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NONDESARGUESIAN PLANES, PARTIAL GEOMETRIES, STRONGLY REGULAR GRAPHS AND CODES ARISING FROM HYPERBOLIC QUADRICS

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ABSTRACT

Spreads of hyperbolic quadrics are used to construct translation planes, partial geometries, strongly regular graphs and codes, all having a rich geometric structure.

1. INTRODUCTION

A hyperbolic quadric Q^+ in $PG(2n-1,q)$ has $(q^n-1)(q^{n-1}+1)/(q-1)$ points, and contains n-1-spaces each having $(q^n-1)/(q-1)$ points. A *spread* of Q^+ is a family Σ of $q^{n-1}+1$ subspaces of Q^+ of dimension n-1 partitioning the points of Q^+. A spread can only exist if n is even (when n is odd, Q^+ does not even contain 3 pairwise disjoint n-1-spaces). The only known examples occur when n=2, when q is even, or when n=4 and $q \equiv 0$ or $2(mod 3)$.

In this paper, we will summarize some of the ways spreads of hyperbolic quadrics have been used recently.

2. TRANSLATION PLANES AND THE CONSTRUCTION OF SPREADS

If X is any subspace of $PG(2n-1,q)$, then $X^\perp$ will denote its polar with respect to Q^+. Consider a spread Σ of Q^+, and assume that q is even. Fix any point $x \notin Q^+$, and form the family $x^\perp \cap \Sigma = \{x^\perp \cap W | W \in \Sigma\}$; clearly, this consists of n-2-spaces partitioning the quadric $x^\perp \cap Q^+$. Since q is even, $x \in x^\perp$ and $x^\perp/x$ is a $PG(2n-3,q)$ equipped with a symplectic (or null) polarity. If $\Sigma(x)$ denotes the projection of $x^\perp \cap \Sigma$ into $x^\perp/x$, then $\Sigma(x)$ is a spread in the more usual sense: $q^{n-1}+1$ subspaces of dimension n-2 which partition the points of $x^\perp/x$. Consequently Σ and x determine a translation plane $A(\Sigma(x))$ arising from a symplectic spread.

Conversely, suppose that n and q are even, and that Σ' is a symplectic spread of $PG(2n-3,q)$. Then Σ' arises from some Q^+, Σ and x, as follows (Dillon [4], Dye [5]). We can regard $PG(2n-3,q)$ as our former $x^\perp/x$, related to Q^+ as before. Each of the $q^{n-1}+1$ members of Σ' is the projection of a unique n-2-space of $x^{\perp} \cap Q^{+}$, and the resulting family Σ'' of n-2-spaces partitions $x^{\perp} \cap Q^{+}$. Fix one of the two families of n-1-spaces of Q^{+}; each n-2-space of Q^{+} is in a unique member of this family. Thus, Σ'' lifts to a family Σ of $q^{n-1}+1$ of these n-1-spaces, any two having at most a point in common. Since no two of these n-1-spaces can have exactly one point in common (as n is even), Σ consists of pairwise disjoint subspaces. Consequently, Σ is a spread of Q^{+}.

The preceding construction produced an essentially unique spread of Q^{+} from a symplectic spread of $x^{\perp}/x$. In particular, $\Sigma(x)$ essentially determines Σ (where "essentially" means that the symplectic geometry on $x^{\perp}/x$ can arise from several quadrics in $PG(2n\text{-}1,q)$, and that we singled out one of the two families of n-1-spaces of Q^{+}). However, more is true. Let Σ and Σ_1 be spreads of Q^{+}, and let x and x_1 be points off Q^{+}. Then any isomorphism from $A(\Sigma(x))$ to $A(\Sigma_1(x_1))$ induces a collineation of $PG(2n\text{-}1,q)$ sending Q^{+} to itself, Σ to Σ_1 and x to x_1 (Kantor [6]). In particular, each collineation of $A(\Sigma(x))$ is the product of a perspectivity with axis at infinity and a collineation of $PG(2n\text{-}1,q)$ preserving Q^{+}, Σ and x. Thus, the collineation groups of many translation planes can be found simultaneously once the group $G(\Sigma)$ of collineations preserving Q^{+} and Σ is known.

Several spreads Σ are described in Kantor [6,7], and some of the resulting translation planes of order q^{n-1} are studied in detail. When n=2, or n=4 and q=2, only desarguesian planes occur. In all other cases, new planes are obtained. Some of the planes are uninteresting, but others have collineation groups behaving in unusual manners (such as flag-transitively, or behaving as decribed in Johnson's paper in these proceedings). Complete arcs and dual ovals in all of these planes were found by Thas [13].

The simplest example of a hyperbolic spread is obtained as follows. The desarguesian plane $AG(2,q^{n-1})$ arises from the unique spread of $PG(1,q^{n-1})$, which is trivially symplectic. This produces a symplectic spread Σ' in $PG(2n\text{-}3,q)$, and hence (if n and q are even) a spread Σ of a hyperbolic quadric in $PG(2n\text{-}1,q)$. This spread is called *desarguesian*, for obvious reasons. It was found by Dillon [4] and Dye [4]; $G(\Sigma)$ was determined by Dye [5] and Cohen and Wilbrink [2]. As we will see in §6, an "affine" version of this spread was discovered much earlier by Kerdock [10].

Other hyperbolic spreads arise from the hermitian curve in $PG(2,q^{2})$, from triality, and from field changes generalizing that of the preceding paragraph.

For coordinate descriptions of many of the known examples we refer to Kantor[6,7].

It seems unlikely that hyperbolic spreads can exist when $n > 4$ and q is odd. On the other hand, if q is even and fixed, the number of inequivalent hyperbolic spreads probably $\to\infty$ as $n\to\infty$.

3. MORE TRANSLATION PLANES; OVOIDS

Let Q^+ be a hyperbolic quadric in $PG(7,q)$, where q is now even or odd. Consider a spread Σ of Q^+. Then Σ belongs to one of the families of 3-spaces of Q^+. Let T be any member of that family not in Σ. If $W \in \Sigma$ then $T \cap W$ is either empty or a line. Set $T \cap \Sigma = \{T \cap W | W \in \Sigma$ and $T \cap \Sigma$ is a line$\}$. Then $T \cap \Sigma$ is a spread of $PG(3,q)$, and hence determines a translation plane of order q^2. (The translation planes obtained from Σ as in §2 when q is even have order q^3, not q^2.) These translation planes are studied in Kantor [8].

Let τ be a triality map. Then τ cyclically permutes the following three sets: the points of Q^+, and the two families of 3-spaces on Q^+. At the same time, τ sends lines of Q^+ to lines of Q^+, while preserving incidence between lines and both points and 3-spaces of Q^+. It follows that Σ^τ is either another spread, or else consists of q^3+1 pairwise noncollinear points of Q^+.

Let Q^+ be a hyperbolic quadric in $PG(2n-1,q)$. An *ovoid* of Q^+ is defined (by Thas [14]) to be a set Ω of q^{n-1}+1 pairwise noncollinear points of Q^+. A simple count shows that each n-1-space on Q^+ contains a unique point of Ω. It follows that, if $x \in Q^+ - \Omega$, then $x^\perp \cap \Omega$ projects onto an ovoid $\Omega(x)$ of the obvious quadric in $x^\perp/x$.

If $n > 4$, ovoids probably do not exist, but this has only been proven in $PG(2n-1,2)$ (Kantor [9, (4.3)]). If n=3, ovoids correspond (under the Klein correspondence) to spreads in $PG(3,q)$, and hence to translation planes of order q^2.

If n=4, Σ is a spread, and Σ^τ is not a spread, then Σ^τ is an ovoid Ω. Moreover, if T is as before then T^τ is a point x, and $T \cap \Sigma$ and $\Omega(x)$ are related by the Klein correspondence. When dealing with coordinates, it is easier to work with $\Omega(x)$ than $T \cap \Sigma$, since points of Ω require 8 coordinates while 3-spaces in Σ are more complicated to describe. The translation planes $A(\Omega(x))$ are studied for all known Σ in Kantor [8]. Unlike the situation in §2, it is not clear how to recover Ω or Σ from $A(\Omega(x))$; moreover, some collineations of the plane need not be related to automorphisms of Ω or Σ.

4. STRONGLY REGULAR GRAPH

Let Σ be a spread of a hyperbolic quadric Q^+ in $PG(2n-1,q)$, where $n \geqslant 4$. Let Ω be the set of all hyperplanes of members of Σ, so that $|\Omega|=|\Sigma|(q^n-1)/(q-1)$ is the number of points of Q^+. If X, $Y \in \Omega$ and $X \neq Y$, write $X{\sim}Y \Leftrightarrow X^\perp \cap Y \neq 0$. Then $(\Omega,\sim)$ is a strongly regular graph having the same parameters as the collinearity graph $(Q^+,\perp)$ of Q^+ (Kantor [9]). If $n=4$, these graphs are isomorphic (an isomorphism being induced by triality). However, if $n > 4$ they are probably never isomorphic. This is known if $q=2$ (see [9]); the proof uses the nonexistence of ovoids of Q^+ when $q=2$. The graphs are also not isomorphic when Σ is the desarguesian spread defined in §2; this was proved in [9] by brute-force calculations.

If $W \in \Sigma$ let W^* be its set of hyperplanes. Then $\Sigma^* = \{W^* | W \in \Sigma\}$ is a partition of Ω into cliques. If $X \in \Omega - W^*$ then X is joined to exactly $(q^{n-1}-1)/(q-1)$ members of W^*, namely, $\{Y \in W^* | X^\perp \cap W \in Y\}$. Thus, each clique W^* inherits from $(\Omega,\sim)$ the structure of a $PG(n-1,q)$ in a natural manner. Further properties of $(\Omega,\sim)$ reminiscent of properties of $(Q^+,\perp)$ are found in [9]. While it is easy to reconstruct Q^+ and Σ given $(\Omega,\sim)$ and Σ^*, it is not known whether Σ^* can be determined from $(\Omega,\sim)$ alone.

Variations on the construction of $(\Omega,\sim)$ are found in [9]. Instead of a spread of a hyperbolic quadric, spreads of other quadrics, of hermitian geometries, and of symplectic geometries could have been used: strongly regular graphs again arise in exactly the same manner, having the same parameters as the underlying geometries. In the symplectic case, new symmetric designs are also obtained having the parameters of $PG(2n-1,q)$.

5. PARTIAL GEOMETRIES

In this section, Q^+ and Σ will be as before, but q will be 2 or 3. Define Ω as in §4.

If $q=2$, let N be the complement of Q^+. If $q=3$, let N consist of the points of the complement of "length" 1 (where "length" refers to the value at x of the quadratic form defining Q^+).

It is straightforward to check that the incidence structure $(N,\Omega,\perp)$ is a partial geometry $pg(\Sigma)$ (DeClerck, Dye, Thas [3,15]). Namely, if $x \in N$ and $X \in \Omega$, then x is perpendicular to $q^{n-1}+1$ members of Ω (one per member of Σ), X is perpendicular to $q^{n-1}+1$ members of N (lying in an affine space), and (if $x \notin X^\perp$) there are exactly $q^{n-2}(q-1)$ pairs (y,Y) with $y \in N$, $Y \in \Omega$, $y \in X^{\perp} \cap Y^{\perp}$ and $x \in Y^{\perp}$.

Let x, $y \in N$, $x \neq y$. Then x, $y \in X^{\perp}$ for some $X \in \Omega$ if and only if $q=2$ and x and y are perpendicular, or $q=3$ and x and y are not perpendicular. Consequently, $PG(2n-1,q)$ and Q^{+} can be recovered from $pg(\Sigma)$. Moreover, projectively inequivalent spreads Σ produce nonisomorphic partial geometries (Kantor [9, (7.3)]). If $n=4$ then Σ is essentially unique (cf. Patterson [12]; Kantor [6, §10]), and hence so is $pg(\Sigma)$. If $q=2$ and $n-1$ is composite then Σ is not unique (Kantor [7,(9.12)]); presumably, the same is true for all even $n > 4$. On the other hand, if $q=3$ and $n > 4$ it seems likely that no Σ exists.

Let X, $Y \in \Omega$, $X \neq Y$. Then X, $Y \in x^{\perp}$ for some $x \in N$ if and only if X and Y are not adjacent in the graph $(\Omega,\sim)$ obtained in §4.

6. CODES

Let Q^{+} and Σ be as usual. In order to define some codes, we will have to introduce coordinates. The vector space underlying Q^{+} can be written $E \oplus F$ with E, $F \in \Omega$. Fix a basis $e_1,\ldots,e_n$ of E, and let $f_1,\ldots,f_n$ be the dual basis of F, so that the underlying bilinear form has $(e_i,f_j) = \delta_{ij}$. We will write matrices with respect to the ordered basis $e_1,\ldots,e_n$, $f_1,\ldots,f_n$.

Let $F' \in \Omega - \{E\}$. Then

$$F' = F \begin{pmatrix} I & O \\ M' & I \end{pmatrix} \qquad (*)$$

for a uniquely determined skew symmetric nXn matrix M' (with zero diagonal), where 0 and I are the zero and identity nXn matrices. (The above $2n$X$2n$ matrix preserves Q^{+}, and is the identity on E.)

Thus, Σ and E determine a family $K=K(\Sigma,E)$ of skew symmetric nXn matrices. This provides an "affine" version of Σ. If F', $F'' \in \Sigma - \{E\}$, $F' \neq F''$, and if M' and M'' are the corresponding matrices, then $F' \cap F'' = 0$ implies that $M'-M''$ is nonsingular. Thus, K is a *Kerdock set*: a set of q^{n-1} skew symmetric nXn matrices (with zero diagonal) such that the difference of any two is nonsingular.

Conversely, any Kerdock set of nXn matrices defines a hyperbolic spread via (*). Note that this spread is uniquely determined; thus, inequivalent spreads produce different Kerdock sets. Note, however, that different Kerdock sets can produce the same spread. A further discussion of equivalence of Kerdock sets can be found in Kantor [6, §5].

Now let q=2, and fix a Kerdock set K of nXn matrices. We may assume that $0 \in K$. If $M \in K$, there is a quadratic form $Q_M(x)$ associated with M:

$$xMy^t = Q_M(x+y) - Q_M(x) - Q_M(y)$$

for all $x, y \in Z_2^n$. Consider the following functions $Z_2^n \to Z_2$:

$$Q_M(x) + L(x) + c \qquad (**)$$

for $M \in K$, L a linear functional, and $c \in Z_2$. There are $2^{n-1}\cdot 2^n \cdot 2 = 2^{2n}$ functions. Let $C = C(K)$ be the set of all sets of zeros of the various functions (**). Clearly, C contains all hyperplanes of $AG(n,2)$ as well as $\emptyset$ and Z_2^n.

If $X, Y \in C$, then their symmetric difference $X\Delta Y$ has size 0, 2^n, 2^{n-1}, $2^{n-1} \pm 2^{\frac{1}{2}n}$, corresponding to whether $X\Delta Y$ is $\{0\}$, Z_2^{2n}, an affine hyperplane, or a quadric or its complement in Z_2^{2n} (cf. Cameron and Seidel [1]). If the vectors in Z_2^{2n} are called $v_1,\ldots,v_N$ with $N = 2^{2n}$, then each subset X of Z_2^n can be written $X = \sum_1^N \alpha_i v_i$ with $\alpha_i \in Z_2$, where $\sum_1^N$ refers to symmetric difference. In this way, C can be regarded as an error-correcting code (MacWilliams and Sloane [11]), having length 2^n, minimum distance $2^{n-1}-2^{\frac{1}{2}n}$, and size 2^{2n}. It is extremal in a sense discussed on p. 667 of [11].

The codes $C(K)$ were first discovered by Kerdock [10], with K a Kerdock set arising from the desarguesian hyperbolic spread described in §2. Since inequivalent spreads exist whenever n-1 is composite, inequivalent codes $C(K)$ exist as well. However, from a coding theoretic point of view, it is not clear how different codes $C(K)$ differ (for a fixed n): those corresponding to desarguesian spreads seem as if they should be "best", but it is not clear what this means.

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