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Jordan Groups

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1. INTRODUCTION

In 1871, Jordan [7] initiated the study of primitive permutation groups Γ of finite degree v such that the pointwise stabilizer of some set B of $k \leqslant v - 2$ points is transitive on the remaining points. One of his principal results was that such a group is 2-transitive. Every $(k + 1)$—transitive group trivially satisfies this condition. In the non-trivial case, where Γ is not $(k + 1)$-transitive, Γ is called a Jordan group (this definition differs slightly from that of Hall [4]). Examples of Jordan groups are the full collineation groups of finite projective and affine spaces—with B a subspace—and the Mathieu groups M_{22}, M_{23} and M_{24}. Given the pair (Γ, B), a geometry or design (see Section 2 for a more precise definition of this term) of points and blocks may be obtained as follows: points are the points permuted by Γ, and blocks are the distinct sets B^γ, $\gamma \in \Gamma$. If $\Gamma = M_v$, the resulting design $\mathscr{D}$ was described by Witt [17] and will be denoted $\mathscr{W}_v$. Hall [4] was the first to construct such designs $\mathscr{D}$ from such pairs (Γ, B). He observed that, with little loss in generality, it may be assumed that Γ is not 3-transitive and that every two distinct points are on a unique block (i.e., blocks behave like lines). When supplemented by a loop theoretic argument of Bruck (see [2], p. 100), Hall's main result is that, if $k = 3$ and Γ is not 3-transitive, then $\mathscr{D}$ is the design of points and lines of $PG(d, 2)$ or $AG(d, 3)$, for some d.

We shall obtain further characterizations of Jordan groups and their designs, under weaker assumptions than those given above. Our approach will be more geometric than Hall's combinatorial approach. The main tools will be induction and the enumeration of fixed points and blocks of elements of Γ. Whereas Hall was interested only in the case where two distinct points are on a unique block, we shall be mostly interested in the case where this is not so. For, there is then more structure available, as in the classical instance of higher dimensional geometries having more structure than 2-dimensional ones.

The main results are summarized in terms of groups in the following

THEOREM 1.1. *Let Γ be a 2-transitive group of degree v. Suppose that, for some set B of k points, $2 \leqslant k \leqslant v-2$, the pointwise stabilizer $\Gamma(B)$ of B is transitive on the set $\mathscr{C}B$ of remaining points. Let Γ_B be the global stabilizer of B. Then Γ is a known group provided that one of the following conditions holds.*

(i) $v \leqslant 6k$.
(ii) *Γ is not k-transitive, and $\Gamma(B)$ has a 2-subgroup transitive on $\mathscr{C}B$ which is either elementary abelian or normal in Γ_B.*
(iii) *Γ is not k-transitive, and Γ_B is 2-transitive on $\mathscr{C}B$.*

Here (i) and (ii) should be compared with results on Jordan groups of Marggraf ([*16*], 34-35) and Nagao and Oyama [*9a*], respectively.

Section 2 consists of definitions. In Section 3, the basic geometric properties of the associated designs are given. The results of this section are probably known, in a non-geometric form. Although additional transitivity properties may be readily obtained by similar methods, our results are stated principally for later applications.

In Section 4, we briefly describe a geometric generalization of Jordan groups. As the proofs are very similar to those in Section 3, or readily reduce to these previous results, they will not be given. This section is not required in the proof of Theorem 1.1. We note, however, that (3.i) *and* (4.i) *state essentially the same facts*, $1 \leqslant i \leqslant 18$.

Sections 5 and 6 are highly geometric. We describe the designs and groups to be characterized. In order that later characterizations of designs may be translated into characterizations of groups, results are given concerning collineation groups of finite projective and affine spaces which follow very easily from results of Wagner [*15*]. Section 6 is basic to our approach, containing the necessary inductive results. In particular, some natural attempts to generalize the Mathieu group M_{22} are shown to yield nothing new.

Sections 7 and 8 contain the proof of Theorem 1.1.

2. DEFINITIONS

The following definition of designs is not the usual one [*2*], but is well-suited for our purposes: a design $\mathscr{D}$ is a set of v points, together with b subsets, called blocks, such that each block has k points, $2 \leqslant k \leqslant v-2$, each point is on $r < b$ blocks, and every two points are on $\lambda < r$ blocks. $\mathscr{D}$ is called degenerate if every set of k points is a block, and non-degenerate otherwise. If x and y are distinct points of $\mathscr{D}$, the line xy is the intersection of the blocks on x and y; there is a unique line containing two distinct points.

The intersection of all the blocks containing 3 non-collinear points which are contained in some block is called a plane. A planar design is a non-degenerate design in which any triple of non-collinear points is contained in a unique block.

If $\mathscr{D}$ is a design, set $\mu = \max\{| X \cap Y | \mid X$ and Y are distinct blocks of $\mathscr{D}\}$. A flat is an intersection $X \cap Y$ with μ points (this differs from the definition given in [*6*]). Clearly $\lambda = 1$ if and only if $\mu = 1$.

A subblock of $\mathscr{D}$ is a non-empty intersection of a non-empty set of blocks; examples are points, lines, planes, flats and blocks.

If S is a set of points of $\mathscr{D}$, $\mathscr{C}S$ is the set of remaining points.

If a design is denoted, for example, by $\mathscr{D}^*$ or $\mathscr{D}^{(0)}$, its parameters are $v^*, k^*, \mu^*,\ldots$, resp. $v^{(0)}, k^{(0)}, \mu^{(0)},\ldots$.

An automorphism group of an incidence structure is called 2-transitive if it is 2-transitive on points.

An automorphism of a design is said to fix a set S of points blockwise if it fixes every block $\supset S$.

If Γ is a permutation group and S is a subset of the set of permuted points, then Γ_S and $\Gamma(S)$ are, respectively, the global and pointwise stabilizers of S. $\Gamma_S|_S \approx \Gamma_S/\Gamma(S)$ is the permutation group induced by Γ_S on S.

All groups will be finite. Isomorphic designs will be regarded as identical, as will similar permutation groups.

3. Jordan Pairs

The following simple result will be used frequently.

Lemma 3.1. (*Wielandt* [*16*], p. 7). *Let Γ be a transitive group on a set S, and let $B_i \subset S$ such that $\Gamma(B_i)$ is transitive on $S - B_i$, $i = 1, 2$. If $| B_1 | \leqslant | B_2 |$, then $B_1^\gamma \subseteq B_2$ for some $\gamma \in \Gamma$.*

We next restate the connection between the groups and designs to be considered which was indicated in the Introduction.

Lemma 3.2. *Let Γ be a 2-transitive group of degree v such that, for some set B of k points, $2 \leqslant k \leqslant v - 2$, $\Gamma(B)$ is transitive on the remaining points. If $\mathscr{D}$ consists of the points permuted by Γ and the distinct sets B^γ, $\gamma \in \Gamma$, then $\mathscr{D}$ is a design with 2-transitive and block-transitive automorphism group Γ such that, for each block X, $\Gamma(X)$ is transitive on $\mathscr{C}X$. Moreover, $\mathscr{D}$ is non-degenerate if and only if Γ is not k-transitive.*

Proof. This is clear from the definition of designs, except possibly for the fact that Γ is k-transitive if $\mathscr{D}$ is degenerate. However, if $\mathscr{D}$ is degenerate, then by induction the pointwise stabilizer of each set of $k - i$ points is $(i + 1)$-transitive on the remaining points, for every $0 \leqslant i \leqslant k$.

If Γ and $\mathscr{D}$ are as in Lemma 3.2, $(\mathscr{D}, \Gamma)$ is called a Jordan pair, and is said to be non-degenerate if and only if $\mathscr{D}$ is. The study of such pairs is, by Lemma 3.2, equivalent to the study of the pairs (Γ, B) which were considered in the Introduction.

Let $(\mathscr{D}, \Gamma)$ be a non-degenerate Jordan pair. We shall list a number of properties of $(\mathscr{D}, \Gamma)$, many of which even hold in the degenerate case. μ was defined in Section 2.

(3.1) $$\mu \leqslant k - 2.$$

Proof. Let X and Y be blocks with $|X \cap Y| = k - 1$. As $\Gamma(X)$ moves $Y - X \cap Y$ to all the points $\notin X$, while $\Gamma(Y)$ moves $X - X \cap Y$ to all the points not in Y, $\Gamma(X \cap Y)$ is 2-transitive on $\mathscr{C}(X \cap Y)$, and Γ is k-transitive by Jordan's theorem ([7]; [*16*], p. 32).

By a result of Marggraf ([*16*], p. 35),

(3.2) $$v \geqslant 2k.$$

(3.3) All lines have the same number h of points.

(3.4) Let $\mu > 1$, let B be a block, and let $\mathscr{D}(B)$ consist of the points of B as points and the flats in B as blocks. Then $\Gamma_B|_B$ acts as an automorphism group of $\mathscr{D}(B)$, and $(\mathscr{D}(B), \Gamma_{B|B})$ is a possibly degenerate Jordan pair with $v(B) = k$ and $k(B) = \mu$.

Proof. $\Gamma_{B|B}$ is 2-transitive by Wielandt ([*16*], p. 36). If X is a block such that $X \cap B$ is a flat, then $\Gamma(X)_B$ is transitive on $B - B \cap X$ by the definition of μ. Then Lemma 3.1 shows that $\Gamma_{B|B}$ is transitive on the flats in B. Clearly $v(B) = k$ and $k(B) = \mu$, and $2 \leqslant h \leqslant k(B) \leqslant v(B) - 2$ by (3.1), (3.3) and $\mu > 1$.

(3.5) If S is a subblock then $\Gamma(S)$ is transitive on $\mathscr{C}S$.

Proof. Clearly $S = \cap \{X \mid X \supseteq S, X \text{ is a block}\}$. If X and Y are such blocks, then $|\mathscr{C}X \cap \mathscr{C}Y| = v - 2k + |X \cap Y| > 0$ by (3.2), so that $\Gamma(S)$ is transitive on $\cup \{\mathscr{C}X \mid X \supseteq S\} = \mathscr{C}S$. If S is a line, (3.5) is essentially due to Hall [*4*]. (3.5) implies

(3.6) If $\mu > 1$, every 3 non-collinear points are on a unique plane. $h = 2$ if and only if Γ is 3-transitive.

(3.7) Let $\mu > 1$, let p be a point, and let $\mathscr{D}[p]$ consist of the lines through p as points and the blocks through p as blocks. Γ_p induces an automorphism group $\hat{\Gamma}_p$ of $\mathscr{D}[p]$. Then $(\mathscr{D}[p], \hat{\Gamma}_p)$ is a non-degenerate Jordan pair with $v[p] = (v - 1)/(h - 1)$, $k[p] = (k - 1)/(h - 1)$ and $\mu[p] = (\mu - 1)/(h - 1)$.

Proof. $\hat{\Gamma}_p$ is 2-transitive by (3.5) applied to a line. As Γ_B is transitive on B, $\hat{\Gamma}_p$ is block-transitive. If B is on p, then $\Gamma_p(B)$ is certainly transitive on $\mathscr{C}B$. The values of $v[p]$, $k[p]$ and $\mu[p]$ are clear from (3.3). $k[p] \geqslant 2$ as $k > \mu \geqslant h$. If $h = 2$ then $v[p] \geqslant k[p] + 2$. If $h > 2$ and $p' \in B - \{p\}$, joining p' with the points $\neq p$ of a line $\not\subseteq B$ through p shows that $v[p'] \geqslant k[p'] + 2$.

If $\mathscr{D}[p]$ is degenerate, then $h[p] = 2$ implies that planes of $\mathscr{D}$ have only 3 points and thus $h = 2$. Then $k[p] = k - 1$, and the $k[p]$-transitivity of $\hat{\Gamma}_p$ implies the k-transitivity of Γ, a contradiction. By (3.5), (3.6) and (3.7) we have

(3.8) If $\mu > 1$, let $\mathscr{D}^{\#}$ consist of the points and planes of $\mathscr{D}$. Then $(\mathscr{D}^{\#}, \Gamma)$ is a Jordan pair, and the following are equivalent: (i) $\mathscr{D}^{\#}$ is degenerate; (ii) $h[p] = 2$; (iii) planes have 3 points; and (iv) Γ is 4-transitive.

(3.9) If S is a subblock properly contained in a block B, then S is an intersection of flats in B, that is, S is a subblock of $\mathscr{D}(B)$ if $\mu > 1$.

Proof. Use induction on μ : if $\mu = 1$ this is trivial. Let $\mu > 1$ and let $p \in S$. Then S is a subblock of $\mathscr{D}[p]$ properly contained in the block B of $\mathscr{D}[p]$. As $\mu[p] < \mu$, S is an intersection of flats of $\mathscr{D}[p]$ contained in B. Since each such flat is the set of lines of $\mathscr{D}$ through p contained in a flat of $\mathscr{D}$, the result follows.

(3.10) The set of subblocks of $\mathscr{D}$ may be partitioned into d classes, the members of the ith class being called i-subblocks ($0 \leqslant i \leqslant d - 1$), such that the following statements hold.

- (i) 0-subblocks are points, 1-subblocks are lines and, if $\mu > 1$, 2-subblocks are planes.
- (ii) For each i, Γ is transitive on i-subblocks.
- (iii) If S_i is an i-subblock, not a block, and $x \notin S_i$, then there is a unique $(i + 1)$-subblock containing S_i and x.
- (iv) If $i \geqslant 2$, and $\mathscr{D}(S_i)$ consists of the points of S_i as points and the $(i - 1)$-subblocks $S_{i-1} \subset S_i$ as blocks, then either $(\mathscr{D}(S_i), \Gamma_{S_i|S_i})$ is a possibly degenerate Jordan pair or $|S_i| = |S_{i-1}| + 1$.
- (v) If S_i is neither a block nor a flat, let $\mathscr{D}[S_i]$ consist of the $(i + 1)$-subblocks $\supset S_i$ as points and the blocks $\supset S_i$ as blocks. $\Gamma(S_i)$ induces an automorphism group $\hat{\Gamma}(S_i)$ of $\mathscr{D}[S_i]$ and $(\mathscr{D}[S_i], \hat{\Gamma}(S_i))$ is a nondegenerate Jordan pair.

Proof. This holds trivially or vacuously if $\mu = 1$ or (except for (v)) if $\mathscr{D}$ is degenerate. Let $\mu > 1$ and use induction on μ. The i-subblocks of the various $\mathscr{D}(B)$ are defined to be i-subblocks of $\mathscr{D}$, and we add an additional class, namely, the blocks of $\mathscr{D}$. By Lemma 3.1 and (3.9), we obtain in this way all subblocks of $\mathscr{D}$. (i) is then trivial. Γ is block-transitive, and $\Gamma_{B|B}$ is transitive on the i-subblocks $\subset B$, proving (ii).

(iii) By (3.5) and Lemma 3.1, S_i is contained in a flat $F = X \cap Y$. Either $x \in X$, or $x \notin X$ and $\Gamma(X)$ has an element moving Y to a block containing x and F. Thus there is a block B containing x and S_i. Either S_i is a block of $\mathscr{D}(B)$, and then $B = S_{i+1}$ is an $(i+1)$-subblock, or induction produces an $(i+1)$-subblock S_{i+1} containing x and S_i. If S'_{i+1} also contains x and S_i, then $S_{i+1} \cap S'_{i+1}$ is a subblock and

$$1 + |S_i| \leqslant |S_{i+1} \cap S'_{i+1}| \leqslant |S_{i+1}|.$$

However, by (iii) applied to $\mathscr{D}(B)$, $|S_{i+1}| > |S_i|$ and there is no subblock T with $|S_{i+1}| > |T| > |S_i|$. Thus we must have $S'_{i+1} = S_{i+1}$.

(iv) By Lemma 3.1, or (iii), $S_i \subseteq B$ for some block B. If $S_i = B$, (3.4) may be used, while if $S_i \subset B$, by (3.9) we may use induction.

(v) If $S_i \subset B$, then (iii) implies that $\Gamma(B)$ is transitive on the $(i+1)$-subblocks $\supset S_i$ and $\not\subset B$. If $S_{i+1} \supset S_i$, (3.5) and (iii) imply that $\Gamma(S_{i+1})$ is transitive on the $(i+1)$-subblocks $\neq S_{i+1}$ containing S_i, so that $\hat{\Gamma}(S_i)$ is 2-transitive. An application of Lemma 3.1 to $\hat{\Gamma}(S_i)$ yields the block-transitivity of $\hat{\Gamma}(S_i)$.

By (3.2),

$$\begin{aligned} v[S_i] &= (v - |S_i|)/(|S_{i+1}| - |S_i|) > (2k - 2|S_i|)/(|S_{i+1}| - |S_i|) \\ &= 2k[S_i],. \end{aligned}$$

As S_{i+1} is not a block, $k > |S_{i+1}|$ and thus $k[S_i] \geqslant 2$. Thus, $(\mathscr{D}[S_i], \hat{\Gamma}(S_i))$ is a Jordan pair. If $i = 0$ this is non-degenerate by (3.7). If $i > 0$ let $S_{i-1} \subset S_i$. Then S_i is a point of $\mathscr{D}[S_{i-1}]$, and $\mathscr{D}[S_i] = (\mathscr{D}[S_{i-1}])[S_i]$. Thus, if $\mathscr{D}[S_{i-1}]$ is non-degenerate then so is $\mathscr{D}[S_i]$ by (3.7).

(3.11) Set $\mathscr{D}^{(-1)} = \mathscr{D}$ and $\mathscr{D}^{(i)} = \mathscr{D}[S_i]$ if S_i is as in (3.10v). There is an integer $m \geqslant 0$ such that $h^{(i)} = 2$ if $i \leqslant m - 2$ and $h^{(i)} > 2$ if $i \geqslant m - 2$. Γ is $(m+2)$- but not $(m+3)$-transitive.

Proof. If $h^{(i)} = 2$ for all i then the same is true for $\mathscr{D}^{(0)}$, and induction shows that $\mathscr{D}$ is degenerate, a contradiction. Thus, m exists. If $m = 0$ then Γ is not 3-transitive by (3.6). If $m > 0$, it is clear that for $\mathscr{D}^{(0)}$ we have $m^{(0)} = m - 1$. As $h = 2$, induction shows that Γ is $(m+2)$- but not $(m+3)$-transitive.

m has several other useful properties.

(3.12) $|S_i| - |S_{i-1}| > 1$ for $i \geqslant m + 1$ and $|S_i| = i + 1$ for $i \leqslant m$.

Proof. If $\mu = 1$ or $\mathscr{D}$ is degenerate this is clear. Let $\mu > 1$. If $m = 0$ then $h^{(i)} > 2$ for each i, (3.12) holds for $\mathscr{D}(B)$, and so $|S_i| - |S_{i-1}| > 1$ by induction, (3.9) and (3.1). If $m > 0$ then $m^{(0)} = m - 1$ as before. Also, $h = 2$, so an i-subblock of $\mathscr{D}^{(0)}$ has one less point than an i-subblock of $\mathscr{D}$, and induction completes the proof.

(3.5), (3.1), and (3.12) imply the following generalization of (3.6) and (3.8).

(3.13) If $i > 0$ is fixed, the points and i-subblocks of $\mathscr{D}$ are the points and blocks of a design $\mathscr{D}_{(i)}$ such that $(\mathscr{D}_{(i)}, \Gamma)$ is a Jordan pair, which is degenerate if $i \leqslant m$ and non-degenerate if $i \geqslant m + 1$.

(3.14) $m + 1 \leqslant \mu$, with equality if and only if $\mathscr{D}(B)$ is degenerate.

Proof. By (3.12), $|S_m| = m + 1$. $m + 2 < k$ by (3.11). Thus $m + 1 \leqslant \mu$. If $m + 1 = \mu$ then the $(\mu + 1)$-transitivity of Γ implies that $\Gamma_B|_B$ is also $(\mu + 1)$-transitive, as claimed. Conversely, if Γ is $(\mu + 1)$-transitive, then $m + 2 \geqslant \mu + 1$ by (3.11).

(3.15) $\Gamma_{S_{m+1}}|_{S_{m+1}}$ is $(m + 2)$-transitive.

Proof. Γ is $(m + 2)$-transitive by (3.11), $|S_m| = m + 1$ by (3.12), and the assertion follows from (3.10iii).

(3.16) If B is a block and T an $(m + 1)$-subblock such that $B \cap T$ is an m-subblock, then $\Gamma_{BT}|_{B\cap T}$ is the symmetric group.

Proof. By (3.13), $\Gamma_{BB\cap T}|_{B\cap T}$ is the symmetric group, and (3.16) follows from $\Gamma_{BB\cap T} = \Gamma_{BT}\Gamma(B)$.

The following result is clear.

(3.17) Let $\mu > 1$ and $p \notin B$. For $x \in B$ set $x^\varphi = px$. Then φ maps $\mathscr{D}(B)$ isomorphically into $\mathscr{D}[p]$, and $(\mathscr{D}(B)^\varphi, (\hat{\Gamma}_p)_{B^\varphi}|_{B^\varphi})$ is a Jordan pair.

(3.18) If $v > 2k$, then Γ_B is faithful on $\mathscr{C}B$.

Proof. Otherwise, as $\Gamma(\mathscr{C}B) \trianglelefteq \Gamma_B$, $\Gamma(\mathscr{C}B)$ is transitive on B. As $v < 2(v - k)$, Lemma 3.2 and (3.2) imply that Γ is $(v - k)$-transitive, a contradiction.

4. Generalized Jordan Pairs

A generalized Jordan pair $(\mathscr{D}, \Gamma)$ consists of a possibly degenerate design $\mathscr{D}$ and a point- and block-transitive automorphism group Γ such that, for each block B and each $p \in B$, Γ_p is 2-transitive on the lines on p and $\Gamma(B)$ is transitive on the lines on p not in B. Although every Jordan pair is also a generalized Jordan pair, the converse is false (see the following Section). All lines of $\mathscr{D}$ have the same number h of points. We may define $\mathscr{D}[p]$ and $\hat{\Gamma}_p$ as in Section 3, and then $(\mathscr{D}[p], \hat{\Gamma}_p)$ is a Jordan pair. It is straightforward to use this fact to repeat the arguments of Section 3 to prove results analogous to our previous results. For example, $\mathscr{D}(B)$ is defined as before, and (3.4) now states that $(\mathscr{D}(B), \Gamma_{B|B})$ is a generalized Jordan pair. We shall not explicitly state the previous results for generalized Jordan pairs, and will use the following convention.

(3.i) *and* (4.i) *state essentially the same facts*, $1 \leqslant i \leqslant 18$.

5. Collineation Groups

The only non-degenerate designs $\mathscr{D}$ known to admit an automorphism group Γ such the $(\mathscr{D}, \Gamma)$ is a generalized Jordan pair are

(i) $PG_t(d, q)$, the design of points and t-spaces of $PG(d, q)$ $(1 \leqslant t \leqslant d-1)$;
(ii) $AG_t(d, q)$, the design of points and t-spaces of $AG(d, q)$ $(1 \leqslant t \leqslant d-1;\ t \geqslant 2$ if $q = 2)$; and
(iii) $\mathscr{W}_v$, $v = 22$, 23, or 24, the Witt "spaces" characterized by the property that $\mathscr{W}_{24}^{(0)} = \mathscr{W}_{23}$, $\mathscr{W}_{23}^{(0)} = \mathscr{W}_{22}$, and $\mathscr{W}_{22}^{(0)} = PG(2, 4)$ [*17*] (a description of $\mathscr{W}_{22}$ is also found in [*6*]).

If $(\mathscr{D}, \Gamma)$ is a generalized Jordan pair, $\mathscr{D}$ will be called of *known type* provided that $\mathscr{D}$ is either one of the above designs, or if

(iv) $\mathscr{D}$ is the degenerate design $AG_1(d, 2)$, and Γ is a collineation group of $AG(d, 2)$.

If $\mathscr{D} = \mathscr{W}_v$ then Γ is the Mathieu group M_v or $\mathrm{Aut}(M_v)$ (Witt [*17*]). In cases (i) and (ii) (and (iv)), Γ has been determined in only a few instances. Before presenting these instances, we describe the known groups. If $\mathscr{D} = PG_t(d, q)$ the only known possibilities are: Γ contains the little projective group; or $\Gamma \approx A_7$ and $\mathscr{D} = PG_1(3, 2)$ (Wagner [*15*]). In either case, $(\mathscr{D}, \Gamma)$ is a Jordan pair.

If $\mathscr{D} = AG_t(d, q)$, the known possibilities are: Γ contains the group, which will be called $ASL(d, q)$, generated by all elations of $AG(d, q)$; or $\mathscr{D} = AG_t(4, 2)$, $t = 1$ or 2, and Γ is the semidirect product of the translation group of $AG(4, 2)$ and the collineation group A_7 of $PG(3, 2)$. Here $(AG_{d-1}(d, q), ASL(d, q))$ is a Jordan pair if and only if $ASL(d, q)$ contains all homologies of $AG(d, q)$, which is not always the case (cf. [*15*], Lemma 4).

Lemma 5.1. ([*2*]). *If $(\mathscr{D}, \Gamma)$ is a generalized Jordan pair, and $\mathscr{D}$ is an affine or projective plane, then $\mathscr{D}$ is desarguesian and Γ contains all elations.*

LEMMA 5.2. *Let Γ be a collineation group of $AG(d, q)$ such that, for all points p, the collineation group of $PG(d-1, q)$ induced by Γ_p contains $PSL(d-1, q)$. Then $\Gamma \geqslant ASL(d, q)$.*

Proof. $\Gamma|_H$ contains the little projective group of the hyperplane H at infinity. As in Wagner ([*15*], p. 421), Γ contains the translation group of $AG(d, q)$. As each element of Γ_p inducing an elation on $PG(d-1, q)$ is an elation of $AG(d, q)$, the result follows.

THEOREM 5.3. *Let $(\mathscr{D}, \Gamma)$ be a non-degenerate generalized Jordan pair with $\mathscr{D} = PG_t(d, q)$ and $t \geqslant d-4$, or $AG_t(d, q)$ and $t \geqslant d-4$. Then $\Gamma \geqslant PSL(d, q)$ or $ASL(d, q)$, or $\mathscr{D} = PG_1(3, 2)$ and $\Gamma \approx A_7$, or $\mathscr{D} = AG_2(4, 2)$ and Γ is the semidirect product of A_7 and the translation group of $\mathscr{D}$.*

Proof. Wagner ([*15*], Theorems 3 and 4) and Lemmas 5.1 and 5.2.

6. INDUCTIVE PRELIMINARIES

The following axiom system is essentially that of Sasaki [12a].

LEMMA 6.1. *Let $\mathscr{A}$ be a finite set of points, together with certain subsets, called lines and planes. Then $\mathscr{A}$ is an affine space if the following conditions hold.*

- (i) *Any* 2 *distinct points are on a unique line.*
- (ii) *Any* 3 *non-collinear points are on a unique plane.*
- (iii) *The points and lines in each plane form an affine plane.*
- (iv) *If* 3 *pairwise disjoint lines are given,* 2 *pairs of which coplane, then so does the third pair.*
- (v) *There are* 4 *non-coplanar points.*

LEMMA 6.2. *Let $\mathscr{D}$ be a planar design with $k = 6$ such that any 4 non-coplanar points determine a set of 22 points which contains precisely 77 blocks, these points and blocks forming a design $\mathscr{W}_{22}$. Then $\mathscr{D}$ is $\mathscr{W}_{22}$.*

Proof. All lines of $\mathscr{D}$ have $h = 2$ points. The $\mathscr{W}_{22}$ determined by 4 non-coplanar points is clearly unique. Let $\mathscr{D}[p]$ be the set of lines on the point p. As $h = 2$, there is a unique plane containing p and any two lines on p. By the Veblen and Young axioms [*14*], $\mathscr{D}[p] = PG_1(d, 4)$ for some $d \geqslant 2$. If $d = 2$ we are finished. Let $d \geqslant 3$. Let S be a 3-space in $\mathscr{D}[p]$. As $h = 2$, S may be regarded as a set of points of $\mathscr{D}$. Any 3 non-collinear points of S are contained in both a unique plane of S and a unique plane of $\mathscr{D}$ contained in it. The $\mathscr{W}_{22}$ determined by 4 non-coplanar points of $S \cup \{p\}$ is thus $\subset S \cup \{p\}$, and the $\mathscr{W}_{22}$'s contained in this set thus form an incidence structure $\mathscr{D}^*$ such that $\mathscr{D}^*[p'] = PG_2(3, 4)$ for all points p'. The number of such $\mathscr{W}_{22}$'s is then found to be $[(4^4 - 1)/3 + 1][(4^4 - 1)/3]/22$, which is impossible (cf. Hughes [5]).

Let $(\mathscr{D}, \Gamma)$ be a generalized Jordan pair. A subsystem $\mathscr{S}$ of $\mathscr{D}$ is a set of points and blocks such that (i) not all points are on a block, (ii) if X and Y are blocks in $\mathscr{S}$ meeting in a flat, and x is a point in $\mathscr{S}$, $x \notin X \cap Y$, then the block containing $X \cap Y$ and x is in $\mathscr{S}$, and (iii) if a block X is in $\mathscr{S}$ so is each $x \in X$. Clearly an intersection of subsystems satisfies (ii) and (iii). A minimal subsystem $\mathscr{M}$ is one without proper subsystems. If X_i is in $\mathscr{M}$, and $x_i \notin X_i (i = 1, 2)$, and if $\gamma \in \Gamma$ maps X_1 to X_2 and x_1 to x_2, then $\mathscr{M}^\gamma = \mathscr{M}$. Also, the intersection of the blocks of $\mathscr{M}$ on x_1 meeting X_1 in a flat which contains a point $y_1 \in X_1$ is the line $x_1 y_1$. Together with the preceding lemmas, this implies

COROLLARY 6.3. *If $(\mathscr{D}, \Gamma)$ is a generalized Jordan pair with $\mu > 1$, and $\mathscr{M}$ and $\mathscr{M}'$ are minimal subsystems of $\mathscr{D}$, then $(\mathscr{M}, \Gamma_{\mathscr{M}|\mathscr{M}})$ is a generalized Jordan pair, and there is an element $\gamma \in \Gamma$ such that $\mathscr{M}^\gamma = \mathscr{M}'$. If $\mathscr{M}$ is of known type, then so is $\mathscr{D}$.*

The properties of axial collineations of finite projective planes used in the next two results may be found in [2] or [3].

LEMMA 6.4. *Let S be a subset of a finite projective plane containing 3 non-collinear points and such that, for some $k \geq 2$, every line containing 2 points of S contains exactly k points of S. Let Π be a collineation group fixing S and such that, if l is any line meeting S in k points and $x \in S \cap l$, then $\Pi(l)$ is transitive on the lines $\neq l$ through x meeting S in k points. Then one of the following statements holds if $|S| > 3$:*

(i) *S is the set of points of a projective subplane.*
(ii) *The centers of the non-trivial elations in Π whose axes meet S in k points are collinear and, together with S, are the points of a projective subplane.*
(iii) *$|S| = 6$, and S and the centers of the non-trivial elations in Π whose axes meet S in k points form a subplane $PG(2, 4)$.*
(iv) *$|S| = 4$, but S is not contained in a subplane $PG(2, 2)$.*
(v) *$|S| = 9$, $k = 3$, but S is not contained in a subplane $PG(2, 3)$.*

Proof. Let $|S \cap l| = k$, and suppose that $\Pi(l)$ has non-trivial elations, all with the same center c. Then $c \notin S$. Let $\Pi(c, l)$ be the group of these elations. If $x \in S \cap l$, then $\Pi(l)$ has a subgroup $\Sigma(l)$ of order $r - 1$ transitive and regular on the $r - 1$ lines $\neq l$ on x meeting S in k points. Every line

$l' \neq l$ on c meeting S contains k points of S. c is the elation center of l' ([2], p. 122). Thus, $|\Pi(c, l)| = p^a$ where p is a prime and p^a is independent of l. As homologies and elations have relatively prime orders, p^a is the largest power of p dividing $|\Sigma(l)| = r - 1$. If m is the axis of a non-trivial elation in Π of p-power order, and $|m \cap S| \neq k$, then m contains c; otherwise, Gleason's Lemma ([2], p. 191) would imply that Π has an element moving l to m. Thus, there is at most one such line m. Then Π contains

$$(p^a - 1)(v/k) + p^b - 1$$

non-trivial elations with center c (where $v = |S|$ and $b \geqslant 0$), none of these fixing a line not on c meeting S in k points. Then

$$[(p^a - 1)(v/k) + p^b] \mid [vr/k - v/k] = (v/k)p^a \cdot (r - 1)/p^a.$$

This yields $p^b = v/k$. As $p^b | k$, we have (ii).

Assume next that the elation subgroup $\Pi^e(l)$ of $\Pi(l)$ contains non-trivial elements with different centers. $\Pi^e(l)$ is an elementary abelian p-group for some prime p. If all centers of non-trivial elements of $\Pi^e(l)$ are in S, then (i) holds by Piper [11], Lemma 5. Suppose some such center $c \notin S$. Then $p|k$, all such centers are not in S, and there are v/k lines on c meeting S in k points. By Piper [11], Lemma 3, there is an integer $a > 0$ such that $|\Pi(c, l)| = p^a$, $|\Pi^e(l)| = p^{2a}$ and $v/k = p^a + 1$. As $v - k = (r - 1)(k - 1)$ and $p^a | k$, this yields $k = 2 = p^a$, $v = 6$. Let c and c' be elation centers for lines meeting S in 2 points such that cc' does not meet S in 2 points. Then Π has precisely $p^{2a} - 1 = 3$ non-trivial elations with center c and axis meeting S in 2 points, which move c' to 3 other centers. There are 6 points in S and $\binom{6}{2}$ centers, a total of 21 points, such that each line meets this set in 0, 1 or at least 5 points. This proves (iii) (cf. [12]).

Finally, suppose $\Pi(l)$ consists entirely of homologies. Then these homologies all have the same center $c \notin S$ ([2], p. 84), and $|\Pi(l)| = r - 1$. A line m through c meeting $S - S \cap l$ meets S in k points. If m meets $S \cap l$ then $|\Pi(l)| | (k - 1)$ and we have (i) again. If m does not meet l then $r - 1 = k$, and we have (iv) if $k = 2$. Let $k > 2$. S is the set of points of an affine plane. The $k - 1$ lines of this plane $\| l$ but $\neq l$ meet at c. As Π has an element moving l to such an l', symmetry and $k - 1 \geqslant 3$ yield $c \in l$, a contradiction. We thus have (v).

The main result of this section is the following

THEOREM 6.5. *If $(\mathscr{D}, \Gamma)$ is a non-degenerate generalized Jordan pair with $\mu > 1$ such that either $\mathscr{D}(B)$ or $\mathscr{D}[p]$ is of known type, then either $\mathscr{D}$ is of known type, or the design $\mathscr{D}_{(1)}$ of points and lines is of known type, not a $\mathscr{W}_v$,* *and blocks are subspaces. Moreover, $\mathscr{D}$ is of known type provided that either $v \leqslant 6k$, $\Gamma(B)$ has a 2-subgroup transitive on $\mathscr{C}B$, or Γ_B is 2-transitive on $\mathscr{C}B$.*

Proof. We examine the various possibilities for $\mathscr{D}(B)$ or $\mathscr{D}[p]$ separately.

Case 1. $\mathscr{D}(B) = PG(d, q)$. Here $\mathscr{D}_{(1)}$ is a projective space by the Veblen and Young axioms [*14*]. Clearly B is a subspace. As Γ_x is 2-transitive on the lines on x it is transitive on the hyperplanes and the subspaces of codimension 2 containing x. The remaining assertions follow immediately.

Case 2. $\mathscr{D}(B) = AG_t(d, q)$. Then $\Gamma_{B|B}$ is not 4-transitive, so that Γ is not 4-transitive. By (4.8), $(\mathscr{D}^{\#}, \Gamma)$ satisfies the given conditions, and we may assume that $\mathscr{D} = \mathscr{D}^{\#}$ is planar, and thus $t = 1 = \mu(B)$.

Let B and C be planes, meeting in a line, and let $l \subset C - B \cap C$ be $\| B \cap C$ in $\mathscr{D}(C)$. Let C^* be the affine subplane of $\mathscr{D}(C)$ containing $B \cap C$ and l. If $\mathscr{D}(C)$ has characteristic q, a Sylow q-subgroup Σ of $\Gamma(B)_{C^*}$ induces a non-trivial group of elations of C^* with axis $B \cap C$; for $\Gamma(B)_{C^*|C^*}$ acts as a Frobenius group on the lines $\neq B \cap C$ of C^* through a point of $B \cap C$ with kernel consisting of elations. In particular, Σ fixes l and

(6.1) $|\Sigma| > |\Sigma(C^*)|$.

We next note that

(6.2) Σ does not fix any line $\not\subset B$ meeting B.

For if Σ fixes such a line m, it also fixes a plane $X \supset m$ meeting B in a line. Then Σ fixes the affine subplane X^* of $\mathscr{D}(X)$ containing m and $B \cap X$. $\Sigma|_{X^*}$ is a q-group with axis $B \cap X$ fixing a line $m \| B \cap X$, and thus $=1$. That is, $\Sigma \leqslant \Gamma(B \cup X^*)$. Since Γ_B has an element moving C to X and C^* to an affine subplane of $\mathscr{D}(X)$ containing $B \cap X$, while $\Gamma(B)_X$ is transitive on such subplanes, $|\Sigma| \leqslant |\Sigma(X^*)| = |\Sigma(C^*)|$, contradicting (6.1).

(6.2) implies

(6.3) Every plane $D \supset l$ meeting B meets B in a line $\| l$ in $\mathscr{D}(D)$. Moreover, $B \cap D \| B \cap C$ in $\mathscr{D}(B)$. For, if $x \in B \cap D$, then Σ fixes the line of $\mathscr{D}(D)$ through x and $\| l$, and the first assertion follows from (6.2). Thus $B \cap D \| C \cap D$ in $\mathscr{D}(D)$ and $B \cap C \| C \cap D$ in $\mathscr{D}(C)$. Interchanging the roles of B and D we obtain $B \cap D \| B \cap C$ in $\mathscr{D}(B)$.

Let $\mathscr{D}^*$ consist of the points of $\mathscr{D}$ and the affine subplanes of the affine spaces $\mathscr{D}(B)$. (6.3) readily implies that $\mathscr{D}^*$ satisfies Lemma 6.1 iv. Thus, $\mathscr{D}^*$ is an affine space, and then $\mathscr{D} = \mathscr{D}^*$. The remaining assertions are handled as in Case 1.

Case 3. $\mathscr{D}(B) = \mathscr{W}_k$. This will be shown impossible. If $\mathscr{D}(B) = \mathscr{W}_{23}$ then for $p \in B$, $\mathscr{D}[p](B) = \mathscr{W}_{22}$. We may thus assume that $\mathscr{D}(B) = \mathscr{W}_{22}$. Then lines have 2 points and planes have 6 points. By (3.10iii), 4 non-coplanar points are contained in a subsystem $\mathscr{W}_{22}$ of the design $\mathscr{D}^{\#}$ of points and planes (see (3.8)). Then $\mathscr{D}^{\#} = \mathscr{W}_{22}$ by Lemma 6.2, a contradiction.

Case 4. $\mathscr{D}[p] = PG_t(d, q)$ *or* $AG_t(d, q)$. If $t > 1$, then for $p \in B$ $\mathscr{D}(B)[p] = PG_{t-1}(d, q)$ or $AG_{t-1}(d, q)$, $\mathscr{D}(B)$ satisfies the given conditions, and $\mathscr{D}_{(1)}$ is known if $\mathscr{D}(B)_{(1)}$ is known by the preceding cases. Thus, we may assume that $t = 1 = \mu(B)$: $\mathscr{D}$ is planar.

If $\mathscr{D}[p] = AG_1(d, 2)$, then the stabilizer in $\hat{\Gamma}_p$ of 3 points fixes the fourth point of their plane in $\mathscr{D}[p]$. As Γ is 4-transitive, a result of Nagao [9] implies that $v = 5$, which is impossible. (We note that the case $\mathscr{D}[p] = AG_2(d, 2)$ can be handled in a more elementary manner by considering a minimal subsystem.) Thus, we may assume that $k > 3$.

Let p, B and φ be as in (4.17). Set $B^* = B^\varphi$. Every line l of $\mathscr{D}^{(0)} = \mathscr{D}[p]$ meets B^* in 0, 1 or h points. Set $\Gamma^{(0)} = \hat{\Gamma}_p$. By (4.17), if $x \in l$ then $\Gamma^{(0)}(l)_{B^*}$ is transitive on the lines $\neq l$ of $\mathscr{D}^{(0)}$ through x which meet B^* in h points. In particular, $\Gamma^{(0)}(l)_{B^*}$ is transitive on the planes of $\mathscr{D}^{(0)}$ containing l and a point of $B^* - B^* \cap l$.

Let E be a plane of $\mathscr{D}^{(0)}$ containing 3 non-collinear points of B^*; E exists as B is not a line of $\mathscr{D}$. Then $S = B^* \cap E$ and $\Pi = \Gamma^{(0)}_{B^*E|E}$ satisfy the conditions of Lemma 6.4 (if E is an affine plane, projectivize it).

Subcase 4.1. *Lemma* 6.4i *holds*. Here every 3 non-collinear points of B^* determine a subplane CB^* of the plane of $\mathscr{D}^{(0)}$ they determine, and $\mathscr{D}(B)$ is a projective space by (4.17) and the Veblen and Young axioms [14]; or, $\mathscr{D}(B)$ is an affine space by (4.17) and Lemma 6.1. Now apply Cases 1 and 2.

Subcase 4.2. *Lemma* 6.4ii *holds*. If $\mathscr{D}^{(0)} = AG_1(d, q)$, then every plane E of $\mathscr{D}^{(0)}$ meets B^* in 0, 1, or h points, or in the points of a subplane of E. Considering the points of B^* and those intersections $B^* \cap E$ with more than h points, we find by (4.17) and Lemma 6.1 that $\mathscr{D}(B)$ is an affine space, and Case 2 applies.

Let $\mathscr{D}^{(0)} = PG_1(d, q)$. If a plane E of $\mathscr{D}^{(0)}$ meets B^* in three non-collinear points, then there is a unique line $m(E)$ of E, and a unique set $\bar{m}(E)$ of $h + 1$ points of $m(E)$, such that $(B^* \cap E) \cup \bar{m}(E)$ are the points of a projective subplane of E. We shall show that, once again, the points of B^* and the intersections $B^* \cap E$ with more than h points form a design $\mathscr{D}^*$ satisfying the conditions of Lemma 6.1, thus reducing to Case 2 again. This is a design by (4.17). Also, Lemma 6.1i, ii, iii and v are clear.

Let l, l_1 and l_2 be lines meeting B^* in h points such that there are planes E_i containing l and l_i with $l \parallel l_i$ in E_i, $i = 1, 2$,; E_i meets B^* in a plane of $\mathscr{D}^*$. Then $\Gamma^{(0)}(l)$ has an element moving E_1 to E_2, and thus $\bar{m}(E_1)$ to $\bar{m}(E_2)$. As $B^* \cap l$ is a line of $B^* \cap E_i$, $B^* \cap l \cap E_i \in \bar{m}(E_i)$. Thus, $\bar{m}(E_1)$ and $\bar{m}(E_2)$ meet at this point y. As $l \parallel l_i$ in E_i, $y \in l_i$. Thus, l_1 and l_2 span a plane of $\mathscr{D}^{(0)}$ meeting B^* in at least $2h$ points, and Lemma 6.1iv holds for $\mathscr{D}^*$.

Subcase 4.3. *Lemma* 6.4iii *holds.* Here $h = 2$, and every 3 points of B^* determine a plane E of $\mathscr{D}^{(0)}$ meeting B^* in 6 points which uniquely determine a subplane $E_1 = PG(2, 4)$ of E. If l is a line of E which is also a line of E_1, then $\Gamma^{(0)}(l)_{E_1}$ is transitive on $E_1 - l$. Thus, $\Gamma^{(0)}_{E_1|E_1} \trianglerighteq PSL(3, 4)$, which moves $B^* \cap E_1$ to a set of subsets of E_1 which, together with the lines of E_1 with p adjoined, determines a $\mathscr{W}_{22}$ (see Witt [*17*], or [*6*]). Thus, relative to the base point p, any three points of $\mathscr{D}$ not coplanar with p determine a unique $\mathscr{W}_{22}$ (recall that $h = 2$). Moreover, the blocks of this $\mathscr{W}_{22}$ are subsets of the blocks of $\mathscr{D}$. As the pointwise stabilizer of a block not on p of some such $\mathscr{W}_{22}$ is transitive on these points p, any three points of $\mathscr{D}$ uniquely determine a set of 6 points of $\mathscr{D}$. Thus, we may apply Lemma 6.2 and $\mathscr{D} = \mathscr{W}_{22}$.

Subcase 4.4. *Lemma* 6.4iv *holds.* Here $h = 2$ and every 3 distinct points x, y, z of B^* determine a plane E of $\mathscr{D}^{(0)}$ meeting B^* in a unique fourth point w. As $\Gamma(B)$ is transitive on $\mathscr{C}(B)$, w is independent of p: 3 distinct points of $\mathscr{D}$ determine a unique fourth point on their plane. Also, even if $\mathscr{D}[p]$ is an affine space, p, x, y, z and B determine w as that point of B-$\{x, y, z\}$ such that the planes determined by p, x, y and p, z, w meet at a point $u \neq p$. Thus, these quadruples of points of $\mathscr{D}$ will form an affine space over GF(2) if we can check Lemma 6.liv for 3 pairs of points of B. Let $\{x, y, z, w\}$ and $\{x, y, z', w'\}$ be distinct quadruples lying in B, and let p, u be as above. Let C be the plane containing $\{p, u, x, y\}$. Then $\Gamma(C)_B$ has an element moving $\{x, y, z, w\}$ to $\{x, y, z', w'\}$ which fixes u. Thus, $\{p, u, z', w'\}$ is a quadruple, and has a point $u \neq p$ in common with $\{p, u, z, w\}$. It follows that $\{z, w, z', w'\}$ is a quadruple. Now apply Case 2.

Subcase 4.5. *Lemma* 6.4v *holds.* Here $h = 3$, $\mathscr{D}[p] = PG(d, 2)$ and $(\mathscr{D}, \Gamma)$ is not a Jordan pair. By (4.5), the points and lines of $\mathscr{D}$, together with Γ, form a generalized Jordan pair. Three non-collinear points lie in an $AG(2, 3)$. As Γ induces a 2-transitive collineation group on this plane, Γ is 2-transitive on points. As in [*2*], pp. 100-101, it follows that the points and lines of $\mathscr{D}$ form an affine space $AG_1(e, 3)$ for some e. Now apply Case 2.

Case 5. $\mathscr{D}[p] = \mathscr{W}_{v[p]}$. Then $h[p] = 2$, so $h = 2$ by (4.8). Thus, $v = v[p] + 1$ and $\mathscr{D} = \mathscr{W}_v$ by Witt [*17*].

The special case $k = h^2 = 9$ of Theorem 6.5 was proved by Hall [*4*] using entirely different methods.

7. The Cases $v \leqslant 6k$

We shall prove the following

THEOREM 7.1. *If $(\mathscr{D}, \Gamma)$ is a non-degenerate generalized Jordan pair with $v \leqslant 6k$, then $\mathscr{D}$ is of known type.*

The proof, which is inductive, is complicated by the fact that the cases $k \leqslant 12$ must be handled separately and by a computational approach. If more information were available concerning slightly larger values of k, the integer 6 could, correspondingly, be increased.

LEMMA 7.2. *If $k \geqslant 9$, the only subgroups of S_k or A_k of index $<5k$ are S_k, A_k, the stabilizer of a point in S_k or A_k, or, if $k \leqslant 10$, the stabilizer in S_k or A_k of an unordered pair of points.*

Proof. See Wielandt ([*16*], p. 42) or Parker ([*10*], Lemma 2).

LEMMA 7.3. *Let Γ be a t-transitive group of degree k, where*

$$t \geqslant \max(4, 1 + k/6).$$

Then either $\Gamma \geqslant A_k$ or $k = 11, 12$ or 24 and $\Gamma = M_k$.

Proof. By Hall ([*3*], p. 80), this is true if $k < 35$. For $k \geqslant 35$, the assertion follows readily from a result of Miller ([*3*], *p*. 69; [*16*], p. 40).

We shall also make use of the Hall-Bruck Theorem, stated in the Introduction.

Proof of Theorem 7.1. Let $(\mathscr{D}, \Gamma)$ be a counterexample with minimal v. Suppose $\mu = 1$. Then $v - 1 = r(k-1)$, $b = v(v-1)/k(k-1)$, $r \geqslant k + 2$, $k \geqslant 4$ and $v \leqslant 6k$ are readily found to be incompatible.

Thus, $\mu > 1$. As $v(p) < v$, the minimality of v and Theorem 6.5 imply that $v(p) > 6k(p)$. By (4.7), we have

(7.1) $v = 6k - e, \qquad 0 \leqslant e \leqslant 4.$

There are $1 + (v - k)/(k - \mu)$ blocks containing a flat, so that

(7.2) $a = (5k - e)/(k - \mu)$ is an integer.

If $a = 5$ then $5\mu = e \leqslant 4$, a contradiction. If $a < 5$ then

$$4 \geqslant e = a\mu + (5 - a)\,k \geqslant 2 + k,$$

a contradiction. Thus, $a \geqslant 6$ and

(7.3) $6\mu = 6(a - 5)\,a^{-1}k + 6ea^{-1} \geqslant k.$

The minimality of v, (4.4) and Theorem 6.5 now imply that $\Gamma_{B|B}$ is μ-transitive. As $\mu > 1$, it follows that $h = 2$, $(\mathscr{D}, \Gamma)$ is a Jordan pair, and by (3.4)

(7.4) $\Gamma_{B|B}$ is $(\mu + 1)$-transitive.

Case 1. $\mu \geqslant 3$. By (7.3), (7.4) and Lemma 7.3, $\Gamma_{B|B}$ is S_k, A_k or M_k.

Subcase 1.1. $k \geqslant 11$ *and* $\Gamma_{B|B} \geqslant A_k$. If $p \notin B$ then $\Gamma_B = \Gamma_{pB}\Gamma(B)$ implies that $\Gamma_{pB|B} \geqslant A_k$. Let $\mathbf{s}$ be a non-empty subset of $\mathscr{C}B$ maximal with the property that $\Delta|_B \geqslant A_k$, where $\Delta = \Gamma_B(\mathbf{s})$. $\mathbf{s} \neq \mathscr{C}B$, as otherwise Δ contains a 3-cycle and $\Gamma \geqslant A_v$. Let $\mathbf{p}$ be an orbit of Δ on $\mathscr{C}B - \mathbf{s}$, and let $p \in \mathbf{p}$. Then $\mathbf{c} = p^{\Gamma(B \cup \mathbf{s})}$ is a (possibly trivial) imprimitivity class of Δ on $\mathbf{p}$. As $\Delta_{\mathbf{c}} = \Delta_p\Gamma(B \cup \mathbf{s})$, the maximality of $\mathbf{s}$ implies that $\Delta_{\mathbf{c}|B} = \Delta_{p|B} \ngeqslant A_k$. Moreover,

$$(7.5) \qquad \begin{aligned} 5k > 5k - e - 1 = v - k - 1 \geqslant v - k - |\,\mathbf{s}\,| \geqslant |\,\mathbf{p}\,| \\ = |\,\Delta\,|_B : \Delta_{\mathbf{c}|B}\,|\,|\,\mathbf{c}\,|. \end{aligned}$$

By Lemma 7.2, $k \mid |\,\Delta|_B : \Delta_{\mathbf{c}|B}|$. As $\mathbf{p}$ is arbitrary, (7.1) implies that $-e - |\,\mathbf{s}\,| \equiv v - k - |\,\mathbf{s}\,| \equiv 0 \pmod k$, and thus $|\,\mathbf{s}\,| \geqslant k - e$. By (7.1) and (7.5),

(7.6) $4k \geqslant v - k - |\,\mathbf{s}\,| \geqslant |\,\Delta|_B : \Delta_{\mathbf{c}|B}\,|\,|\,\mathbf{c}\,|$.

By Lemma 7.2, $\Delta_{\mathbf{c}}$ fixes a point $x \in B$, and $|\,\Delta : \Delta_{\mathbf{c}}\,| = k$ or $2k$.

Define $\mathbf{c}^*$ as follows. If $|\,\Delta : \Delta_{\mathbf{c}}\,| = k$, set $\mathbf{c}^* = \mathbf{c}$. Suppose that $|\,\Delta : \Delta_{\mathbf{c}}\,| = 2k$. This means that $\Delta|_B = S_k$ and $|\,\Delta_{x|B} : \Delta_{\mathbf{c}|B}\,| = 2$. If $\sigma \in \Delta_x - \Delta_{\mathbf{c}}$, set $\mathbf{c}^* = \mathbf{c} \cup \mathbf{c}^\sigma$.

Thus, $\mathbf{c}^*$ is an imprimitivity class of Δ on $\mathbf{p}$ such that $|\,\mathbf{p}\,|/|\,\mathbf{c}^*\,| = k$ and $\Delta_x = \Delta_{\mathbf{c}^*}$. That is, Δ acts on B as it does on the system $\mathscr{I}$ of imprimitivity classes of Δ on $\mathbf{p}$ determined by $\mathbf{c}^*$. Let $\gamma \in \Delta$ act on B as a 5-cycle and have order a power of 5. Then γ fixes $k - 5$ points of B and $k - 5$ elements of $\mathscr{I}$. Since $|\,\mathbf{c}^*\,| = |\,\mathbf{p}\,|/k \leqslant (v - k - |\,\mathbf{s}\,|)/k \leqslant 4$ by (7.6), γ fixes each such element of $\mathscr{I}$ pointwise, thus fixing $(k - 5) \cdot |\,\mathbf{p}\,|/k$ points of $\mathbf{p}$. As $\mathbf{p}$ is arbitrary, γ fixes a total of

$$(k - 5)(v - k - |\,\mathbf{s}\,|)/k + (k - 5) + |\,\mathbf{s}\,|$$

points of $\mathscr{D}$, thus displacing only $5(v - |\,\mathbf{s}\,|)/k \leqslant 25$ points by (7.6). As $\mu \geqslant 3$, (7.4) and a result of Bochert ([*16*], p. 42) imply that γ displaces at least $\frac{1}{2}v - 1$ points. Then $52 \geqslant v = 6k - e$, where $k \geqslant 11$ and $e \leqslant 4$, a contradiction.

Subcase 1.2. $k = 24$ *and* $\Gamma_{B|B} = M_{24}$. Then (7.3) and (7.4) imply that $\mu = 4$ and $e = 0$. In the notation of (3.11), $v^{(2)} = 141$, $k^{(2)} = 21$, $\mu^{(2)} = 1$, and thus $r^{(2)} = 7 < k^{(2)}$, a contradiction.

Subcase 1.3. *$k = 12$ and $\Gamma_{B|B} = M_{12}$*. By (7.4), $\mu = 3$ or 4. By (7.2), $(12 - \mu)|(60 - e)$, so $\mu = 4 = e$ and $v = 68$ by (7.1). As in Subcase 1.2 we obtain the contradiction $r^{(2)} < k^{(2)}$.

Subcase 1.4. *$k = 11$ and $\Gamma_{B|B} = M_{11}$*. Here (7.4) yields $\mu = 3$, and then (7.1) and (7.2) are incompatible.

Subcase 1.5. *$k = 9$ or 10.* Set $g = (2, k)^{-1}$. Define $\mathbf{s}, \Lambda, \mathbf{p}$ and $\mathbf{c}$ as in Subcase 1.1. (7.5) holds, and by Lemma 7.2, (7.6) becomes

$$(7.7) \quad (5 - g)\, k \geqslant v - k - |\,\mathbf{s}\,| \geqslant |\,\Lambda|_B : \Lambda_{\mathbf{c}|B}\,|\,|\,\mathbf{c}\,|\,.$$

If $\Lambda_{\mathbf{c}}$ fixes a point of B, we may proceed as before to obtain $52 \geqslant v = 6k - e$, $0 \leqslant e \leqslant 4$. This contradicts (7.1) and (7.2).

Thus, $\Lambda_{\mathbf{c}}$ displaces all points of B. By Lemma 7.2, $\Lambda_{\mathbf{c}|B} = \Lambda_{\{x,y\}|B}$ for some distinct points $x, y \in B$. Then Λ acts on the images of $\mathbf{c}$ under Λ as a group of degree $\binom{k}{2}$. However, by (7.7) and the definition of g, $v - k - |\,\mathbf{s}\,| \leqslant \binom{k}{2}$. Thus, $|\,\mathbf{c}\,| = 1$ and Λ acts on $\mathscr{C}B - \mathbf{s}$ as S_k or A_k does on the unordered pairs of points of B. If $\delta \in \Lambda$ has order a power of 3 and is a 3-cycle on B, it fixes $k - 3$ points of B, $\binom{k-3}{2}$ points of $\mathscr{C}B - \mathbf{s}$ and $|\,\mathbf{s}\,| = gk - e$ points of $\mathbf{s}$. By (7.1), δ displaces

$$6k - (k - 3) - (k - 3)(k - 4)/2 - gk$$

points of $\mathscr{D}$. As $\mu \geqslant 3$, (7.4) and a result of Bochert ([*16*], p. 42) imply that δ displaces at least $[\frac{1}{2}(v - 1)] \geqslant 3k - 2$ points, this is a contradiction.

Subcase 1.6. *$k \leqslant 8$.* This is found to be impossible by straightforward computation using (7.1), (7.2), (7.4), the parameters of the designs $\mathscr{D}^{(i)}$ of (3.11), and the Hall-Bruck Theorem and Theorem 6.5 whenever some $k^{(i)} = 3$. For example, if $v = 48$, $k = 8$, $\mu = 4$ (possible by (7.2)), then $b^{(0)}$ is $(33 \cdot 23)47/7$.

Case 2. $\mu = 2$. In this case, $v^{(0)} = v - 1$, $k^{(0)} = k - 1$, $\mu^{(0)} = \mu - 1$ and (by Theorem 6.5)

$$r^{(0)} = (6k - e - 2)/(k - 2) \geqslant k^{(0)} + 2 = k + 1.$$

Then $k \leqslant 6$. Also, $k^{(0)} > 3$ by the Hall-Bruck Theorem. Thus, $k = 5$ or 6. A straightforward check shows that $r^{(0)}$ and $b^{(0)}$ are integers only if $v = 26$, $k = 5$, $\lambda = r^{(0)} = 8$ and $r = b^{(0)} = 50$.

To eliminate this case, note that $|\,\mathscr{C}B\,| = 21$, so there is a $\gamma \in \Gamma(B)$ of order 3. As $\mu = 2$ and $\lambda = 8$, γ fixes at least one block $\neq B$ through each pair of points of B, thus fixing at least $\binom{k}{2} = 10$ such blocks. As these blocks contain a total of at least $10(k - \mu) > 21$ points, two of them must have a common point in $\mathscr{C}B$. Then $\mu = 2 < 3 = |\,\gamma\,|$ implies that γ fixes points of

$\mathscr{C}B$. If $\mathbf{f}$ is the set of fixed points of γ, then $|\mathbf{f}| \geq 6$. If a block contains 3 points of $\mathbf{f}$ it is fixed pointwise, as $k - 3 < 3$. Thus, $\mathbf{f}$ is the set of points of a proper subsystem of $\mathscr{D}$, contradicting Corollary 6.3. This completes the proof of Theorem 7.1.

We note that Subcase 1.5 could be handled in the same way as Subcase 1.6, except that $v = 54$, $k = 9$, $\mu = 6$ does not lead to a numerical contradiction; that is, there could conceivably be a $6 - (54, 9, 1)$ design, in the notation of Hughes [5].

8. Subgroups of $\Gamma(B)$

Lemma 8.1. *Let $(\mathscr{D}, \Gamma)$ be a Jordan pair and let Σ be a subgroup of $\Gamma(B)$ transitive on $\mathscr{C}B$, where B is a block of $\mathscr{D}$.*

(i) *If h is odd, an involution σ in the center of Σ fixes a point of B linewise.*

(ii) *If Σ is abelian, then for each $x \in B$, there are precisely $h - 1$ elements of Σ fixing x linewise.*

Proof. (i) The transitivity of Σ implies that σ is fixed point free on $\mathscr{C}B$. If $y \in \mathscr{C}B$, then σ fixes yy^σ. Since h is odd, yy^σ must meet B. As Σ is transitive on the lines on $B \cap yy^\sigma$ not contained in B, σ fixes this point linewise.

(ii) Σ is regular on $\mathscr{C}B$. If l is a line meeting B at x, then $|\Sigma_l| = h - 1$. As Σ centralizes Σ_l, it follows that Σ_l fixes x linewise.

Theorem 8.2. *Let $(\mathscr{D}, \Gamma)$ be a non-degenerate Jordan pair. If Σ is a 2-subgroup of $\Gamma(B)$ transitive on $\mathscr{C}B$, where B is a block of $\mathscr{D}$, then the following statements are equivalent.*

(i) *$\mathscr{D}$ is of known type.*

(ii) *Σ is elementary abelian.*

(iii) *Σ is normal in Γ_B.*

Proof. Clearly (i) $\Rightarrow$ (ii), (iii). To prove (ii) $\Rightarrow$ (i) or (iii) $\Rightarrow$ (i), we may assume that $\mu = 1$ ((3.7) and Theorem 6.5). Then $h = k > 2$, $k - 1$ $(v - k)/(r - 1)$ is even, and Lemma 8.1i applies.

(ii) $\Rightarrow$ (i). If $x, y \in \mathscr{C}B$, $x \neq y$, then there is an element $\sigma \in \Sigma$ such that $x^\sigma = y$. By Lemma 8.1i, xy meets B. Thus, $\mathscr{D}$ is a projective plane.

(iii) $\Rightarrow$ (ii). Let Λ be the subgroup of Σ generated by the involutions in the center of Σ. Lemma 8.1i, the transitivity of $\Gamma_{B|B}$, and (iii) imply that each point of B is fixed linewise by the same number $g > 1$ of elements of Λ. Let $\mathbf{p}$ be an orbit of Λ on $\mathscr{C}B$. If $x, y \in \mathbf{p}$, $x \neq y$, then xy meets B by Lemma 8.1i;

conversely, if $p \in B$ and $x \in \mathbf{p}$, then $| px \cap \mathbf{p} | = g$. Thus, each point of $\mathbf{p}$ is on $(| A | - 1)/(g - 1)$ lines meeting $\mathbf{p}$ in g points, each of which meets B:

$$(| A | - 1)/(g - 1) = k = 1 + (v - k)/(r - 1).$$

As $| A |$, g and $v - k$ are powers of 2, $k = g + 1$. The points of $\mathbf{p} \cup B$ are thus the points of a subsystem of $\mathscr{D}$ which is a projective plane. The Veblen and Young axioms [*14*] now readily imply that $\mathscr{D}$ is a projective plane.

LEMMA 8.3. *Let Γ be an $(a + 1)$-transitive group on a set S, $| S | = n$, $a \geqslant 3$. Let $A \subset S$, $| A | = a$. If Γ_A is 2-transitive on $S - A$, and $\Gamma(A)$ has an elementary abelian q-subgroup normal in Γ_A and transitive on $S - A$, where q is an odd prime, then either*

(i) $\Gamma \geqslant A_n$, $n - a = 3$; *or*

(ii) $n = 11$ *or* 12, $n - a = 9$ *and* $\Gamma = M_n$.

Proof. Γ is transitive on the set of $a + 2$ element subsets of S. If $a + 2 \leqslant n/2$, then Γ is $(a + 2)$-transitive by Livingstone and Wagner [*8*]. By Suzuki [*13*], either $\Gamma \geqslant A_n$ or $\Gamma = M_n$ ($n = 11$ or 12). If $a + 2 > n/2$, then the $(a + 1)$-transitivity of Γ implies that $\Gamma \geqslant A_n$ or $\Gamma = M_n$ ($n = 11$ or 12), by Lemma 7.3.

If $\Gamma \geqslant A_n$, then $\Gamma(A)|_{S-A}$ has no non-trivial abelian normal subgroup unless $| S - A | \leqslant 4$. As $q > 2$, $n - a = 3$. If $\Gamma = M_n$ it is clear that $| S - A | = 9$.

It is more difficult to use part (ii) of Lemma 8.1 than it is to use (i). The difficulty is caused by the fact that (ii) does not imply that the product of elements of Σ fixing different points of B linewise also fixes a point of B linewise. This does, however, hold under the hypotheses of the following

THEOREM 8.4. *If $(\mathscr{D}, \Gamma)$ is a non-degenerate generalized Jordan pair such that Γ_B is 2-transitive off of the block B, then $\mathscr{D}$ is of known type.*

Proof. Let $(\mathscr{D}, \Gamma)$ be a counterexample with minimal v. $h = 2$, as otherwise all lines meet all blocks and $\mathscr{D}$ is a projective space ([*2*], p. 74; [*6*]). Thus, $(\mathscr{D}, \Gamma)$ is a Jordan pair with $\mu > 1$. If $p \in B$, then Γ_{pB} is not 2-transitive on $\mathscr{C}B$ by (3.7) and the minimality of v. Thus,

(8.1) If $x, y \in \mathscr{C}B$, $x \neq y$, then Γ_{Bxy} is intransitive on B.

$\Gamma(B)|_{\mathscr{C}B}$ is imprimitive by Jordan's Theorem ([*3*], *p*. 66; [*16*], p. 34), so that it has a unique elementary abelian q-subgroup $\Sigma \trianglelefteq \Gamma_B$ transitive and regular on $\mathscr{C}B$ (Burnside [*1*], p. 199). Here the prime q is odd by Theorem 8.2. Define m by (3.11); $m \geqslant 1$ as $h = 2$. If S_{m-1} is an $(m - 1)$-subblock $\subseteq B$, Σ induces an automorphism group on $\mathscr{D}[S_{m-1}]$ (see (3.10v)). As $h^{(m-1)} > 2 = h^{(m-2)}$, Lemma 8.1ii implies that there are non-trivial elements of Σ fixing an m-subblock of B blockwise but no smaller subblock blockwise. Since all elements of $\Sigma - \{1\}$ are conjugate in Γ_B, it follows that

(8.2) Every element $1 \neq \sigma \in \Sigma$ fixes a certain number f of m-subblocks $S(\sigma) \subset B$ blockwise. Here $f \geqslant 1$ is independent of σ, and $|S(\sigma)| = m + 1$. (For the final assertion, see (3.12).)

(8.3) If $x \notin B$ and $1 \neq \sigma \in \Sigma$, then the set of $S(\sigma)$'s is $\{S \mid S$ is an m-subblock $\subset B$, and there is an $(m + 1)$-subblock T containing S, x and $x^\sigma\}$.

Proof. If a block contains x and $S(\sigma)$ then it contains x^σ so that $S(\sigma)$ belongs to the stated set. Conversely, if S is in the stated set, and T is an $(m + 1)$-subblock containing S, x and x^σ, then σ fixes T by (3.10iii). As Σ centralizes σ, σ fixes $T^{\sigma'}$ for all $\sigma \in \Sigma$. By (3.10iii), each block $\neq B$ containing S is a union of these $T^{\sigma'}$, and S is fixed blockwise, as claimed.

Case 1. *$k > 2\mu$ and $f > 1$* (f is defined in (8.2)). Let $x \notin B$, and let T be the $(m + 1)$-subblock containing $S(\sigma)$, x and x^σ for some $1 \neq \sigma \in \Sigma$ and some $S(\sigma)$ as in (8.2). Then $\Gamma(T)_B$ is transitive on $B - S(\sigma)$, as in the proof of (3.5). For the same reason, if $S(\sigma) \supset S(\sigma) \cap S'(\sigma) \neq \phi$, then $\Gamma_{Bxx^\sigma}(S(\sigma) \cap S(\sigma'))$ is transitive on $B - (S(\sigma) \cap S'(\sigma))$. Using (8.1), we find in this manner that the f $S(\sigma)$'s have a non-empty intersection $W(\sigma)$. Set $w = |W(\sigma)|$.

(8.4) $1 \leqslant w < m + 1$, and $m \geqslant 2$.

For $S(\sigma) \supset W(\sigma) \neq \phi$. Suppose $m = 1$. Then $w = 1$, and $S(\sigma)$ is a line. T is thus a plane. As $h = 2$, x, x^σ and $W(\sigma)$ uniquely determine T, and there cannot be two distinct subblocks $S(\sigma)$, whereas $f > 1$.

As $\Gamma_{Bxx^\sigma}(W(\sigma))$ is transitive on $B - W(\sigma)$, and $\Gamma_{Bxx^\sigma}(S(\sigma))$ is transitive on $B - S(\sigma)$, (8.3) and Lemma 3.1 imply that

(8.5) If $1 \neq \sigma \in \Sigma$, and $x \notin B$, then Γ_{Bxx^σ} permutes the $S(\sigma)$'s transitively.

Let $S(\sigma)$ and T be as above. Let $t = |T|$. Let $x, y \in T - S(\sigma)$, $x \neq y$. There is a $\gamma \in \Gamma_{Bx}$ with $y^\gamma = x^\sigma$. Then $x, x^\sigma \in T^\gamma$ and $B \cap T^\gamma$ is an m-subblock. By (8.3) and (8.5), there is a $\gamma' \in \Gamma_{Bxx^\sigma}$ with $T^\gamma = T^{\gamma'}$. Then $\gamma\gamma'^{-1} \in \Gamma_{BTx}$ takes y to x^σ. Thus, $\Gamma_{BT|T-B\cap T}$ is 2-transitive. Also, $\Sigma_T \trianglelefteq \Gamma_{BT}$ is transitive and regular on $T - B \cap T$. By (3.15) and Lemma 8.3 (with $a = m + 1 \geqslant 3$ by (8.4)), $t - (m + 1) = 3$ or 9.

Claim: $\binom{m+1}{w} \mid [t - (m + 1) - 1]$. For $|\Sigma_T - \{1\}| = t - (m + 1) - 1$ as $|S(\sigma)| = m + 1$. Each $\sigma' \in \Sigma_T - \{1\}$ determines a unique set $W(\sigma') \subset B \cap T$, by (8.3). By (3.15), each subset W of $B \cap T$ with w points has the form $W(\sigma')$ for g elements $\sigma' \in \Sigma_T - \{1\}$, where g is independent of W. Thus, $t - (m + 1) - 1 = g\binom{m+1}{w}$.

We now have $\binom{m+1}{w} \mid 2$ or 8, where $t = 11$ or 12 in the latter case and thus $m + 1 = 2$ or 3. This contradicts (8.4).

Case 2. $f = 1$. Here, each $\sigma \in \Sigma - \{1\}$ fixes a unique m-subblock $S(\sigma) \subset B$ blockwise. Once again, let $x \notin B$ and let T be the $(m + 1)$—subblock containing $S(\sigma)$, x and x^σ.

Claim: $S(\sigma)$ is a flat. Otherwise, T is not a block and $\mathscr{D}[S(\sigma)]$ is defined (see (3.10v)). Within $\mathscr{D}[S(\sigma)]$, T is a point and B is a block not on T. As $h = 2$, (3.10v) implies that $\Gamma_B(T)$ is transitive on those flats of $\mathscr{D}$ containing $S(\sigma)$ and contained in B. If X is a block $\supset T$ meeting B in a flat, then as in Case 1, Γ_{BX} is 2-transitive on $X - B \cap X$. By (3.4) and the minimality of v, $\Gamma_{X|X}$ is $(\mu + 1)$-transitive. As we are assuming that $m + 1 < \mu$, this contradicts (3.14).

Thus, T is a block, and once again Γ_{BT} is 2-transitive on $T - B \cap T$. Also, $\Sigma_T \trianglelefteq \Gamma_{BT}$ is transitive on $T - B \cap T$. By Lemma 8.3, there are two possibilities:

(i) $k - \mu = 3$, $\Gamma_{B|B} \geqslant A_k$; or

(ii) $k - \mu = 9$, $k = 11$ or 12, $\Gamma_{B|B} = M_k$.

Each $\sigma \in \Sigma - \{1\}$ determines a unique flat $S(\sigma)$. Conversely, each flat F in B is fixed blockwise by $k - \mu - 1$ elements of $\Sigma - \{1\}$. As $\Gamma_{B|B}$ is μ-transitive, $v - k - 1 = \binom{k}{\mu}(k - \mu - 1)$. In case (ii), this yields

$$v - k = 1 + 8\binom{11}{2} \quad \text{or} \quad 1 + 8\binom{12}{3},$$

whereas $v - k$ is a power of 3. Thus, (i) holds and

$$(8.6) \quad v - k - 1 = 2\binom{k}{3}.$$

If $x \notin B$, then $\Gamma_{xB|B} \geqslant A_k$. Each $y \in \mathscr{C}B - \{x\}$ determines a unique block Y such that $x, y \in Y$ and $|B \cap Y| = \mu$, by (8.3). Then $|Y - B \cap Y| = 3$, and Γ_{xyB} fixes $Y - B \cap Y = \{x, y, y'\}$ pointwise. Thus, $\{y, y'\}$ is an imprimitivity class of $\Gamma_{xB|\mathscr{C}B-\{x\}}$. Moreover, $\Gamma_{x\{y,y'\}B}$ fixes $B - B \cap Y = \{z, z', z''\}$. By (8.6), $\Gamma_{x\{y,y'\}B} = \Gamma_{x\{z,z',z''\}B}$. Thus, Γ_{xB} acts on the sets $\{y, y'\}$ as S_k or A_k acts on the set of triples of distinct points of B. Let $\gamma \in \Gamma_{xB}$ be a 3-cycle on B and have order a power of 3. γ fixes $k - 3$ points of B, and fixes $1 + \binom{k-3}{3}$ such triples. Thus, γ fixes $1 + \binom{k-3}{3}$ pairs $\{y, y'\}$ and then fixes each such pair pointwise: γ fixes

$$(k - 3) + 1 + 2 + (k - 3)(k - 4)(k - 5)/3$$

points. However, $v = k + 1 + k(k - 1)(k - 2)/3$ by (8.6). Thus, γ displaces $3(k^2 - 5k + 7)$ points. $\mu = k - 3 \geqslant 2$. As $v - k$ is a power of 3, (8.6) implies that $k \geqslant 10$. Since Γ is $(k - 2)$-transitive, γ displaces at least $v/2$ points ([*16*], p. 42). This is a contradiction.

Case 3. $k \leqslant 2\mu$ *and* $f > 1$. By Lemma 3.2, (3.2), Theorems 6.5 and 7.1, and Lemma 8.3, $\Gamma_{B|B} \geqslant A_k$ and $k - \mu = 3$. By (3.14), $m + 1 = \mu$ and each $S(\sigma)$ is a flat.

Let $1 \neq \sigma \in \Sigma$, $x \notin B$, and set $L = \{x, x^{\sigma}, x^{\sigma^{-1}}\}$. If X is a block $\supset L$ meeting B in a flat, then $L = X - B \cap X$ and, by (8.3), $B \cap X$ is an $S(\sigma)$. If $S'(\sigma) \neq S(\sigma)$, then each element of $\Gamma(X)_B$ maps $S'(\sigma)$ to another $S''(\sigma)$ (by (8.3)). Thus, each point of B is in some $S'(\sigma)$.

By (3.16), $\Gamma_{BX} = \Gamma_{BLS(\sigma)}$ is transitive on $S(\sigma)$. Except possibly when $k = 2\mu$, any two $S(\sigma)$ meet. Thus, unless $k = 2\mu, f = 2$ and $S(\sigma) \cap S'(\sigma) = \phi$, it follows that Γ_{BL} is transitive on B. L may be regarded as a line of $AG(d, 3)$, where $v - k = 3^d$, and by (3.18), Γ_B is then faithfully a collineation group of $AG(d, 3)$. If $p \in B$, we have shown that Γ_{pB} is line-transitive on $AG(d, 3)$. $|\Gamma_{pB}|$ is even as $k > 3$. Thus, there is an element of Γ_{pB} fixing a line L' of $AG(d, 3)$, fixing a point $x \in L'$, and interchanging the remaining points of L'. Then Γ_{pB} is a 2-transitive on $\mathscr{C}B$, a contradiction (cf. (8.1)).

It remains only to consider the case $\mu + 3 = k = 2\mu, f = 2$, $S(\sigma) \cap S'(\sigma) = \phi$. Then $k = 6$. Each flat F in B determines a second flat $B - F$, and these are the $S(\sigma)$ for $k - \mu - 1$ elements $\sigma \in \Sigma - \{1\}$. Thus, $v - k - 1 = \frac{1}{2}\binom{k}{\mu}(k - \mu - 1)$ (cf. (8.6)). As $k = 6$ and $\mu = 3$, $v - k = 21$ is not a power of 3. This contradiction proves the Theorem.

If $(\mathscr{D}, \Gamma) = (\mathscr{W}_{23}, M_{23})$, (8.2) still holds. This leads to the following fact: if B is a block of $\mathscr{W}_{23}$, and $1 \neq \sigma \in \Gamma(B)$, then σ fixes 7 sets of 3 points of B blockwise, and together with the 7 points of B these form a projective plane. Similarly, involutions in M_{24} fixing a block of $\mathscr{W}_{24}$ pointwise determine an $AG_2(3, 2)$ in the block.

Proof of Theorem 1.1. (i) follows from Lemma 3.2, Theorems 7.1 and 5.3, Lemma 7.3, and the fact that all $(k + 1)$-transitive groups of degree $v \leqslant 6k \leqslant 12$ are known. (ii) and (iii) follow from Lemma 3.2 and Theorems 8.2, 8.4 and 5.3.

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