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TWO EXCEPTIONAL 3-ADIC AFFINE BUILDINGS

1. Introduction

This note contains constructions of some discrete chamber-transitive automorphism groups of affine buildings for certain 5- and 6-dimensional orthogonal groups over the field $\mathbb{Q}_3$ of 3-adic numbers. Such groups are very rare. Similar constructions can be found in [4], [7], [9] and [17], and the groups obtained by such constructions have been classified in [6] in the case of Witt index $\geqslant 2$ in arbitrary dimension for an arbitrary local field.

One motivation for such constructions arises from the fact that, upon passage modulo suitable normal subgroups, they produce finite building-like geometries having chamber-transitive automorphism groups. Such geometries have, in turn, been characterized group-theoretically (cf. [8], [11], [12], [13] and their references).

We have described the examples in a very concrete manner as arithmetic groups. Consequently, all finite homomorphic images can be immediately obtained, in view of the results in [5].

2. Affine buildings of 5- and 6-dimensional orthogonal groups

We will need descriptions of two affine buildings. These are based on [3] and [14].

Consider a vector space V over $\mathbb{Q}_3$. Let $\mathbb{Z}_3$ denote the ring of 3-adic integers. If S is any basis of V let $L = \langle S\rangle_{\mathbb{Z}_3}$ be the lattice ($\mathbb{Z}_3$-module) generated by S. Write $[L] = \{aL \mid 0 \neq a \in \mathbb{Q}_3\}$. Note that all members of a lattice-class $[L]$ have the same stabilizer in $\mathrm{GL}(V)$.

Now assume that V has dimension 6, with basis $e_1, e_2, f_1, f_2, u_5, u_6$, and is equipped with the inner product $f = (,)$ defined by $(e_i, e_j) = (f_i, f_j) = 0$, $(e_i, f_j) = \delta_{ij}, (e_i, u_k) = (f_i, u_k) = (u_5, u_6) = 0, (u_5, u_5) = 1$ and $(u_6, u_6) = 3$ for all i, j, k. Then the *affine building* of the corresponding orthogonal group $\mathrm{O}(f, \mathbb{Q}_3)$ is the simplicial complex Δ whose vertices (of respective *types* 0, 1 and 2) are the

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lattice-classes $[\Lambda_0]^g, [\Lambda_1]^g, [\Lambda_2]^g$, $g \in O(f, \mathbb{Q}_3)$, where Λ_i is defined by

$$\begin{aligned} \Lambda_0 &= \langle e_1, e_2, f_1, f_2, u_5, u_6 \rangle_{\mathbb{Z}_3} \\ \Lambda_1 &= \langle e_1/3, e_2, f_1, f_2, u_5, u_6/3 \rangle_{\mathbb{Z}_3} \\ \Lambda_2 &= \langle e_1/3, e_2/3, f_1, f_2, u_5, u_6/3 \rangle_{\mathbb{Z}_3}. \end{aligned}$$

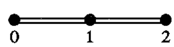
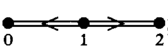
One *chamber* (i.e., maximal simplex) of Δ is $\{[\Lambda_0], [\Lambda_1], [\Lambda_2]\}$, with all others obtained by applying elements of $O(f, \mathbb{Q}_3)$. Note that the transformation θ taking

$$\begin{aligned} &e_1 \to 3f_2 \to -3e_1 \\ &e_2 \to -3f_1 \to -3e_2 \\ &u_5 \to u_6 \to -3u_5 \end{aligned}$$

sends Λ_1 to its dual lattice

$$\Lambda_1^\# := \{x \in V \mid (x, \Lambda_1) \subseteq \mathbb{Z}_3\} = \langle e_1, e_2, 3f_1, f_2, u_5, u_6 \rangle_{\mathbb{Z}_3},$$

and sends $\Lambda_2 \to \Lambda_0 \to 3\Lambda_2$. Here, θ satisfies $(u^\theta, v^\theta) = 3(u, v)$ for all $u, v \in V$, and $\theta^2 = -3$. Note that the stabilizer of $[\Lambda_1]$ in $O(f, \mathbb{Q}_3)$ coincides with that of $[\Lambda_1^\#]$.

The building Δ has the diagram  (or, more precisely, ; cf. [14]). This has, among others, the following meaning. Star($[\Lambda_0]$), the star of $[\Lambda_0]$, is the finite building of the orthogonal group $O(5, 3)$. Namely, the bilinear form $(u + 3\Lambda_0, v + 3\Lambda_0) \bmod 3$ induced on the GF(3)-space $\bar{\Lambda}_0 := \Lambda_0/3\Lambda_0$ has radical $\bar{\Lambda}_0^\perp$ generated by $u_6 + 3\Lambda_0$. Moreover,

(#) If Λ is a lattice in V, then $[\Lambda]$ represents a vertex of Δ_6 adjacent to $[\Lambda_0]$ if and only if some member of $[\Lambda]$, say Λ itself, satisfies the following conditions: $3\Lambda_0 \subset \Lambda \subset \Lambda_0$, $\Lambda/3\Lambda_0 \supset \bar{\Lambda}_0^\perp$, and $\Lambda/3\Lambda_0$ is totally singular.

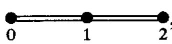
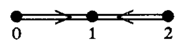
(In fact, $3\Lambda_1/3\Lambda_0$ and $3\Lambda_2/3\Lambda_0$ are the totally singular subspaces spanned by $e_1 + 3\Lambda_0$, $u_6 + 3\Lambda_0$ and $e_1 + 3\Lambda_0, e_2 + 3\Lambda_0, u_6 + 3\Lambda_0$, respectively.) Similarly, Star($[\Lambda_1]$) and Star($[\Lambda_2]$) are a generalized digon and the $O(5, 3)$ building, respectively. (In fact, Star($[\Lambda_2]$) = Star($[\Lambda_0]$)$^\theta$.)

Clearly, $O(f, \mathbb{Q}_3)$ acts on Δ_6 with induced automorphism group $PO(f, \mathbb{Q}_3) = O(f, \mathbb{Q}_3)/\langle -1 \rangle$.

The second building Δ' we will consider is that of the commutator subgroup $\Omega(f', \mathbb{Q}_3)$ of the orthogonal group $O(f', \mathbb{Q}_3)$ of the form f' obtained by restricting f to the orthogonal complement $V' := \langle e_1, e_2, f_1, f_2, u_5 \rangle$ of u_6.

This building is defined as above, this time using the three lattices

$$\Lambda'_0 = \langle e_1, e_2, f_1, f_2, u_5 \rangle_{\mathbb{Z}_3}$$
$$\Lambda'_1 = \langle e_1/3, e_2/3, f_1, f_2, u_5 \rangle_{\mathbb{Z}_3}$$
$$\Lambda'_2 = \langle e_1/3, e_2, 3f_1, f_2, u_5 \rangle_{\mathbb{Z}_3}.$$

The diagram is , (or, more precisely, ), with respective stars the O(5, 3) building, a generalized digon, and the O(5, 3) building. This time $O(f', \mathbb{Q}_3)$ is not type-preserving: the reflection in $(e_1 - 3f_1)^\perp$ fixes $[\Lambda'_1]$ and interchanges $[\Lambda'_0]$ and $[\Lambda'_2]$.

3. The 6-dimensional discrete groups

Let V_6 be the 6-dimensional vector space $\mathbb{Q}^6$, with standard basis $u_1, u_2, u_3, u_4, u_5, u_6$, equipped with the bilinear form f defined by $(u_i, u_j) = \delta_{ij}$ except that $(u_6, u_6) = 3$. Let V_6 be embedded in $V = \mathbb{Q}_3^6$ in the natural manner, and let f also denote the form induced on V.

We begin by relating the present notation to that of Section 2. Namely, let $\alpha \in \mathbb{Z}_3$ be the root of the equation $x^2 + 2 = 0$ such that $1 + \alpha \notin 3\mathbb{Z}_3$ (i.e., $1 + \alpha$ is a unit) and $1 - \alpha \in 3\mathbb{Z}_3$. Let $e_1, e_2, f_1, f_2, u_5, u_6$ be the basis of V defined by

$$e_1 = \tfrac{1}{2}(1 + \alpha)(u_1 + u_2 + \alpha u_3)$$
$$f_1 = \tfrac{1}{2}(1 + \alpha)^{-1}(u_1 + u_2 - \alpha u_4)$$
$$e_2 = \tfrac{1}{2}(u_1 - u_2 + \alpha u_4)$$
$$f_1 = \tfrac{1}{2}(u_1 + u_2 - \alpha u_4).$$

Then these basis vectors behave exactly as described in Section 2. In particular, V has witt index 2 – while, clearly, V_6 is anisotropic.

Let Δ_6 denote the building for the group $\Omega(f, \mathbb{Q}_3)$, and define $[\Lambda_0]$, $[\Lambda_1]$, $[\Lambda_2]$ using the vectors $e_1, e_2, f_1, f_2, u_5, u_6$ just as in Section 2. The radical of $\bar{\Lambda}_0 = \Lambda_0/3\Lambda_0$ is spanned by $u_6 + 3\Lambda_0$.

PROPOSITION 1. *$G_6 := O(f, \mathbb{Q}_3) \cap GL(6, \mathbb{Z}[\frac{1}{3}])$ is discrete and chamber-transitive on Δ_6.*

Proof. By a simple calculation, $\Lambda_0 = \langle u_1, \ldots, u_6 \rangle_{\mathbb{Z}_3}$. The stabilizer $(G_6)_0$ of the vertex $[\Lambda_0]$ consists of rational transformations preserving the $\mathbb{Z}$-lattice $\langle u_1, \ldots, u_6 \rangle_{\mathbb{Z}}$, and hence preserves its set $\{\pm u_1, \ldots, \pm u_5\}$ of squared length 1 vectors, so that $(G_6)_0$ is a monomial group of the form $2^6 S_5$. In particular, this group induces a chamber-transitive group on Star($[\Lambda_0]$).

Now consider the transformation θ defined by the matrix

$$\begin{pmatrix} 0 & -1 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -3 & 0 \end{pmatrix}.$$

Then $(u^\theta, v^\theta) = 3(u, v)$ for all $u, v \in V$, and $\theta^2 = -3$. It follows easily that $[\Lambda_0^\theta]$ has the properties in (#) (cf. Section 2; in fact, $\Lambda_0^\theta/3\Lambda_0 = 3\Lambda_2/3\Lambda_0$ is the totally singular 3-space spanned by $(1,1,1,0,0,0) + 3\Lambda_0, (1,-1,0,1,0,0) + 3\Lambda_0$ and $u_6 + 3\Lambda_0$), and hence is a vertex of Δ_6 of type 2 adjacent to $[\Lambda_0]$. Moreover, θ normalizes G_6 (since θ^{-1} has all entries in $\mathbb{Z}[\frac{1}{3}]$). Now $(G_6)_0^\theta$ is chamber-transitive on the star of $[\Lambda_0]^\theta$. Connectedness of Δ_6 implies that the subgroup $\langle (G_6)_0, (G_6)_0^\theta \rangle$ of G_6 is chamber-transitive on Δ_6. Finally, G_6 is discrete since the stabilizer of any vertex is finite. □

Since we now have two chamber-transitive automorphism groups G_6 and $\langle (G_6)_0, (G_6)_0^\theta \rangle$ of Δ_6 having the same vertex stabilizer, namely $(G_6)_0$, we deduce the

COROLLARY. *The stabilizer $(G_6)_0$ of the vertex $[\Lambda_0]$ is $2^6 S_5$, and $G_6 = \langle (G_6)_0, (G_6)_0^\theta \rangle$.*

The vertices of Δ_6 can all be represented by $\mathbb{Z}$-lattices, instead of by $\mathbb{Z}_3$-lattices as was done in the above proof and in Section 2. Namely, it is straightforward to check that $[\Lambda_i] = [\Sigma_i \otimes_{\mathbb{Z}} \mathbb{Z}_3]$ for $i = 1, 2, 3$, where

$$\Sigma_0 := \langle u_1, \ldots, u_6 \rangle_{\mathbb{Z}}, \Sigma_2 := \Sigma_0^\theta,$$

and

$$\Sigma_1 := \langle u_1 + u_2 - 2u_3, u_2 + u_3 - 2u_1 \rangle_{\mathbb{Z}} \perp \langle 3u_4, 3u_5 \rangle_{\mathbb{Z}} \perp \\ \perp \langle u_1 + u_2 + u_3, u_6 \rangle_{\mathbb{Z}}.$$

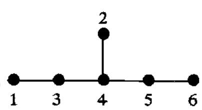
Here, $\Sigma_1/3\Sigma_0$ is the totally singular 2-space of $\Sigma_0/3\Sigma_0$ spanned by $u_1 + u_2 + u_3 + 3\Sigma_0$ and $u_6 + 3\Sigma_0$, while $\Sigma_2/3\Sigma_0$ is the totally singular 3-space spanned by $u_1 + u_2 + u_3 + 3\Sigma_0, u_1 - u_2 + u_4 + 3\Sigma_0$ and $u_6 + 3\Sigma_0$. Moreover, from the description of Σ_1 it follows that *the stabilizer $(G_6)_1$ of the vertex $[\Lambda_1]$ is $D_{12} \times D_8 \times D_8$*. Note that $(G_6)_0 \cap (G_6)_1 = S_3 \times D_8 \times 2$ and $(G_6)_1 = [(G_6)_0 \cap$

$(G_6)_1][(G_6)_2 \cap (G_6)_1]$. Since θ maps $(1/3)\Sigma_1$ onto the dual lattice $\Sigma_1^{\#} := \{v \in \mathbb{Q}^6 | (v, \Sigma_1) \subseteq \mathbb{Z}\}$ of Σ_1 it normalizes $(G_6)_1$.

Next we turn to a new basis for V_6, as well as a new chamber-transitive automorphism group of Δ_6.

Consider the following six vectors:

$$\begin{aligned} z_1 &= \tfrac{1}{2}(-u_1 - u_2 - u_3 - u_4 + u_5 + u_6) \\ z_2 &= u_4 + u_5 \\ z_3 &= u_4 - u_5 \\ z_4 &= u_1 - u_4 \\ z_5 &= u_2 - u_1 \\ z_6 &= u_3 - u_2. \end{aligned}$$

These vectors satisfy the relations 

for a fundamental system of roots of a root system of type E_6 (cf. [2, p. 268]). Note that, with Λ_0 as before, we have $\Lambda_0 = \langle z_1, z_2, z_3, z_4, z_5, z_6 \rangle_{\mathbb{Z}_3}$.

Let H_6 denote the stabilizer of

$$L := \langle z_1, z_2, z_3, z_4, z_5, z_6 \rangle_{\mathbb{Z}[\frac{1}{3}]}$$

in $O(f, \mathbb{Q}_3)$. (Alternatively, H_6 can be defined as $O(e_6, \mathbb{Q}_3) \cap GL(6, \mathbb{Z}[\frac{1}{3}])$, where e_6 denotes the quadratic form for the E_6 root lattice, and matrices are written with respect to the z_i.)

PROPOSITION 2. *H_6 is a discrete chamber-transitive automorphism group of Δ_6.*

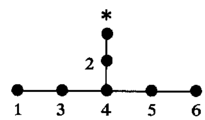
Proof. This time, $(H_6)_0 = W(E_6) \times \langle -1 \rangle$. The transformation θ sends L to itself, and hence normalizes H_6. Thus, the argument in Proposition 1 can be repeated in the present situation. □

COROLLARY. *The stabilizer $(H_6)_0$ of the vertex $[\Lambda_0]$ is $W(E_6) \times \langle -1 \rangle$. Moreover, $H_6 = \langle (H_6)_0, (H_6)_0^{\theta} \rangle$.*

As before, the vertices of Δ_6 can all be represented by $\mathbb{Z}$-lattices. This time $[\Lambda_i] = [\Gamma_i \otimes_{\mathbb{Z}} \mathbb{Z}_3]$ for $i = 1, 2, 3$, with $\Gamma_0 := \langle z_1, \ldots, z_6 \rangle_{\mathbb{Z}}$, $\Gamma_2 := \Gamma_0^{\theta}$, and $\Gamma_1 := \langle z_1, z_3 \rangle_{\mathbb{Z}} \perp \langle z_5, z_6 \rangle_{\mathbb{Z}} \perp \langle z_2, z_* \rangle_{\mathbb{Z}}$, where $-z_* = z_1 + 2z_2 + 2z_3 + 3z_4 + 2z_5 + z_6$ is the highest root in the E_6 root system generated by $z_1, z_2, z_3, z_4,$

z_5, z_6 [Bo, p. 268],

so that the following relations hold.



Note that $\Gamma_1/3\Gamma_0$ is the totally singular 2-space of $\bar{\Gamma}_0 = \Gamma_0/3\Gamma_0$ spanned by $z_5 - z_6 + 3\Gamma_0$ and $z_1 - z_3 + z_5 - z_6 + 3\Gamma_0$, while $\Gamma_2/3\Gamma_0$ is the totally singular 3-space spanned by $z_5 - z_6 + 3\Gamma_0, z_2 + z_3 + z_5 + 3\Gamma_0$ and $z_1 - z_3 + z_5 - z_6 + 3\Gamma_0$. Moreover, we see that the *stabilizer* $(H_6)_1$ *of the vertex* $[\Lambda_1]$ *has the form* $(3^3 \times 2^3)2^3S_3$. Note that $(H_6)_0 \cap (H_6)_1 = 2 \times (3^3(2 \times S_4))$ and $(H_6)_1 = [(H_6)_0 \cap (H_6)_1][(H_6)_2 \cap (H_6)_1]$.

There is another interesting graph automorphism φ. First, let t be the following product of reflections lying in $W(E_6)$: $t = r_1 r_3 \cdot r_5 r_6 \cdot r_2 r_*$ (where r_i denotes the reflection in $z_i^\perp$).Then t lies in the center of a Sylow 3-subgroup of $W(E_6)$ (namely, $\langle t \rangle$ is the long root group of O(5, 3) such that $[\bar{\Gamma}_0/\bar{\Gamma}_0^\perp, \langle t \rangle]$ is spanned by the projections of $z_5 - z_6$ and $z_2 + z_3 + z_5$ into $\bar{\Gamma}_0/\bar{\Gamma}_0^\perp$), and $t^2 + t + 1 = 0$. Write $\varphi = 1 + 2t$. Then $\varphi^2 = -3$, and $(u^\varphi, v^\varphi) = 3(u, v)$ for any vectors u and v. It is straightforward to check that $\Gamma_2 = \Gamma_0^\varphi$, and that $(\frac{1}{3})\Gamma_1^\varphi = \Gamma_1^\#$, so that φ normalizes $(H_6)_1$. Note also that θ normalizes $(H_6)_1$.

4. Congruence subgroups

In order to relate G_6 and H_6, and at the same time construct some additional chamber-transitive automorphism groups of the building Δ_6 as well as to construct finite morphic images of Δ_6, we introduce congruence subgroups of these two groups.

View H_6 as a group $O(e_6, \mathbb{Q}_3) \cap GL(6, \mathbb{Z}[\frac{1}{3}])$ of matrices. For any integer $m > 1$ not divisible by 3, let $H_6(m)$ (the 'level m congruence subgroup' of H_6) denote the normal subgroup consisting of those matrices $\equiv I \pmod m$. This is just the centralizer in H_6 of L/mL. Similarly, the group $G_6(m)$ is defined to be the centralizer in G_6 of M/mM, where $M := \langle u_1, \ldots, u_6 \rangle_{\mathbb{Z}[1/3]}$.

PROPOSITION 3. $H_6 = H_6(2) \cdot (H_6)_0$, *where* $H_6(2) \cap (H_6)_0 = \langle -1 \rangle$. *Thus,* $H_6(2)$ *acts regularly on the set of vertices of* Δ_6 *of type* 0. *It also acts regularly on the set of vertices of type* 2.

Proof. Obviously H_6 acts on $L/2L$, preserving the quadratic form $\frac{1}{2}(u + 2L, v + 2L) \bmod 2$ induced on that GF(2)-space with respect to which $L/2L$ is nondegenerate with Witt index 2. The group $(H_6)_0$ induces $W(E_6) \cong O^-(6, 2)$ on that space. This proves that $H_6 = H_6(2) \cdot (H_6)_0$. Since it is clear that $H_6(2) \cap (H_6)_0 = \langle -1 \rangle$, this proves the desired regularity on vertices of type 0.

Now note that θ normalizes $H_6(2)$ (since θ normalizes H_6 and L). This proves the last assertion. □

Note that the graph automorphism $\varphi = 1 + 2t$ defined at the end of Section 3 centralizes $L/2L$ and hence also $H_6/H_6(2)$.

We now determine the intersection $G_6 \cap H_6$. Let ${}^{\circ}G_6$ be the subgroup $\{G_6 \cap \Omega(f, \mathbb{Q}_3)\}\langle -1, (12)\rangle$, where (12) denotes the reflection in $(u_1 - u_2)^{\perp}$ acting as a transposition on $u_1, \ldots, u_5$. Then ${}^{\circ}G_6$ has index 2 in G_6, and is normalized by θ (by a straightforward calculation). Write $({}^{\circ}G_6)_0 = {}^{\circ}G_6 \cap (G_6)_0$.

LEMMA 4. ${}^{\circ}G_6 = G_6 \cap H_6 = H_6(2)\cdot({}^{\circ}G_6)_0$.

Proof. Abbreviate $D := ({}^{\circ}G_6)_0$. Then $D = \mathrm{W}(D_5) \times \langle -1\rangle$, and D is chamber-transitive on Star($[\Lambda_0]$). Moreover, $D < (H_6)_0$ (it is the $\mathrm{W}(D_5) \times \langle -1\rangle$ in $\mathrm{W}(E_6) \times \langle -1\rangle$ induced on the space $\langle z_2, \ldots, z_6\rangle$). Thus, $\langle D, D^{\theta}\rangle$ is a chamber-transitive subgroup of both ${}^{\circ}G_6$ and $G_6 \cap H_6$, since θ normalizes both of these groups.

We already know that H_6 induces $\mathrm{O}^-(6,2)$ on $L/2L$. Since $\langle D, D^{\theta}\rangle$ induces a proper subgroup of $\mathrm{O}^-(6,2)$ containing a parabolic subgroup 2^4S_5, it is contained in $H_6(2)D$. However, $H_6(2)$ is regular on the vertices of Δ_6 of type 0, so that no proper subgroup of $H_6(2)D$ containing D can be chamber-transitive. Thus, $H_6(2)D = \langle D, D^{\theta}\rangle$.

In particular, $H_6(2)D \leqslant G_6 \cap H_6$. In fact, $H_6(2)D = G_6 \cap H_6$ since we have seen that $H_6(2)D$ is a maximal subgroup of H_6. Finally, ${}^{\circ}G_6$ is a chamber-transitive group with $({}^{\circ}G_6)_0 = (H_6(2)D)_0$, so that ${}^{\circ}G_6 = G_6 \cap H_6$. □

We have just constructed a chamber-transitive subgroup (namely, $G_6 \cap H_6$) of H_6. In order to construct even smaller chamber-transitive subgroups, consider a subgroup $\mathbf{X} = 2^4A_5$ or 2^4F_{20} of $G_6 \cap H_6 = 2^4S_5$ (where F_{20} denotes a Frobenius group of order 20 in S_5: the normalizer of a Sylow 5-subgroup). Then clearly $H_6(2)\mathbf{X}$ *is chamber-transitive on* Δ_6. Note that $H_6(2)\mathbf{X} = \langle \mathbf{X}, \mathbf{X}^{\varphi}\rangle$, where φ is the graph automorphism defined at the end of Section 3. (For, since φ centralizes $H_6/H_6(2)$ the chamber-transitive groups $H_6(2)\mathbf{X}$ and $\langle \mathbf{X}, \mathbf{X}^{\varphi}\rangle$ project onto the same subgroup of $H_6/H_6(2)$, so that the intersection of each with $(H_6)_0$ is contained in $\mathbf{X}$.)

We now turn to the slightly more complicated case of G_6 in order to construct additional chamber-transitive groups. Let $M := \langle u_1, \ldots, u_6\rangle_{\mathbb{Z}[\frac{1}{3}]}$, so that $G_6(2)$ is the centralizer in G_6 of $M/2M$. Here, $M/2M$ inherits the quadratic form $x_1^2 + \cdots + x_5^2 + 3x_6^2 \bmod 2$, which is the square of a linear form whose kernel is the projection $\bmod 2M$ of the sublattice N of M defined by $\Sigma x_i \equiv 0 (\mathrm{mod}\ 2)$. Moreover, $N/2M$ inherits the quadratic form $\frac{1}{2}(x_1^2 + \cdots +$

$x_5^2 + 3x_6^2)$ mod 2, whose radical is the singular 1-space with representative Σu_i satisfying $\frac{1}{2}(\Sigma u_i, \Sigma u_i) = 4 \equiv 0 (\mathrm{mod}\, 2)$. Thus, G_6 induces a subgroup G_6/K of the orthogonal group $\mathrm{O}^-(4,2) \cong S_5$ of the resulting nonsingular 4-space (for a normal subgroup K of G_6). Since $({}^\circ G_6)_0$ has a subgroup S_5 acting faithfully on $M/2M$, it follows that $G_6/K \cong S_5$ and hence that $G_6 = K \cdot ({}^\circ G_6)_0$.

LEMMA 5. $G_6 = \{K \cap {}^\circ G_6\} \cdot (G_6)_0$, where $K \cap {}^\circ G_6 = K \cap H_6$ *contains* $H_6(2)$ *and is transitive on the vertices of type* 0.

Proof. Note that $K = \{K \cap {}^\circ G_6\} \langle r_6 \rangle$ and $(G_6)_0 = ({}^\circ G_6)_0 \langle r_6 \rangle$, where r_6 denotes the reflection in $u_6^\perp$. Then $G_6 = K \cdot ({}^\circ G_6)_0 = \{K \cap {}^\circ G_6\} \cdot (G_6)_0$.

It remains only to show that $K \cap {}^\circ G_6$ contains $H_6(2)$. First note that $2(L + M) = \langle \Sigma u_i \rangle + 2M$ and $(L \cap M) + 2M = N$, in view of the definition of the z_i. Thus, $H_6(2)$ centralizes $N/(\langle \Sigma u_i \rangle + 2M)$, and hence lies in K. We already know that it lies in ${}^\circ G_6$. □

We can now construct several groups acting chamber-transitively on Δ_6. Let Y denote any of the following subgroups of $(G_6)_0 = 2^6 S_5$, each containing –1:

$2^5 S_5$ (two classes, one of which is $({}^\circ G_6)_0$ and hence contains the reflection (12), while the other contains $(12)r_6$);

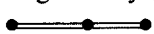
$2^5 A_5$ (this is inside $({}^\circ G_6)_0$);

$2^6 A_5$;

$2^6 F_{20}$;

$2^5 F_{20}$ (two classes, one of which lies in $({}^\circ G_6)_0$).

Then $\{K \cap {}^\circ G_6\} \cdot$ Y *is chamber-transitive on* Δ_6. Of course, some of these chamber-transitive groups were already encountered earlier, since $H_6(2) \leqslant K \cap {}^\circ G_6$.

Next, let m be an integer >2 not divisible by 3. Then $\Delta_6/G_6(m)$ is a finite morphic image of Δ_6 (cf. [15]), and is a geometry (by [1]) with chamber-transitive group $G_6/G_6(m)$ and diagram . If $m = p$ is a prime >3 then $G_6/G_6(p)$ is a subgroup of $\mathrm{O}^\pm(6,p)$ properly containing $\Omega^\pm(6,p)$ (by [16], since the group induced on the vector space M/pM is irreducible). The sign $\pm$ is $+$ if and only if -3 is a square in GF(p), i.e., if and only if $p \equiv 1 (\mathrm{mod}\, 3)$.

Similar remarks hold for $H_6/H_6(m)$ as well – and, for that matter, for each of the chamber-transitive automorphism groups of Δ_6 obtained above. However, *all of the resulting finite quotients of* Δ_6 *are isomorphic for a given* m. Namely, each inclusion map between two of these groups induces an isomorphism of finite geometries.

Incidentally, $G_6/G_6(2) \cong 2^4 S_5$. Namely, we saw above that $G_6/G_6(2)$ has S_5 as a homomorphic image. Also, θ acts nontrivially on $N/2M$. From this it follows readily that $\langle \theta \rangle G_6/G_6(2) \cong 2^5 S_5$.

Using the spinor norm, it is easy to see that $G_6' = \langle (G_6)_0, (G_6)_0^\theta \rangle'$ is the chamber-transitive group generated by $((G_6)_0)'$ (isomorphic to $2^4 A_5$) and its image under θ. In particular, G_6' is the intersection of G_6 with the kernel of the restriction of the spinor norm to $\mathrm{SO}(f, \mathbb{Q}_3)$. By [5], every finite homomorphic image of G_6' is a homomorphic image of some group $G_6'/G_6'(m)$. Thus, all finite quotients of Δ_6 on which G_6' induces a chamber-transitive group are obtained by factoring out suitable normal subgroups of $G_6'/G_6'(m)$. Similar statements hold if H_6' is used in place of G_6'.

Finally, we note that the group $\langle G_6, H_6 \rangle$ is chamber-transitive on Δ_6 but is not discrete. For, neither group G_6, H_6 contains the other; and $W(E_6)$ is a maximal finite subgroup of $O(f_6, \mathbb{Q})$ generated by reflections. In fact, the group $\langle (H_6)_0, r_6 \rangle$, which stabilizes $[\Lambda_0]$, is known to contain $O(e_6, \mathbb{Z}[\frac{1}{2}])'$ as a normal subgroup [7].

5. The 5-dimensional discrete groups

In this section we will briefly sketch results analogous to those of Sections 3 and 4 for the 5-dimensional subspace $V_5 := \langle u_1, u_2, u_3, u_4, u_5 \rangle$ of the space V_6 studied in those sections. Note that $V' := V_5 \otimes_{\mathbb{Q}} \mathbb{Q}_3$ is spanned by the vectors e_1, e_2, f_1, f_2, u_5.

Let f' denote the restriction of f to V_5 or V'. The corresponding affine building Δ_5 of $\Omega(f', \mathbb{Q}_3)$ is defined in Section 2.

This time we will use the group G_5 consisting of all elements of $O(f', \mathbb{Q}_3) \cap \mathrm{GL}(5, \mathbb{Z}[1/3])$ having spinor norm in $\{\pm 1\}(\mathbb{Q}_3)^2$, where matrices are written with respect to the u_i. As in Section 2, let $\Lambda_0' = \langle e_1, e_2, f_1, f_2, u_5 \rangle_{\mathbb{Z}_3}$, and note that $\Lambda_0' = \langle u_1, u_2, u_3, u_4, u_5 \rangle_{\mathbb{Z}_3}$. Let r denote the reflection in $(u_1 + u_2 + u_3)^\perp = (e_1 - 3f_1)^\perp$.

Precisely as in Section 3, we find that G_5 *is discrete and chamber-transitive on* Δ_5, and *the stabilizer* $(G_5)_0$ *of* $[\Lambda_0']$ *is the monomial group* $2^5 S_5$. *Moreover* $G_5 = \langle (G_5)_0, (G_5)_0^r \rangle$.

We have $[\Lambda_i'] = [\Sigma_i' \otimes_{\mathbb{Z}} \mathbb{Z}_3]$ for $i = 1, 2, 3$, where

$$\Sigma_0' := \langle u_1, \ldots, u_5 \rangle_{\mathbb{Z}}, \Sigma_2' := \Sigma_0'^r,$$

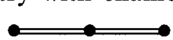
and

$$\Sigma_1' := \langle u_1 + u_2 + u_3 \rangle_{\mathbb{Z}} \perp \langle u_1 - u_2 + u_4 \rangle_{\mathbb{Z}} \perp \langle u_1 - u_3 - u_4 \rangle_{\mathbb{Z}} \perp \\ \perp \langle u_2 - u_3 + u_4 \rangle_{\mathbb{Z}} \perp \langle 3u_5 \rangle_{\mathbb{Z}}.$$

Here, $(\Sigma_0' + \Sigma_2')/3\Sigma_0'$ is the totally singular 1-space of $\Sigma_0'/3\Sigma_0'$ spanned by $u_1 + u_2 + u_3 + 3\Sigma_0'$, while $\Sigma_1'/3\Sigma_0'$ is the totally singular 2-space spanned by

$u_1 + u_2 + u_3 + 3\Sigma'_0$ and $u_1 - u_2 + u_4 + 3\Sigma'_0$. Moreover, $(G_5)_1 \cong (2^3 S_4) \times 2$, while $(G_5)_0 \cap (G_5)_1 = GL(2,3) \times 2$ and $(G_5)_1 = [(G_5)_0 \cap (G_5)_1][(G_5)_2 \cap (G_5)_1]$. Since r fixes Σ'_1 it normalizes $(G_5)_1$.

Let $L' = \langle u_1, \ldots, u_5 \rangle_{\mathbb{Z}[1/3]}$. Then G_5 acts on $L'/2L'$, and hence also acts on the kernel N' of the linear form $(u + 2L', u + 2L') \bmod 2$. The quadratic form $\frac{1}{2}(u + 2L', u + 2L') \bmod 2$ induced on N' is nondegenerate, with Witt index 1 and orthogonal group $O^-(4,2) \cong S_5$ induced by $(G_5)_0$. It follows that $G_5 = G_5(2) \cdot (G_5)_0$, where $E := G_5(2) \cap (G_5)_0$ consists of 2^5 diagonal matrices. *If* $\mathbf{X} = A_5$ *or* $F_{20} < S_5$ *then* $G_5(2)\mathbf{X}$ *is chamber-transitive on* Δ_5. Since $r \equiv \mathrm{I}$ (mod 2) it centralizes $G_5/G_5(2)$, and hence $G_5(2)\mathbf{X} = \langle \mathrm{EX}, (\mathrm{EX})^r \rangle$. Note that $G'_5 \times \langle -1 \rangle = G_5(2)((G_5)_0)' = G_5(2)A_5$.

This time $\Delta_5/G_5(m)$ is a finite geometry with chamber-transitive automorphism group $G_5/G_5(m)$ and diagram  whenever $m > 2$ is not divisible by 3. If $m = p$ is prime, then $G_5/G_5(p) \cong O(5,p)$. (Namely, $(12) \in G_5$, so that [16] applies.) Once again, by [5] this produces all finite quotients of Δ_5 on which G'_5 induces a chamber-transitive group.

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