + 1$ for some integer a. Then $(p^a + 1) \mid (p^b + 1)$, so that b/a is an odd integer and $(p^b + 1)_2 = (p^a + 1)_2$. Since $G_{\alpha\beta} \leqq N(P)$, $|N(P)|_2 = (p^a + 1)_2 \, |G_{\alpha\beta}|_2 = (p^b + 1)_2 \, |G_{\alpha\beta}|_2 = |G|_2$.

Lemma 10. (i) *$C_0(P) = \langle Z(Q^g) \mid \alpha^g \in \Delta \rangle$ is a normal subgroup of $C(P)$ and $N(P)$ which is transitive on Δ.*

(ii) *$C_0(P)^{\Delta}$ contains a Sylow 2-subgroup of $(N(P)^{\Delta})'$.*

(iii) *$C_0(P) \cap W$ is a p-group.*

(iv) *$C_0(P) \cap W \leqq Z(C_0(P))$.*

Proof. (i) If $\alpha^g \in \Delta$, then we may assume that $g \in N(P)$ (Lemma 4). Thus, $P = P^g \leqq Q^g$ and $Z(Q^g) \leqq C(P)$. By Lemma 2, $Z(Q^g)$ is semiregular on $\Delta - \alpha^g$. Thus, $C_0(P)$ is transitive on Δ, $C_0(P) \leqq C(P)$, and $C_0(P) \trianglelefteq N(P)$.

(ii) This follows by considering the structure of groups satisfying the conclusion of Theorem C' with $|\Omega|$ even.

(iii) This follows from Lemma 8 as $C_0(P) \cap W \leqq C(P) \cap W$.

(iv) $C_0(P) \cap W \trianglelefteq C_0(P)$ and, if $\alpha^g \in \Delta$, then $C_0(P) \cap W \leqq G_{\alpha^g}$ and $C_0(P) \cap W$ normalizes $Z(Q^g)$. Thus, $[C_0(P) \cap W, Z(Q^g)] \leqq C_0(P) \cap W \cap Z(Q^g) = 1$ since W fixes Δ pointwise and $Z(Q^g)$ is semiregular on $\Delta - \alpha^g$. This proves (iv).

We now complete the proof of Theorem C′ by considering four cases. We use repeatedly properties of the groups being characterized by Theorem C′. We refer the reader to § 3 of [9] and to § 2 of [11] for these properties. Moreover, we use the Thompson transfer lemma (see Lemma 2.3 of [9]) and the Burnside fusion lemma ([6], p. 203).

Let S_1 be a Sylow 2-subgroup of $C_0(P)$ such that $(S_1)_{\alpha\beta}$ is a Sylow 2-subgroup of $C_0(P)_{\alpha\beta}$, and let S_2 be a Sylow 2-subgroup of W. Since $C_0(P)\trianglelefteq N(P)$, $W\trianglelefteq N(P)$, and $C_0(P)\cap W$ is a p-group, $S_1S_2=S_1\times S_2$. Let S be a Sylow 2-subgroup of $N(P)$ such that $S_1\times S_2\leqq S$. By Lemma 9, S is a Sylow 2-subgroup of G, and by Lemma 10 (ii), $S/S_1\times S_2$ is abelian. Also, $S_1=S\cap C_0(P)\trianglelefteq S$ and $S_2=S\cap W\trianglelefteq S$.

Case 1. $N(P)^{\Delta}$ contains a regular normal subgroup.

Here $k=2^a\geqq 4$ for some integer a and $C_0(P)^{\Delta}$ contains a regular normal subgroup. Since $C_0(P)\cap W$ is contained in $Z(C_0(P))$ and has odd order, S_1 is an elementary abelian 2-group characteristic in $C_0(P)$. Then $2^a=|S_1|=|\Delta|=p^b+1$ for some integer b, so that a is prime and $2^a=p+1$. If $a>2$, then by a result of Huppert [10], $S_1\times S_2=S$. If $a=2$ then $|S:S_1\times S_2|\leqq 2$.

Let u be an involution in S_1. Then $\Delta(u)\cap\Delta=\emptyset$ and $P\leqq C(u)$. If $\Delta(u)\neq\emptyset$ then P acts on $\Delta(u)$, and since $|\Delta(u)|-1$ is a power of p (Lemmas 5(i) and 7), $\Delta\cap\Delta(u)\neq\emptyset$, a contradiction. Thus, each involution in S_1 is a regular involution.

Suppose that $a>2$, so that $S=S_1\times S_2$ and S_2 is a Sylow 2-subgroup of $G_{\alpha\beta}$. By a result of Bender [1], $S_2\neq 1$. Let t be an involution in S_2, so that $S_1\leqq C(t)$. Since S_1 is semiregular on Ω, $S_1^{\Delta(t)}\approx S_1$. By the minimality of G and the structure of groups satisfying the conclusion of Theorem C′, $C(t)^{\Delta(t)}$ contains a regular normal subgroup. Thus, $2^c=|\Delta(t)|=p^d+1$ for some integers c and d, and $2^c=p+1$ as before. Then $2^c=2^a$ and $\Delta(t)=\Delta$. By Lemma 6 (ii), t is the unique involution in S_2. By [9], Lemma 2.6, t is conjugate to some involution in $S_1\times\langle t\rangle-\{t\}$, and from Burnside's fusion lemma it follows that $S_2=\langle t\rangle$. Now Thompson's transfer lemma and Lemma 3 imply that t is conjugate to some involution u in S_2. This is a contradiction since u is a regular involution.

Now suppose that $a=2$ and $|\Delta|=4$. Here $p=3$ and $|Z(Q)|=3=|\Delta|-1$. As before, there is an involution t in $G_{\alpha\beta}$ (Bender [1]). If $S_2=1$ then $S_1\langle t\rangle=S$ and, by Thompson's transfer lemma and Lemma 3, t is conjugate to an involution in S_1. This is a contradiction as before. Thus, $S_2\neq 1$.

Suppose that S_2 contains a Klein group $\langle t,u\rangle$. Since S_1 centralizes $\langle t,u\rangle$ and contains only regular involutions, $C(t)^{\Delta(t)}$ contains a regular normal subgroup or a normal subgroup isomorphic to $PSL(2,p^e)$ for some integer e. Also, $Z(Q)\trianglelefteq G_\alpha$ and $t\in S_2\leqq C(C_0(P))$, so that $Z(Q)^{\Delta(t)}\trianglelefteq$

$C(t)_\alpha^{\Delta(t)}$. As $|Z(Q)|=3$, $|\Delta(t)|=4$. Similarly, $|\Delta(u)|=4=|\Delta(tu)|$. Thus $\Delta(t)=\Delta(u)=\Delta(tu)=\Delta$, contradicting Lemma 6. Hence, S_2 is cyclic or generalized quaternion.

Let $\langle t\rangle=\Omega_1(S_2)$ and suppose that there is a conjugate v of t with $v\in N(P)-C_0(P)W$. Then we may assume that $v\in G_{\alpha\beta}$. By Lemma 5(iii) v and t are conjugate in $G_{\alpha\beta}$. However, $G_{\alpha\beta}\leqq N(P)$, $t\in W\trianglelefteq N(P)$ and $v\notin W$, a contradiction. Thus, each conjugate of t contained in S is in $S_1\times S_2$. Suppose that $S>S_1\times S_2$. Then S/S_2 is dihedral of order 8, and it follows that $Z(S)\cap S_1=\langle u\rangle$ for some regular involution u. By [9], Lemma 2.6, t is conjugate to tu. However, $\Omega_1(Z(S))=\langle u\rangle\times\langle t\rangle$, so that Burnside's fusion lemma implies that t, u and tu are conjugate, a contradiction. Thus, $S=S_1\times S_2$ and $\Omega_1(Z(S))=S_1\times\langle t\rangle$. By [9], Lemma 2.6, $Z(S)-\langle t\rangle$ contains a conjugate of t, and by Burnside's fusion lemma $S_2=\langle t\rangle$. Once again Thompson's lemma yields a contradiction.

Case 2. $N(P)^\Delta$ contains a normal unitary subgroup.

Here $C_0(P)^\Delta\approx PSU(3,q)$. Suppose that $\langle t,u\rangle$ is a Klein group contained in S_2. By Lemma 6, we may assume that $\Delta(t)\supset\Delta(\langle t,u\rangle)\supseteqq\Delta$. As $\langle t,u\rangle$ centralizes $C_0(P)$, $C_0(P)^{\Delta(t)}$ is a unitary group centralized by $u^{\Delta(t)}$. This contradicts [9], Lemma 3.2. Thus, S_2 is cyclic or generalized quaternion.

We claim that $S_2=1$. For, otherwise, set $\langle t\rangle=\Omega_1(S_2)$. By [9], Lemma 3.2, S_1 is a quasidihedral or wreathed group. Let $\langle u\rangle=\Omega_1(Z(S_1))$, so that $\Omega_1(Z(S))=\langle u\rangle\times\langle t\rangle$. If t is conjugate to an involution $v\in S-(S_1\times S_2)$ then, by [9], Lemma 3.2, we may assume $v\in G_{\alpha\beta}$. By Lemma 5(iii), v and t are conjugate in $G_{\alpha\beta}$. As $G_{\alpha\beta}\leqq N(P)\leqq N(W)$, $t\in W$, and $v\notin W$, this is a contradiction. Thus, each conjugate of t lying in S is contained in $S_1\times S_2$. By [9], Lemma 2.6, t is conjugate to some involution in $S_1\times S_2-\{t\}$. As $C_0(P)^{\Delta(t)}$ has only one class of involutions, t is conjugate to u or ut. By Burnside's fusion lemma, t, u and tu are conjugate. Since u fixes at least 2 points of Δ, u is conjugate to an element u' of $C_0(P)_{\alpha\beta}$. As above, t and u' are conjugate in $G_{\alpha\beta}\leqq N(P)$, whereas $u'\in C_0(P)$ and $t\in W$, a contradiction. Thus, $S_2=1$ as claimed.

By [9], Lemma 3.2, $S=S_1S_0$ with $S_1\cap S_0=1$ and S_0 cyclic. If $S_0\neq 1$, then, by [9], Lemma 2.3, the involution v in S_0 is conjugate to the involutions in S_1. We may assume that $v\in G_{\alpha\beta}$ and v is conjugate to an involution $u\in C_0(P)_{\alpha\beta}$. Then v and u are conjugate in $G_{\alpha\beta}$, whereas u^Δ and v^Δ are not conjugate. Consequently, $S_1=S$.

Let t be an involution in $C_0(P)_{\alpha\beta}$. As $C_0(P)\cap W\leqq Z(C_0(P))$ is of odd order, t is the unique involution in $C_0(P)_{\alpha\beta}$, and consequently t is the unique involution in $G_{\alpha\beta}$. With the notation of Lemma 4(ii) of § 1, $(n-k^*)/k^*(k^*-1)=c-d$, where $k^*=|\Delta(t)|$. All involutions in G are conjugate, so that $d=0$. If t' is an involution interchanging α and β, then

$t' \in N(P)$. Since $C_0(P)$ contains each involution in $N(P)$, $t' \in C_0(P)$. Here $C_0(P)^\Delta$ contains $q-1$ involutions interchanging α and β. Since $C_0(P) \cap W \leqq Z(C_0(P))$ has odd order, $C_0(P)$ contains $q-1$ involutions $t' = (\alpha\,\beta) \ldots$. Thus, $c = q-1$.

As $t \in C_0(P) \leqq C(P)$, P acts on $\Delta(t)$ and $P^{\Delta(t)} \trianglelefteq C(t)_{\alpha\beta}^{\Delta(t)}$. Since $|\Delta(t)|-1$ is a power of p the minimality of G implies that $P^{\Delta(t)} = 1$. Thus, $\Delta(t) \subset \Delta$. Since $C_0(P)^\Delta \approx PSU(3, q)$, $|\Delta| = q^3+1$ and $k^* = |\Delta(t)| = q+1$. Thus, $(n-(q+1))/(q+1)\,q = q-1$ and $n = q^3+1 = |\Delta|$, a contradiction.

Case 3. $N(P)^\Delta$ contains a normal subgroup of Ree type.

Let $|\Delta| = q^3+1$. Here $C_0(P)^\Delta \trianglelefteq N(P)^\Delta$ so that either $C_0(P)^\Delta$ is a group of Ree type or $|\Delta| = 28$ and $C_0(P)^\Delta \approx PSL(2, 8)$. By [9], Lemma 3.3, $C_0(P)^\Delta$ contains a Sylow 2-subgroup of $N(P)^\Delta$, so that $S = S_1 \times S_2$. Moreover, S_1 is elementary abelian of order 8 and there is an element $g \in C_0(P)$ such that $|g| = 7$, $g \in N(S_1)$ and $\langle g \rangle$ is transitive on the involutions in S_1. If u is an involution in S_1 then $u \in C(P)$ and P acts on $\Delta(u)$. We may assume that $u \in G_{\alpha\beta}$, so that $P^{\Delta(u)} \trianglelefteq C(u)_{\alpha\beta}^{\Delta(u)}$. Since $|\Delta(t)|-1$ is a power of p we must have $P^{\Delta(u)} = 1$. Thus, $\Delta(u) \subset \Delta$.

Let t be an involution in S_2. Then $S_1\langle g \rangle \leqq C(t)$ and, by the preceding paragraph, $S_1\langle g \rangle^{\Delta(t)} \approx S_1\langle g \rangle$. By considering the groups satisfying the conclusion of Theorem C', it follows that either $C(t)^{\Delta(t)}$ contains a regular normal subgroup $L^{\Delta(t)}$ with $S_1^{\Delta(t)} \leqq L^{\Delta(t)}$ or $C(t)^{\Delta(t)}$ contains a normal subgroup of Ree type. The first case cannot occur since $S_1 \cap G_{\alpha\beta} \neq 1$. If the second case occurs then $|\Delta(t)| = q_1^3+1$ for some integer q_1, and, if u is an involution in S_1, then $|\Delta(u) \cap \Delta(t)| = q_1+1$. Since $\Delta(u) \subset \Delta$ and $\Delta \subseteq \Delta(t)$, $q_1+1 = |\Delta(u)| = q+1$ and $\Delta(t) = \Delta$. By Lemma 6 (ii) $\langle t \rangle = \Omega_1(S_2)$.

Thus, if $S_2 \neq 1$ then $\Omega_1(S) = S_1 \times \langle t \rangle \leqq Z(S)$. In this case, there is a conjugate v of t lying in $S_1 \times \langle t \rangle - \{t\}$ ([9], Lemma 2.6). By Burnside's fusion lemma we must have $S_2 = \langle t \rangle$. Now Thompson's transfer lemma implies that t is conjugate to some involution u in S_1. However, $|\Delta(u)| = q+1$ and $|\Delta(t)| = q^3+1$, a contradiction. Therefore, $S_2 = 1$ and $S = S_1$.

Now just as in the preceding case, we proceed as in § 1, Lemma 4 (ii), in order to show that $n = |\Omega| = q^3+1$. Consequently, $\Delta = \Omega$, a contradiction.

In view of the minimality of G, the proof of Theorem C' will be completed once we eliminate the following case.

Case 4. $N(P)^\Delta$ contains a normal subgroup isomorphic to $PSL(2, q)$, $q > 3$.

Here $C_0(P)^\Delta \approx PSL(2, q)$. Let u be an involution in S_1. Then $|\Delta \cap \Delta(u)| = 0$ or 2. Since $u \in C_0(P) \leqq C(P)$, P acts on $\Delta(u)$ where $|\Delta(u)|-1$ is -1 or a power of p. By the minimality of G, $P^{\Delta(u)} = 1$ and $\Delta(u) \subset \Delta$. Thus, if $q \equiv 3 \pmod 4$ u is a regular involution, while if $q \equiv 1 \pmod 4$ u fixes exactly 2 points of Ω.

If $q \equiv 1 \pmod 4$, then there is an involution u in $C_0(P)_{\alpha\beta}$. As $C_0(P) \cap W \leqq Z(C_0(P))$ and $C_0(P) \cap W$ has odd order, u is the unique involution in $C_0(P)_{\alpha\beta}$. Also, $C_0(P)_{\alpha\beta} \trianglelefteq G_{\alpha\beta}$, so that u is central in $G_{\alpha\beta}$. Since u is a 2-involution, Theorem B yields a contradiction.

Thus, $q \equiv 3 \pmod 4$. As usual, there are involutions fixing at least 2 points (Bender [1]).

Suppose that $S_2 = 1$. Then $|S:S_1| \leqq 2$ and there is a subgroup S_0 such that $S = S_1 S_0$ and $S_1 \cap S_0 = 1$. Here S_1 contains only regular involutions, so that $S_1 < S$. Let $S_0 = \langle t \rangle$, where t is an involution. Then t fixes 2 points of Δ and hence is not a regular involution. Thompson's transfer lemma now yields a contradiction. Consequently, $S_2 \neq 1$.

Let t be an involution in S_2. Then $C(t)^{\Delta(t)}$ contains $S_1^{\Delta(t)}$, so that $C(t)^{\Delta(t)}$ contains regular involutions. As $C_0(P) \leqq C(t)$ and $C_0(P)^{\Delta(t)} \neq 1$, $C(t)^{\Delta(t)}$ contains a normal subgroup isomorphic to $PSL(2, q')$ for some q'. Clearly, t centralizes $Z(Q)$, $Z(Q) \trianglelefteq G_\alpha$, and $Z(Q)$ is semiregular on $\Omega - \alpha$. Thus, $Z(Q)^{\Delta(t)} \trianglelefteq C(t)_\alpha^{\Delta(t)}$ and $Z(Q)^{\Delta} \trianglelefteq N(P)_\alpha^{\Delta}$. It follows that $q = |Z(Q)| = q'$, $\Delta = \Delta(t)$, and, by Lemma 6(ii), $\langle t \rangle = \Omega_1(S_2)$.

By the usual arguments, t is conjugate to an involution v in $S_1 \times S_2 - \{t\}$ but is conjugate to no involution in $S - (S_1 \times S_2)$. As the involutions in S_1 are regular involutions, t is conjugate to ut for some involution u in S_1. All involutions in $C_0(P)$ are conjugate, so we may assume that $u \in Z(S) \cap S_1$. If $\langle u \rangle = Z(S) \cap S_1$ then $\langle u \rangle \times \langle t \rangle = \Omega_1(Z(S))$, and by Burnside's fusion lemma, t, tu and u are conjugate. This is a contradiction. Thus, $\langle u \rangle < Z(S) \cap S_1$, and it follows that $S = S_1 \times S_2$ and S_1 is a Klein group. Since t and ut are conjugate, Burnside's lemma implies that $S_2 = \langle t \rangle$. Now Thompson's transfer lemma shows that t is conjugate to some involution in S_1. This is a final contradiction.

Corollary 1. *Let G be a finite group having a faithful split BN-pair of rank 2 at characteristic p. Let P be a maximal parabolic subgroup containing B and set $K = \bigcap_{g \in P} B^g$. Then P/K has a faithful split BN-pair of rank 1 at characteristic p, or $p = 2$ and P/K is the solvable 2-transitive group of degree 9 and order $9 \cdot 8 \cdot 2$.*

Proof. Let G, P, B, and K be as in the statement of the corollary. Then P/K is a 2-transitive permutation group on the cosets of B/K. Let s be a fundamental reflection in P, so that $P = \langle B, s \rangle$. Then $B/K = (XK/K)(HK/K)$ and $XK/K \trianglelefteq B/K$. As HK/K is normalized by s, HK/K fixes the cosets B and Bs and consequently XK/K is a normal p-subgroup of B/K transitive on the cosets of B in P other than B. Thus, P/K satisfies the hypotheses of Theorem 3.

Consequently, either XK/K is regular on the cosets of B in P other than B or P/K has a regular normal subgroup of order q^2 for q a Mersenne prime. However, XK/K is a p-group, so that $q^2-1=p^a$ for some integer a. Thus, $p=2$ and $q=3$. Then $|XK/K|=8$ or 16, and if $|XK/K|=8$ then XK/K is regular on the remaining cosets of B. If $|XK/K|=16$ then B/K is a subgroup of $GL(2,3)$ containing a Sylow 2-subgroup of $GL(2,3)$ as a normal subgroup. Thus, $XK/K=B/K$, and the result follows.

Corollary 2. *Let G be a finite group having a faithful split BN-pair of rank 2 at characteristic p. Let s be a fundamental reflection in the Weyl group $W=N/H$. Then either*

(i) $X\cap X^s \trianglelefteq X$, *or*

(ii) $p=2$ *and* $\langle B,s\rangle$, *in its 2-transitive representation on the cosets of B in $\langle B,s\rangle$, is the 2-transitive permutation group of degree 9 and order* $9\cdot 8\cdot 2$.

Proof. Let $P=\langle B,s\rangle$ and $K=\bigcap_{g\in P} B^g$. From Corollary 1 it follows that either (ii) holds or XK/K is regular on the cosets of B in P other than B itself. Now $(X\cap X^s)K/K$ fixes the cosets B and Bs, so that $X\cap X^s\leqq K$. However, K is a subgroup of B and has a normal Sylow p-subgroup X_0. Thus, $s\in N(X_0)$ and $X_0\leqq X\cap X^s$. Then $X\cap X^s=X_0\trianglelefteq P$, so that $X\cap X^s\trianglelefteq X$.

§ 4. 2-Involutions

Using Theorem C we can now strengthen the main part of Theorem B.

Theorem D. *Let G be a 2-transitive permutation group on a finite set Ω. Suppose that for α,β in Ω, $\alpha\neq\beta$, $G_{\alpha\beta}$ contains a 2-involution z which commutes with no other conjugate of z lying in $G_{\alpha\beta}$. Then there is an odd prime power q such that G acts on Ω as a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ in its usual 2-transitive representation.*

As in the proof of Theorem B, Theorem D follows easily from Glauberman's Z^*-theorem [5] and the following result.

Theorem D′. *Let G be a 2-transitive permutation group on a finite set Ω. Suppose that G contains a 2-involution and that for α in Ω, G_α contains a normal subgroup Q of odd order and Q is transitive on $\Omega-\alpha$. Then there is an odd prime power q such that G acts on Ω as a subgroup of $P\Gamma L(2,q)$ containing $PSL(2,q)$ in its usual 2-transitive representation.*

Proof of Theorem D′. Let G be a minimal counterexample to Theorem D′. Let $\alpha,\beta\in\Omega$, $\alpha\neq\beta$, and let z be a 2-involution in $G_{\alpha\beta}$. Then $G_{\alpha\beta}$ contains a Klein group $\langle t,z\rangle$ (Hering [8]). Also, the Feit-Thompson theorem [4] implies that Q is solvable. Clearly, $Q=C_Q(z)\,C_Q(t)\,C_Q(tz)$, and, since z is a 2-involution, $C_Q(z)\leqq Q_{\alpha\beta}$.

We first show that neither t nor tz is a 2-involution. For if t is a 2-involution, then $C_Q(t) \leqq Q_{\alpha\beta}$ and $Q = Q_{\alpha\beta} C_Q(tz)$. Then $C_Q(tz)$ is transitive on $\Omega - \alpha$ and $|\Delta(tz)| = |\Omega|$, a contradiction. Similarly, tz is not a 2-involution.

As in § 3, Lemma 5, $C(t)^{\Delta(t)}$ and $C(tz)^{\Delta(tz)}$ are 2-transitive and $C_Q(t)^{\Delta(t)-\alpha}$ and $C_Q(tz)^{\Delta(tz)-\alpha}$ are transitive. Moreover, $z^{\Delta(t)}$ and $z^{\Delta(tz)}$ are 2-involutions. By the minimality of G, $C_Q(t)^{\Delta(t)}$ contains a minimal normal subgroup of $C(t)_\alpha^{\Delta(t)}$ which is transitive on $\Delta(t) - \alpha$; a similar statement holds for $C_Q(tz)^{\Delta(tz)}$.

Let A be a minimal normal subgroup of G_α contained in Q. Then A is an elementary abelian p-group. By a recent result of O'Nan [12] A is semiregular on $\Omega - \alpha$, so that z inverts A and $A = C_A(t)\, C_A(tz)$. Precisely as in the proof of Theorem B, we may assume that $A = C_A(t)$ is regular on $\Delta(t) - \alpha$ and $C_A(tz) = 1$.

We claim that $O_{p'}(Q) = 1$. For otherwise, let B be a minimal normal subgroup of G_α contained in $O_{p'}(Q)$. As in the preceding paragraph, we find that $C_B(tz)$ is regular on $\Delta(tz) - \alpha$. Then $C_Q(t) = AC_Q(t)_\beta$ and $C_Q(tz) = BC_Q(tz)_\beta$ imply that $Q = C_Q(z)\, AC_Z(t)_\beta BC_Q(tz)_\beta = ABQ_{\alpha\beta}$, so that AB is transitive on $\Omega - \alpha$. Also, it is easy to see that

$$|AB| = (|\Delta(t)| - 1)(|\Delta(tz)| - 1) = n - 1,$$

so that AB is regular on $\Omega - \alpha$. Since $AB \trianglelefteq G_\alpha$, [9] yields a contradiction. Thus, $O_{p'}(Q) = 1$.

We next claim that $A = O_p(Q)$. For suppose that L/A is a minimal normal subgroup of G_α/A with $L \leqq Q$ and L/A a p-group. First assume that $C_L(tz) > C_L(tz)_{\alpha\beta}$. Then $C_L(tz)^{\Delta(tz)} \neq 1$ and $C_L(tz)^{\Delta(tz)} \trianglelefteq C(tz)_\alpha^{\Delta(tz)}$, so that $C_L(tz)^{\Delta(tz)}$ is transitive on $\Delta(tz) - \alpha$. As $C_L(z) = C_L(z)_{\alpha\beta}$, it follows that $Q = LQ_{\alpha\beta}$ and L is transitive on $\Omega - \alpha$. By Theorem C', this contradicts the minimality of G. Thus, $C_L(tz) = C_L(tz)_{\alpha\beta}$ and $L = C_L(t)\, C_L(tz)\, C_L(z) = AL_{\alpha\beta}$. As A is semiregular on $\Omega - \alpha$, $A \cap L_{\alpha\beta} = 1$. Since $A \leqq Z(L)$, $L = A \times L_{\alpha\beta}$. Again a result of O'Nan [12] yields a contradiction.

Set $H = Q\langle z \rangle$, so that H is solvable, $O_{p'}(H) = 1$, $O_p(H) = A$, and consequently $A = C_H(A)$ ([7], Lemma 1.2.3). Since z inverts A, $zA \in Z(H/A)$. Therefore, $H = AC_H(z)$ and $Q = AC_Q(z) \leqq AQ_{\alpha\beta}$. Thus, A is regular on $\Omega - \alpha$, again contradicting [9].

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