EXPANDED, SLICED AND SPREAD SPREADS

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ABSTRACT

This paper, written in honor of Ted Ostrom, summarizes recent work on affine translation planes arising from spreads of symplectic and orthogonal geometries.

This summary of parts of [4,5] also includes some details not explicitly in those papers. We will first describe properties of a number of translation planes, then explain how they all arise, and conclude with open problems.

I. SOME TRANSLATION PLANES

All the translation planes will have order $q^{2n-1} > 8$, where q is even and $n \geq 2$. In many cases, the full collineation groups are known—and "easily" computed—a situation relatively unusual in the literature on translation planes.

(1) *Flag-transitive planes.* There are at least $q/(2 \log_2 q)$ flag-transitive planes of order $q^{2n-1} > 8$, for each $n \geq 2$ (their spreads will be given below).

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In each case, the full translation complement has order $(q - 1)(q^{2n-1} + 1)\cdot(2n - 1)\ell$, where ℓ is a divisor of $\log_2 q$; it is a subgroup of $GF(q^{4n-2})^* \rtimes \mathrm{Aut}\, GF(q^{4n-2})$, and has a cyclic subgroup of order $q^{2n-1} + 1$ which is transitive on the line at infinity. There are exactly $q - 1$ homologies with center 0.

Let $F = GF(q^{4n-2})$ and $K = GF(q)$. Let $T: GF(q^{2n-1}) \longrightarrow K$ be the trace map, and let W be the kernel of T. Fix $r \in GF(q^2) - K$. Then $\{\theta W + \theta rK \mid \theta \in F,\ |\theta| \mid q^{2n-1} + 1\}$ is one of the required spreads for each r.

(2) *One large cycle*. Let $q > 2$. There are at least $(q - 2)/(2 \log_2 q)$ planes which will be called *large-cycle* planes (quasifields will be given below). These are planes of order q^{2n-1} having a cyclic collineation group of order $q^{2n-1} - 1$ fixing 0 and two points x and y at infinity, while transitively permuting the points of $xy - \{x,y\}$, $0x - \{0,x\}$ and $0y - \{0,y\}$. The full translation complement fixes x and y, has order $(q^{2n-1} - 1)(2n - 1)\ell$ for a divisor ℓ for $\log_2 q$, and is a subgroup of $GF(q^{2n-1})^* \rtimes \mathrm{Aut}\, GF(q^{2n-1})$. There are exactly $q - 1$ homologies with center 0.

Let $F = GF(q^{2n-1})$ and $K = GF(q)$. Let $T: F \longrightarrow K$ be the trace map. Fix $k \in K - GF(2)$. For $x,y \in F$, define

$$x*y = x^2 y + kxT(xy).$$

Let $x \longrightarrow \bar{x}$ denote the inverse of $x \longrightarrow (x*1)/(k + 1)$ and $y \longrightarrow y'$ the inverse of $y \longrightarrow (1*y)/k + 1)$. Finally, let

$$x \circ y = (\bar{x}*y')/(k + 1).$$

Then $(F,\circ)$ is one of the desired quasifields.

(3) *SL(2,q) planes*. There are planes of order q^3 where $\log_2 q$ is odd and not 1. The full translation complement is $(GL(2,q) \times Z_{q+1}) \rtimes \mathrm{Aut}\, GF(q)$. The group Z_{q+1} fixes pointwise a desarguesian subplane of order q, and $\Gamma L(2,q)$ is induced on this subplane. (Note that $AG(2,q^3)$ has no such collineation of order $q + 1$ if $q > 8$.)

The group $SL(2,q)$ has orbit lengths $q + 1$ and $q^3 - q$ at infinity. There is a partition of the line at infinity into $q^2 - q + 1$ sets of size $q + 1$, each fixed pointwise by a collineation of order $q + 1$ whose fixed subplane is desarguesian of order q.

The underlying vector space has dimension 6 over $K = GF(q)$. The group $SL(2,q)$ splits it into the sum of invariant 2- and 4-spaces. More precisely, set $F = GF(q^2)$, and consider the K-space F^3. Let $\omega^3 = 1 \neq \omega$. If $\alpha \in F$

set $\overline{\alpha} = \alpha^q$. Then the spread consists of the q + 1 subspaces

$$\{(a\sigma^2, \gamma\sigma, \gamma\overline{\sigma}) \mid a \in K, \gamma \in F\}$$

as σ ranges over F*, and the $q^3 - q$ subspaces

$$\{(\gamma\alpha^2 + \overline{\gamma}\beta^2, a\omega\alpha + \overline{\gamma}\beta, a\omega\beta + \overline{\gamma\alpha}) \mid a \in K, \gamma \in F\}$$

as $\begin{pmatrix}\alpha & \overline{\beta} \\ \beta & \overline{\alpha}\end{pmatrix}$ ranges over $SU(2,q) \cong SL(2,q)$.

(4) *An SL(2,8) plane.* This is a plane of order 8^3. The full translation complement is $GF(8)^*(Z_3 \times P\Gamma L(2,8))$. The Z_3 fixes pointwise a desarguesian subplane of order 8, and $\Gamma L(2,8)$ is induced on that subplane. The group SL(2,8) has orbit lengths 8+1 and $8^3 - 8$ at infinity.

This plane is neigher desarguesian nor the one of order 8^3 appearing in (3).

The only descriptions I have for the spread are worthless from a computational point of view: it seems impossible to use them in order to attempt to generalize this plane to others of order q^3. On the other hand, it seems likely that such a generalization exists.

(5) *$(q + 1)^2$ planes.* These are planes of order q^3, where $\log_2 q$ is odd and not 1. Their full translation complements have the form $G = (GF(q)^* \times Z_{q+1} \times Z_{q+1}) \rtimes \mathrm{Aut}\, GF(q)$. There are three subgroups Z_{q+1} fixing pointwise Desarguesian subplanes of order q. Here, G acts irreducibly but imprimitively on the underlying 6-dimensional vector space. I have no explicit descriptions for the required spreads or quasifields.

(6) *$q^2 \pm q + 1$-planes.* These are planes of order $q^3 > 8$ admitting a cyclic group of order $q^2 \pm q + 1$. The full collineation group has order $(q - 1)2(q^2 \pm q + 1)\log_2 q^3$. The $q^2 + q + 1$-plane has one orbit at infinity of length 2 and all others of lengths divisible by $q^2 + q + 1$. The $q^2 - q + 1$-plane has all its orbits at infinity of lengths divisible by $q^2 - q + 1$.

Set $F = GF(q^3)$ and $K = GF(q)$. Let $T: F \longrightarrow K$ be the trace map. If $x, y \in F$, set

$$x*y = T(xy^{q+q^2})y^{q+q^2} + x^q y^{q^2} + x^{q^2} y^q.$$

(Note that $(x*y)*y^{-1} = x$ if $y \neq 0$.) Let $y \longrightarrow y'$ denote the inverse of $y \longrightarrow 1*y$, and set

$x \circ y = x * y'$.

Then $(F, \circ)$ is a quasifield for the $q^2 + q + 1$-plane.

In order to construct the $q^2 - q + 1$-plane, start with $F = GF(q^6)$, $K = GF(q^2)$, $T: F \longrightarrow K$, $x*y$ and $x \circ y$ as above. Set $\bar{x} = x^{q^3}$. Then $(x,y) \longrightarrow (\bar{y}, \bar{x})$ is a Baer involution whose fixed point subplane is the desired $q^2 - q + 1$-plane. These fixed points are the points $(x, \bar{x})$, $x \in F$, while the fixed lines through 0 are all the lines $y = x*m$ with $m\bar{m} = 1$.

(7) Semifield planes. There is a semifield of order q^{2n-1} (to be constructed below) for each $n \geq 2$ and even q, which is not a field if $q^{2n-1} > 8$. The full translation complement of the corresponding plane has order $(q - 1)q^{2n-1}\log_2 q^{2n-1}$.

Let $F = GF(q^{2n-1})$ and $K = GF(q)$. Let $T: F \longrightarrow K$ be the trace map. For $x, y \in F$, define

$$x*y = x^2 y + xT(y) + T(xy).$$

Let $x \longrightarrow \bar{x}$ be the inverse of $x \longrightarrow x*1$, and set

$$x \circ y = \bar{x}*y.$$

Then $(F, \circ)$ is the desired semifield. It is noncommutative. Its left nucleus is $GF(2)$ and its right nucleus is K.

(8) $q(q + 1)$-planes. These are planes of order q^3 where $\ell = \log_2 q$ is odd and not 1. Their full translation complements have order $(q - 1)q(q + 1)\ell$. There is a normal cyclic subgroup of order $q + 1$ whose fixed point subplane is $AG(2,q)$. There is a normal subgroup of order q consisting of elations with axis in the aforementioned subplane.

(9) Planes with very small groups. If $2n - 1$ is composite, there is a plane of order 2^{2n-1} whose translation complement has order 1. Such a plane can best be described as "dull."

(1'), (2'), (5'), (6'). There are flag-transitive, large-cycle, $(q + 1)^2$ and $q^2 \pm q + 1$-planes other than those whose quasifields or spreads appeared in (1), (2), and (6). These planes have full translation complements acting essentially the same as for the aforementioned planes, except perhaps for fewer field automorphism occurring. The corresponding spreads or quasifields are extremely messy, even though it is not difficult to prove that these planes exist.

II. SYMPLECTIC AND ORTHOGONAL SPREADS

Let V' be a 2m-dimensional vector space over GF(q), equipped with a symplectic form (u,v). A *symplectic spread* of V' is a spread Σ' each of whose members is a totally singular m-space. (Thus, if $M \in \Sigma$ then $M^{\perp} = M$.)

Every one of the planes in part I arises from a symplectic spread in a 4n-2-dimensional vector space. The only translation planes of concern to us in this paper arise from symplectic spreads.

If Σ' is a symplectic spread, let $A(\Sigma')$ denote the corresponding affine plane. The first basic result concerning symplectic spreads is the following theorem.

Isomorphism Theorem. Let Σ_i be a symplectic spread of an Sp(2m,q) space $V_i (i = 1,2)$, where q is even. Let g be an isomorphism from $A(\Sigma_1)$ to $A(\Sigma_2)$. Then there is a perspectivity t of $A(\Sigma_2)$ with axis the line at infinity, and a semilinear transformation $s:V_1 \longrightarrow V_2$, such that

(i) $g = st$,

(ii) $\Sigma_1^s = \Sigma_2$, and

(iii) $(u^s,v^s)_2 = a(u,v)_1^{\tau}$ for some $a \in GF(q)$,

some $\tau \in \mathrm{Aut}\, GF(q)$, and all $u,v \in V_1$.

Proof. We may assume that $0^g = 0$. Then g is a semilinear transformation from V_1 to V_2.

For each subspace W of V_1, set $W^{\theta} = W^{\perp}$. Then θ is a polarity of PG(2m - 1,q). Define ϕ for V_2 in a similar manner. Note that θ induces the identity on Σ_1, while ϕ induces the identity on Σ_2. Also, θ^g is a polarity of PG(2m-1,q) inducing the identity on Σ_2. Then $\phi\theta^g$ is a collineation of PG(2m - 1,q) inducing the identity on Σ_2, and hence induces a homology of $A(\Sigma_2)$. In particular, $\phi\theta^g$ has odd order, and hence $(\theta^g)^h = \phi$ for some homology $h \in \langle\phi\theta^g\rangle$ of $A(\Sigma_2)$ with center 0. Also, $\Sigma_1^{gh} = \Sigma_2^h = \Sigma_2$. Consequently, s = gh behaves as required.

The Isomorphism Theorem asserts that affine plane isomorphisms essentially arise from transformations preserving the underlying symplectic geometries. In particular, the translation complement of $A(\Sigma')$ can be factored as the product of its homologies (with center 0) and its intersection with the symplectic group.

While the above theorem tells us when two planes of a certain sort are isomorphic, it gives us no information concerning the construction of planes. For this we need to consider orthogonal geometries.

Let V be a 4n-dimensional vector space over GF(q), equipped with a quadratic form Q(v) and associated nonsingular bilinear form (u,v). Thus,

$$Q(av) = a^2Q(v)$$
$$Q(u + v) = Q(u) + Q(v) + (u,v)$$

for all $u,v \in V$ and $a \in GF(q)$. We assume that *totally singular* 2n-spaces exist: 2n-spaces W such that Q(W) = 0. This means that Q is equivalent to the form $\sum_{i=1}^{2n} x_{2i-1}x_{2i}$ defined on $GF(q)^{4n}$. There are exactly $(q^{2n+1} + 1)(q^{2n} - 1)/(q - 1)$ singular points. An *orthogonal spread* of V is a family Σ of $q^{2n-1} + 1$ totally singular 2n-spaces which partitions the singular points. Such an orthogonal spread is *not* the same as the usual kind of spread: it is only a maximal partial spread in the language of translation planes.

Note that we have assumed that dim V ≡ 0(mod 4). If dim V ≡ 2(mod 4) then orthogonal spreads of totally singular ½ dim V-spaces do not exist.

Now assume that q is even. Take any nonsingular point b of V. Then $b^\perp$ is an orthogonal space of dimension 4n - 1, and $b \in b^\perp$. (This is where characteristic 2 is needed.) The space $b^\perp/b$ is no longer an orthogonal space, but it does inherit the symplectic form (u,v) in the natural manner. Moreover, the natural projection $b^\perp \longrightarrow b^\perp/b$ induces a bijection between the totally singular subspaces of $b^\perp$ and the totally isotropic subspaces of $b^\perp/b$.

Let Σ be an orthogonal spread of V. If $W \in \Sigma$ then $\dim b^\perp \cap W = 2n - 1$, and the family $\{b^\perp \cap W | W \in \Sigma\}$ partitions the singular points of $b^\perp$. Projecting into $b^\perp/b$, we obtain a family of $q^{2n-1} + 1$ totally isotropic 2n - 1-spaces partitioning the points of $b^\perp/b$. Thus, we obtain a spread (in the usual sense), called the *slice* Σ(b) of Σ.

Each nonsingular point b of V produces a slice Σ(b), and hence a translation plane A(Σ(b)). In order to compare these planes, we need to see how to go from a plane to an orthogonal spread. This is the content of the next result (where "equivalence" refers to the existence of a suitable type of orthogonal or symplectic transformation).

Extension Theorem. Let Σ' be a spread of a 4n - 2-dimensional symplectic space over GF(q), where q is even. Then there is a unique (up to equivalence) orthogonal spread Σ* in a 4n-dimensional space such that Σ' is equivalent to a slice of Σ*.

Proof (Dillon [2], Dye [3]). Let V and b be as before. We may assume that Σ' is a spread of $b^{\perp}/b$. Pulling back to $b^{\perp}$, we obtain a family Σ'' of totally singular 2n-1-spaces of $b^{\perp}$.

There are two classes of totally singular 2n-spaces of V, two totally singular 2n-spaces belonging to the same class if and only if their intersection has even dimension. Pick one of these classes. Each element W of Σ'' belongs to a unique member W* of this class. If $W_1, W_2 \in \Sigma''$ then $\dim W_1^* \cap W_2^*$ is even and at most 1. Thus, $\Sigma^* = \{W^* | W \in \Sigma'\}$ is a family of $q^{2n-1} + 1$ totally singular 2n-spaces, no two having a common point. The union of Σ^* has $(q^{2n-1} + 1)(q^{2n} - 1)/(q - 1)$ points, so Σ^* is an orthogonal spread.

Uniqueness follows from the fact that the orthogonal transvection with axis $b^{\perp}$ interchanges the two classes of totally singular 2n-spaces.

The process of passing from Σ' to Σ^* will be called *spreading*.

Given a symplectic spread, we can spread it and then slice the result, thereby obtaining many new symplectic spreads. Any equivalence between symplectic spreads induces an equivalence between their spread spreads (by the Extension Theorem). In particular, inequivalent orthogonal spreads never produce isomorphic translation planes (by the Isomorphism Theorem). Moreover, if Σ is an orthogonal spread of V and $G(\Sigma)$ denotes the group of projective semilinear orthogonal transformations preserving Σ, then $A(\Sigma(b_1))$ and $A(\Sigma(b_2))$ are non-isomorphic whenever b_1 and b_2 are in different $G(\Sigma)$-orbits. Similarly, $G(\Sigma)_b$ *is the group induced by* Aut $A(\Sigma(b))$ *on the line at infinity*. This explains the introductory remark concerning the ease of determining collineation groups.

Additional symplectic and orthogonal spreads can be obtained as follows. Let V' be a 2m-dimensional symplectic space over $GF(q^e)$, where $e > 1$, and let Σ' be a symplectic spread of V'. Let (u,v) be the form on V', and let $T: GF(q^e) \longrightarrow GF(q)$ be the trace map. Then T(u,v) turns V' into a 2me-dimensional symplectic space over GF(q). Since totally isotropic subspaces remain totally isotropic, we obtain a symplectic spread of the GF(q)-space V'. Of course, the resulting spread defines the same translation plane $A(\Sigma')$ as before. However, if q is even and me is odd we can spread this GF(q) spread in order to obtain a spread in an orthogonal 2me + 2-space over GF(q). This procedure is called *expanding* the spread Σ' into 2me + 2 dimensions.

We now have enough machinery in order to begin discussing the examples in part I.

The most obvious symplectic spreads are the desarguesian ones. If V' is a 2-dimensional symplectic space over $GF(q^{2n-1})$, where q is even and $n \geq 3$, we can form the corresponding spread Σ' of 1-spaces. Now expand this into an orthogonal spread Σ of a 4n-dimensional GF(q) space V. This is the spread first constructed by Dillon [2] and Dye [3], and is called a *desarguesian spread* in [4,5]. Its group $G(\Sigma)$ was determined by Dye [3] and Cohen and Wilbrink [1]. Namely, $G(\Sigma)$ is A_9 if $q^3 = 8$ and $P\Gamma L(2,q^{2n-1})$ otherwise.

EXAMPLES (1), (2), (7). These constitute all the nondesarguesian slices of a desarguesian spread Σ. Since the orbits of $G(\Sigma)$ on nonsingular points are known (Dye [3]), the Isomorphism and Extension Theorems permit us to distinguish between planes and to determine (most of) their collineation groups. Namely, $G(\Sigma)_b$ is the group induced on the line at infinity by the full translation complement. This only leaves homologies to be determined, and this is done by brute force in [5].

Of course, the quasifields or spreads in (1), (2), and (7) can only be found after a coordinate description of Σ has been obtained. We refer the reader to [5] for such a description.

EXAMPLES (3), (5), (8). These arise from an 8-dimensional orthogonal spread Σ, which is closely related to the unitary group GU(3,q). Thus, in some sense the planes in (3) and (5) stem from $PG(2,q^2)$.

Let $F = GF(q^2)$ and $K = GF(q)$. If $\alpha \in F$ write $\bar{\alpha} = \alpha^q$. Let V be the 8-dimensional K-space consisting of all 3×3 matrices $M = (\mu_{ij})$ over F which have trace 0 and satisfy $M^t = \bar{M}$, where $\bar{M} = (\bar{\mu}_{ij})$. Set $Q(M) = \sum_{i<j} (\mu_{ii}\mu_{jj} + \mu_{ij}\mu_{ji})$. Then Q is a quadratic form, and V has totally singular 4-spaces if and only if $\log_2 q$ is odd (or, equivalently, if and only if $\bar{\omega} \neq \omega$ when $\omega^3 = 1 \neq \omega$). One such 4-space W consists of all the matrices

$$\begin{pmatrix} a & a & \bar{\gamma} \\ a & a & \bar{\gamma} \\ \gamma & \gamma & 0 \end{pmatrix} + \begin{pmatrix} 0 & b\omega & 0 \\ b\bar{\omega} & b & 0 \\ 0 & 0 & b \end{pmatrix}$$

with $\gamma \in F$ and $a,b \in K$.

If $A \in GF(3,q)$ then $A^{-1} = \overline{A}^{t}$, and hence $(A^{-1}MA)^{t} = \overline{A}^{-1}\overline{MA}$. Thus, $GU(3,q)$ acts on V, inducing $PGU(3,q)$ there. In fact $\Gamma U(3,q)$ acts, and preserves Q (semilinearly).

Note that transvections (elations) in $GU(3,q)$ have the form $I + X$ with I the identity matrix and $X \in V$, where $X^2 = 0$. A full group of q transvections has the form $I + KX$ with $X \neq 0$. One such X is $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and the normalizer of $I + KX$ in $\Gamma U(3,q)$ stablizes the above 4-space W. *The images of W under GU(3,q) form the desired spread* Σ.

If $q = 2$ then Σ is equivalent to the desarguesian spread. If $q > 2$ then $G(\Sigma) \trianglerighteq PGU(3,q)$, and hence Σ is nondesarguesian. In particular, none of its slices $\Sigma(b)$ can be desarguesian or among examples (1), (2), or (7). Moreover, $G(\Sigma)_b$ is the group induced at infinity by the full translation complement of $A(\Sigma(b))$.

When b is spanned by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A(\Sigma(b))$ is just example (3). The group $SL(2,q) \times Z_{q+1}$ consists of all matrices $\begin{pmatrix} B & 0 \\ 0 & \delta \end{pmatrix}$ with $B\overline{B}^{t} = I$, $\det B = 1$ and $\delta\overline{\delta} = 1$.

When b is spanned by $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a+1 \end{pmatrix}$ with $a \in K - GF(2)$, $A(\Sigma(b))$ is example (5). Note that this matrix is diagonalizable; the $Z_{q+1} \times Z_{q+1}$ consists of diagonal matrices.

When b is spanned by $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A(\Sigma(b))$ is example (8).

As noted in [4, (7.6)], a suitably chosen slice of Σ produces a $q^2 - q + 1$-plane.

Regardless of which slice $\Sigma(b)$ is used, the full translation complement acts at infinity as $G(\Sigma)_b$ acts on the unital associated with the unitary group. Thus, for example, the partition into $q^2 - q + 1$ sets of size $q + 1$ appearing in example (3) arises from the partition of the unital into the lines fixed by a homology.

REMARK. The proof given in [4] that Σ is a spread was valid in odd characteristic as well. In our situation, a simpler proof is as follows.

If $X = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ then $\{M \in V | MX = XM = 0\}$ consists of all matrices

$M = \begin{pmatrix} a & a & \bar{\gamma} \\ a & a & \gamma \\ \gamma & \gamma & 0 \end{pmatrix}$. Note that $M^2 = \gamma\bar{\gamma}X$, so that M uniquely determines X. Also $Q(M) = 0$. Thus, we have $q^3 + 1$ totally singular 3-spaces, any two meeting trivally. Also, W is a totally singular 4-space containing the above 3-space. Since any two images of W under GU(3,q) meet in at most a 1-space, it follows that Σ is a spread.

EXAMPLE (4). The relevant orthogonal spread Σ was discovered by Dye [3]. His construction shows that $G(\Sigma) \geq A_9 \times Z_3$ (in fact, equality holds). Thus, Σ cannot be one of the spreads already encountered. The Z_3 contains a field automorphism, and hence fixes pointwise an orthogonal space over GF(2) upon which A_9 acts. If b is chosen to be fixed by the Z_3 then $G(\Sigma)_b = P\Gamma L(2,8) \times Z_3$.

However, I do not know of any simple description of Σ. Dye used triality. I can describe Σ in a complicated manner which does not take proper advantage of the group PSL(2,8) and which seems relatively useless if one wants to study the above slice. No other slice of Σ seems interesting.

EXAMPLE (6). There is another spread Σ of an orthogonal 8-space over GF(q) such that $G(\Sigma) = P\Gamma L(2,q^3)$ if $q > 2$. Unlike the desarguesian case, this group acts irreducibly on the 8-space. Moreover, there are just two orbits of nonsingular points, and the resulting slices produce the planes in example (6).

Once again $G(\Sigma)$ permits us to distinguish these planes from the others we have seen.

EXAMPLES (1'), (2'), (5'), (6'). These are obtained by expanding and suitably slicing spreads already arising in (1), (2), (5), or (6).

EXAMPLE (9). Expand and then suitably slice example (1).

3. OPEN PROBLEMS

(i) Generalize example (4) to SL(2,q) planes whenever $\log_2 q$ is odd.

(ii) Find all translation planes of order q^3 admitting a collineation group SL(2,q) having orbits of lengths $q + 1$ and $q^3 - q$ at infinity.

(iii) Prove directly that the 4-spaces listed in example (3) form a spread.

(iv) Find more examples of symplectic spreads. Find new ways of modifying one example in order to obtain others.

(v) Study the expansion process further. Symplectic spreads can be expanded and sliced repeatedly. The resulting planes should tend to be new, while having progressively smaller groups.

In particular, the flag-transitive examples (1) can be expanded over and over while continuing to produce flag-transitive planes. Similar remarks apply to large-cycle, $(q + 1)^2$ and $q^2 \pm q + 1$-planes. Prove that large numbers of new planes arise in this manner.

(vi) Find a direct way to pass from one slice of an orthogonal spread to another one without using the spread and slice procedures: find a replacement process analogous to net replacement.

(vii) Find a way to construct symplectic spreads in 4n-dimensional symplectic spaces.

(viii) Prove the Isomorphism Theorem when q is odd.

(ix) Find an internal criterion for a translation plane to be symplectic.

REFERENCES

1. A. M. Cohen and H. A. Wilbrink, The stabilizer of Dye's spread on a hyperbolic quadric in PG(4n - 1,2) within the orthogonal group (to appear).

2. J. Dillon, On Pall partitions for quadratic forms (unpublished manuscript).

3. R. H. Dye, Partitions and their stabilizers for line complexes and quadrics, *Annali di Mat. (4)114* (1977), 173-194.

4. W. M. Kantor, Spreads, translation planes and Kerdock sets. I (to appear).

5. W. M. Kantor, Spreads, translation planes and Kerdock sets. II (to appear).